M1 General Physics **2021-2022**
Major PNU

Particles

Final exam: session 2

March 24th 2022

Documents allowed

Notes:
- Space coordinates may be freely denoted as (x, y, z) or (x^1, x^2, x^3).
- One may always assume that fields are rapidly decreasing at infinity.

1 Muon decay

A muon is a heavy lepton, of mass $m_\mu = 105$ MeV. Its mean life-time is $\tau = \frac{1}{\lambda} = 2.2\,10^{-6}$s. Muons are created in the upper atmosphere when cosmic rays collide with air molecules.

1. Consider a muon of energy 17 GeV. What fraction of the light velocity does it carry, as seen by an observer on Earth?

Solution

From $E_\mu = \gamma_\mu\, m_\mu$ one gets $\gamma_\mu \simeq 162$. From the expression

$$\gamma_\mu = \frac{1}{\sqrt{1 - \frac{v_\mu^2}{c^2}}}$$

one gets

$$v_\mu = c\frac{\sqrt{\gamma_\mu^2 - 1}}{\gamma_\mu} \simeq 0.99998\, c.$$

---

2. What is the mean life-time of such a muon, again as seen by an observer on Earth?

Solution

From the time dilation formula, one has $\tau_{\text{Earth}} = \gamma_\mu\, \tau_\mu \simeq 3.56 \times 10^{-4}$.

---

3. Out of a million particles produced at altitude 50 km with the above energy, how many will reach the Earth before decaying?

Solution

The muons travel a distance $d = 50$ km, at a velocity $0.99998\,c$. This takes a time $t_{\rm Earth} = d/v_\mu = 5\times 10^4/(0.99998\times 3\times 10^8) \simeq 1.67\times 10^{-4}$s. Using the formula describing the decay of muons in the Earth frame

$$N(t) = N(0) e^{-t_{\rm Earth}/\tau_{\rm Earth}}$$

one thus gets, at ground level,

$$N_{\rm ground} = 10^6 \times e^{-1.67\times 10^{-4}/3.56\times 10^{-4}} \simeq 6.3\times 10^5.$$

---

4. Compare this result with the one obtained in a non-relativistic treatment. Comment.

Solution

In a non-relativistic treatment, there is no time dilation, therefore

$$N_{\rm non-relativistic}(t) = N(0) e^{-t_{\rm Earth}/\tau_\mu}$$

and one thus gets, at ground level,

$$N_{\rm ground,\ non-relativistic} = 10^6 \times e^{-1.67\times 10^{-4}/2.2\times 10^{-6}} \simeq 1.2\times 10^{-27}$$

i.e 0!!

---

2 Nonrelativistic Lagrangian

The complex scalar field $\psi(\vec{r},t)$ in the nonrelativistic approximation has a Lagrangian

$$\mathcal{L} = \frac{i\hbar}{2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) - \frac{\hbar^2}{2m}|\vec{\nabla}\psi|^2 - U|\psi|^2 \qquad (1)$$

where $m > 0$ is the particle mass, $\hbar$ the reduced Planck constant and $U(\vec{r},t)$ is the potential field in which the particle propagates.

1. Derive the two equations of motion for $\psi(\vec{r},t)$ and $\psi^*(\vec{r},t)$ from the Lagrangian (1) and interpret them.

Solution

Treating ψ and ψ^* as two independent fields, the two Euler Lagrange equations read

$$\frac{\delta\mathcal{L}}{\delta\psi} = \frac{d}{dt}\frac{\delta\mathcal{L}}{\delta\partial_t\psi} + \partial_i\frac{\delta\mathcal{L}}{\delta\partial_i\psi}$$

and

$$\frac{\delta\mathcal{L}}{\delta\psi^*} = \frac{d}{dt}\frac{\delta\mathcal{L}}{\delta\partial_t\psi^*} + \partial_i\frac{\delta\mathcal{L}}{\delta\partial_i\psi^*}\,.$$

First,

$$\frac{\delta \mathcal{L}}{\delta \psi} = -\frac{i\hbar}{2}\partial_t \psi^* - U\psi^* \quad \text{and} \quad \frac{\delta \mathcal{L}}{\delta \psi^*} = \frac{i\hbar}{2}\partial_t \psi^* - U\psi \,.$$

Second,

$$\frac{\delta \mathcal{L}}{\delta \partial_t \psi} = \frac{i\hbar}{2}\psi^* \quad \text{i.e.} \quad \frac{d}{dt}\frac{\delta \mathcal{L}}{\delta \partial_t \psi} = \frac{i\hbar}{2}\partial_t \psi^*,$$

and

$$\frac{\delta \mathcal{L}}{\delta \partial_t \psi^*} = -\frac{i\hbar}{2}\psi \quad \text{i.e.} \quad \frac{d}{dt}\frac{\delta \mathcal{L}}{\delta \partial_t \psi^*} = -\frac{i\hbar}{2}\partial_t \psi \,.$$

Third,

$$\frac{\delta \mathcal{L}}{\delta \vec{\nabla} \psi} = -\frac{\hbar^2}{2m}\vec{\nabla}\psi^* \quad \text{i.e.} \quad \partial_i \frac{\delta \mathcal{L}}{\delta \partial_i \psi} = \vec{\nabla}\frac{\delta \mathcal{L}}{\delta \vec{\nabla} \psi} = -\frac{\hbar^2}{2m}\Delta\psi^*,$$

and

$$\frac{\delta \mathcal{L}}{\delta \vec{\nabla} \psi^*} = -\frac{\hbar^2}{2m}\vec{\nabla}\psi \quad \text{i.e.} \quad \partial_i \frac{\delta \mathcal{L}}{\delta \partial_i \psi^*} = \vec{\nabla}\frac{\delta \mathcal{L}}{\delta \vec{\nabla} \psi^*} = -\frac{\hbar^2}{2m}\Delta\psi,$$

so that finally one gets the two complex conjugated equations

$$-i\hbar\partial_t \psi^* = -\frac{\hbar^2}{2m}\Delta\psi^* + U\psi^*$$

and

$$i\hbar\partial_t \psi = -\frac{\hbar^2}{2m}\Delta\psi + U\psi$$

which is the Schrödinger equation and its complex conjugated form.

---

2. Hamiltonian
a. Construct the Hamiltonian function density $\mathcal{H}$ using the Lagrangian.

Solution

One should perform a Legendre transformation of the Lagrangian density, i.e.

$$\mathcal{H} = \frac{\delta \mathcal{L}}{\partial_t \psi}\partial_t \psi + \frac{\delta \mathcal{L}}{\partial_t \psi^*}\partial_t \psi^* - \mathcal{L} = \frac{\hbar^2}{2m}|\vec{\nabla}\psi|^2 + U|\psi|^2$$

---

b. Calculate the total field energy and comment.

Solution

The total field energy is

$$H=\int\mathcal{H}\,d^3x=\int\psi\left(-\frac{\hbar^2}{2m}\nabla^2+U\right)\psi\,d^3x\,,\tag{2}$$

where we have used a partial integration on the assumption that $\psi\to 0$ as $r\to\infty$. It coincides with the quantum mechanical average value of the particle's energy.

---

c. Propose the quantum interpretation of the result thus obtained, in the case of a time independent potential.

Solution

If the potential energy $U(\vec{r})$ is time independent and the particle is in a state with a definite energy, that is, $\hat{H}\psi=E\psi$ with the usual quantum mechanical Hamiltonian operator

$$\hat{H}=-\frac{\hbar^2}{2m}\nabla^2+U\,,$$

obviously one gets $H=E$. Thus, the energy calculated by the formulas of field theory as the integral over the entire three-dimensional space of the energy density coincides with the energy of a quantum particle.

---

3. The Schrödinger equation for the wave function $\psi(\vec{r},t)$ of a spin-free nonrelativistic particle of charge q has the form

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\left[\vec{\nabla}-\frac{iq}{\hbar c}\vec{A}(\vec{r},t)\right]^2\psi+q\,\varphi(\vec{r},t)\,\psi\tag{3}$$

where $\vec{A}(\vec{r},t)$ and $\varphi(\vec{r},t)$ are the electromagnetic potentials (specified real functions).

a. Guess the Lagrangian leading to (3).
Hint: rely on the covariant derivative $\vec{D}=\vec{\nabla}-\frac{iq}{\hbar c}\vec{A}$ and on the guess of the potential U.

Solution

Obviously, one should consider the Lagrangian obtained from (1) by performing the minimal replacement $\vec{\nabla}\to\vec{D}$, and $U=q\varphi$, i.e.

$$\begin{aligned}\mathcal{L}&=\frac{i\hbar}{2}\left(\psi^*\partial_t\psi-\psi\partial_t\psi^*\right)-\frac{\hbar^2}{2m}\left|\vec{\nabla}\psi-\frac{iq}{\hbar c}\vec{A}\psi\right|^2-q\varphi|\psi|^2\\&=\frac{i\hbar}{2}\left(\psi^*\partial_t\psi-\psi\partial_t\psi^*\right)-\frac{\hbar^2}{2m}\left(\vec{\nabla}\psi^*+\frac{iq}{\hbar c}\vec{A}\psi^*\right)\cdot\left(\vec{\nabla}\psi-\frac{iq}{\hbar c}\vec{A}\psi\right)-q\varphi\,\psi^*\psi\,.\end{aligned}$$

---

b. Show in detail that the equation of motion of this Lagrangian is indeed the Schrödinger equation (3).

Solution

First,

$$\frac{\delta\mathcal{L}}{\delta\psi} = -\frac{i\hbar}{2}\partial_t\psi^* + \frac{\hbar^2}{2m}\frac{iq}{\hbar c}\left(\vec{\nabla}\psi^* + \frac{iq}{\hbar c}\vec{A}\psi^*\right)\cdot\vec{A} - q\varphi\,\psi^*$$

and

$$\frac{\delta\mathcal{L}}{\delta\psi^*} = \frac{i\hbar}{2}\partial_t\psi^* - \frac{\hbar^2}{2m}\frac{iq}{\hbar c}\left(\vec{\nabla}\psi - \frac{iq}{\hbar c}\vec{A}\psi\right)\cdot\vec{A} - q\varphi\,\psi\,.$$

Second,

$$\frac{\delta\mathcal{L}}{\delta\partial_t\psi} = \frac{i\hbar}{2}\psi^* \quad \text{i.e.} \quad \frac{d}{dt}\frac{\delta\mathcal{L}}{\delta\partial_t\psi} = \frac{i\hbar}{2}\partial_t\psi^*,$$

and

$$\frac{\delta\mathcal{L}}{\delta\partial_t\psi^*} = -\frac{i\hbar}{2}\psi \quad \text{i.e.} \quad \frac{d}{dt}\frac{\delta\mathcal{L}}{\delta\partial_t\psi^*} = -\frac{i\hbar}{2}\partial_t\psi\,.$$

Third,

$$\frac{\delta\mathcal{L}}{\delta\vec{\nabla}\psi} = -\frac{\hbar^2}{2m}\left(\vec{\nabla}\psi^* + \frac{iq}{\hbar c}\vec{A}\psi^*\right) \quad \text{i.e.} \quad \partial_i\frac{\delta\mathcal{L}}{\delta\partial_i\psi} = \vec{\nabla}\frac{\delta\mathcal{L}}{\delta\vec{\nabla}\psi} = -\frac{\hbar^2}{2m}\Delta\psi^* - \frac{\hbar^2}{2m}\frac{iq}{\hbar c}\vec{\nabla}\cdot\left(\vec{A}\psi^*\right),$$

and

$$\frac{\delta\mathcal{L}}{\delta\vec{\nabla}\psi^*} = -\frac{\hbar^2}{2m}\left(\vec{\nabla}\psi - \frac{iq}{\hbar c}\vec{A}\psi\right) \quad \text{i.e.} \quad \partial_i\frac{\delta\mathcal{L}}{\delta\partial_i\psi^*} = \vec{\nabla}\frac{\delta\mathcal{L}}{\delta\vec{\nabla}\psi^*} = -\frac{\hbar^2}{2m}\Delta\psi + \frac{\hbar^2}{2m}\frac{iq}{\hbar c}\vec{\nabla}\cdot\left(\vec{A}\psi\right),$$

Combining these results, the Euler-Lagrange equations read

$$-\frac{i\hbar}{2}\partial_t\psi^* + \frac{\hbar^2}{2m}\frac{iq}{\hbar c}\left(\vec{\nabla}\psi^* + \frac{iq}{\hbar c}\vec{A}\psi^*\right)\cdot\vec{A} - q\varphi\,\psi^* = \frac{i\hbar}{2}\partial_t\psi^* - \frac{\hbar^2}{2m}\Delta\psi^* - \frac{\hbar^2}{2m}\frac{iq}{\hbar c}\vec{\nabla}\cdot\left(\vec{A}\psi^*\right)$$

and

$$\frac{i\hbar}{2}\partial_t\psi - \frac{\hbar^2}{2m}\frac{iq}{\hbar c}\left(\vec{\nabla}\psi - \frac{iq}{\hbar c}\vec{A}\psi\right)\cdot\vec{A} - q\varphi\,\psi = -\frac{i\hbar}{2}\partial_t\psi - \frac{\hbar^2}{2m}\Delta\psi + \frac{\hbar^2}{2m}\frac{iq}{\hbar c}\vec{\nabla}\cdot\left(\vec{A}\psi\right)$$

Besides, since

$$\begin{aligned} \left[\vec{\nabla} - \frac{ie}{\hbar c}\vec{A}(\vec{r},t)\right]^2\psi &= \left[\vec{\nabla} - \frac{iq}{\hbar c}\vec{A}(\vec{r},t)\right]\left[\vec{\nabla} - \frac{iq}{\hbar c}\vec{A}(\vec{r},t)\right]\psi \\ &= \Delta\psi - \frac{q^2}{\hbar^2c^2}\vec{A}^2\psi - \frac{iq}{\hbar c}\vec{\nabla}(\vec{A}\psi) - \frac{iq}{\hbar c}\vec{A}\cdot\vec{\nabla}\psi \end{aligned}$$

the Schrödinger equation can be rewritten as

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\left[\Delta\psi - \frac{q^2}{\hbar^2c^2}\vec{A}^2\psi - \frac{iq}{\hbar c}\vec{\nabla}(\vec{A}\psi) - \frac{iq}{\hbar c}\vec{A}\cdot\vec{\nabla}\psi\right] + q\,\varphi\,\psi$$

while its complex conjugated form reads

$$-i\hbar\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\left[\Delta\psi^* - \frac{q^2}{\hbar^2 c^2}\vec{A}^2\psi^* + \frac{iq}{\hbar c}\vec{\nabla}(\vec{A}\psi^*) + \frac{iq}{\hbar c}\vec{A}\cdot\vec{\nabla}\psi^*\right] + q\,\varphi\,\psi^*.$$

These two equations are exactly the Euler-Lagrange equations obtained above.