**M1 General Physics** **2021-2022**
**Major PNU**

*Particles*

**Mid-term exam**

October 26th 2021

Documents allowed

**Relativistic Doppler effect and aberration of light**

*Notes:*
- **The subject is deliberately long**. It is not requested to reach the end to get a good mark!
- Space coordinates may be freely denoted as $(x, y, z)$ or $(x^1, x^2, x^3)$.
- Any drawing, at any stage, is welcome, and will be rewarded!

# 1 Particle of arbitrary mass

We consider two frames $K$ and $K'$. The frame $F'$ moves with respect to the frame $F$ with a velocity $\vec{v}_{F'/F} = \vec{\beta}c$ along the $z$ axis.

1. Assuming that a particle has a velocity $\vec{v}$ in the frame $F$, show that the velocity $\vec{v}\,'$ in the frame $F'$ is given by

$$v^{3\prime} = \frac{v^3 - \beta\, c}{1 - \beta \frac{v^3}{c}}\,, \tag{1}$$

$$v^{1\prime} = \frac{1}{\gamma} \frac{v^1}{1 - \beta \frac{v^3}{c}}\,, \tag{2}$$

and similarly for $v^{2\prime}$.
*Hint:* consider the differential of a boost, and use the fact that

$$\forall i \in \{1, 2, 3\}, \quad v^i = c\frac{dx^i}{dx^0} \quad \text{and} \quad v^{i\prime} = c\frac{dx^{i\prime}}{dx^{0\prime}}\,.$$

*Solution*

A pure boost of rapidity $\phi$ along $z$ can be written as

$$\begin{cases} x^{0\prime} = \gamma\, x^0 - \gamma\beta\, x^3 \\ x^{3\prime} = -\gamma\beta\, x^0 + \gamma\, x^3. \end{cases}$$

We thus have, through differentiation,

$$\begin{cases} dx^{0\prime} = \gamma\, dx^0 - \gamma\beta\, dx^1 \\ dx^{3\prime} = -\gamma\beta\, dx^0 + \gamma\, dx^3 \end{cases}$$

which gives, since $dx^0 = cdt$ and $dx^{0\prime} = cdt'$,

$$v^{3\prime} = c\frac{dx^{3\prime}}{dx^{0\prime}} = c\frac{-\gamma\beta dx^0 + \gamma dx^3}{\gamma dx^0 - \gamma\beta dx^3} = \frac{v^3 - \beta\, c}{1 - \beta\frac{v^3}{c}}\,.$$

Besides,

$$v^{1\prime} = c\frac{dx^{1\prime}}{dx^{0\prime}} = c\frac{dx^1}{\gamma dx^0 - \gamma\beta dx^3} = \frac{1}{\gamma}\frac{v^1}{1 - \beta\frac{v^3}{c}}\,,$$

and similarly for $v^{2\prime}$.

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2. Consider a particle of velocity $\vec{v}$ in a frame $K$, with spherical coordinates $(v, \theta, \varphi)$ with respect to the above $z$-axis used to defined the boost from $F$ to $F'$. We now observe the particle in the frame $K'$. Its velocity $\vec{v}^{\,\prime}$ is described using the spherical coordinates $(v', \theta', \varphi')$ defined in the same $(x, y, z)$ reference system.

(a) Why does it make sense to use the same system of coordinates for both $F$ and $F'$ frames?

*Solution*

The boost from $F$ to $F'$ does not change the axis directions $x$, $y$ and $z$.

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(b) It is convenient to separate the transverse and the longitudinal velocity with respect to the direction $z$ of the boost, as $\vec{v} = \vec{v}_\parallel + \vec{v}_\perp$ where $\vec{v}_\parallel$ is along the $z-$axis and $\vec{v}_\perp$ is in the $xy$ plane. Express $\vec{v}_\parallel^{\,\prime}$ and $\vec{v}_\perp^{\,\prime}$ in terms of $\vec{v}_\parallel$, $\vec{v}_\perp$, $\beta$, $\gamma$ and $c$.

*Solution*

From the question 1, we get

$$\begin{aligned} \vec{v}_\perp^{\,\prime} &= \frac{1}{\gamma}\frac{\vec{v}_\perp}{1 - \frac{v_\parallel}{c}\beta} \\ v_\parallel' &= \frac{v_\parallel - \beta c}{1 - \frac{v_\parallel}{c}\beta} \end{aligned}$$

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(c) Change of the magnitude of the velocity.

(i) Show that

$$\left(1 - \frac{v'^2}{c^2}\right)\left(1 - \frac{v_\parallel}{c}\beta\right) = \left(1 - \frac{v^2}{c^2}\right)\left(1 - \beta^2\right). \tag{3}$$

*Solution*

Since

$$
\begin{aligned}
\vec{v}_\perp' &= \frac{1}{\gamma}\frac{\vec{v}_\perp}{1-\frac{v_\parallel}{c}\beta} \\
v_\parallel' &= \frac{v_\parallel - \beta c}{1-\frac{v_\parallel}{c}\beta}
\end{aligned}
$$

we have

$$
v'^{\,2} = \frac{1}{\left(1-\frac{v_\parallel}{c}\beta\right)^2}\left[v_\perp^2(1-\beta^2) + (v_\parallel - \beta c)^2\right]
$$

so that

$$
1-\frac{v'^{\,2}}{c^2} = \frac{1}{\left(1-\frac{v_\parallel}{c}\beta\right)^2}\left[\left(1-v_\parallel\frac{\beta}{c}\right)^2 - \frac{v_\perp^2}{c^2}(1-\beta^2) - \left(\frac{v_\parallel}{c}-\beta\right)^2\right]
$$

and thus

$$
\left(1-\frac{v'^{\,2}}{c^2}\right)\left(1-\frac{v_\parallel}{c}\beta\right)^2 = 1-\beta^2 - \frac{v_\parallel^2}{c^2}(1-\beta^2) - \frac{v_\perp^2}{c^2}(1-\beta^2) = \left(1-\frac{v^2}{c^2}\right)(1-\beta^2)\,.
$$

---

(ii) Show that if $v \leqslant c$, this remains true in any frame boosted by a velocity $\beta c \leqslant c$. What about the special case of $v = c$?

*Solution*

From Eq. (3) one immediately sees that $v' \leqslant c$ if $\beta \leqslant 1$ and $v \leqslant c$. In particular, if $v = c$, then $v' = c$ : the speed of light does not depend on the frame.

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(d) Transformation of the direction of the particle velocity.

(i) Write the expressions of $\vec{v}_\perp$ and $v_\parallel$ as functions of $v$, $\varphi$, $\theta$ in the frame $F$, and provide the corresponding expressions in the frame $F'$.

*Solution*

In spherical coordinates, on can write in the frame $F$:

$$
\begin{aligned}
\vec{v}_\perp &= v\,\sin\theta\,[\cos\varphi\,\vec{u}_x + \sin\varphi\,\vec{u}_y]\,, \\
v_\parallel &= v\,\cos\theta\,,
\end{aligned}
$$

and similarly in the frame $F'$:

$$
\begin{aligned}
\vec{v}_\perp' &= v\,\sin\theta'\,[\cos\varphi'\,\vec{u}_x + \sin\varphi'\,\vec{u}_y]\,, \\
v_\parallel' &= v\,\cos\theta'\,.
\end{aligned}
$$

---

(ii) What is the relation between $\varphi$ and $\varphi'$?

*Solution*

The boost changes the magnitude of $v_x$ and $v_y$ with the same amount, indeed $\vec{v}'_\perp$ and $\vec{v}_\perp$ are collinear, therefore $\varphi$ is invariant.

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(iii) Give the expression of $\sin\theta'$, $\cos\theta'$ as functions of $v$, $v'$, $\beta$ and $\gamma$. Check that

$$\tan\theta' = \frac{1}{\gamma}\frac{v\,\sin\theta}{v\,\cos\theta - \beta c}\,. \tag{4}$$

*Solution*

From the expressions obtained for $\vec{v}'_\parallel$ and $\vec{v}'_\perp$ one gets

$$\begin{aligned}
\sin\theta' &= \frac{1}{\gamma}\frac{v}{v'}\frac{\sin\theta}{1 - \frac{v}{c}\beta\cos\theta} \\
\cos\theta' &= \frac{v}{v'}\frac{\cos\theta - \beta c/v}{1 - \frac{v}{c}\beta\cos\theta}
\end{aligned}$$

from which the expression given for $\tan\theta'$ is obvious.

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## 2 Photon

We now consider the case of a photon. We again consider the two frames $F$ and $F'$ related by the same above boost along the $z-$axis.

1. Write the expression of $\sin\theta'$, $\cos\theta'$ and $\tan\theta'$. One should in particular check that

$$\tan\theta' = \frac{1}{\gamma}\frac{\sin\theta}{\cos\theta - \beta}\,. \tag{5}$$

The change of $\theta$ to $\theta'$ is known under the name of aberration of light.

*Solution*

The speed of light does not depend on the frame. One should just replace $v$ by $c$ in the expressions obtained in the previous question, which give

$$\begin{aligned}
\sin\theta' &= \frac{1}{\gamma}\frac{\sin\theta}{1 - \beta\cos\theta} \\
\cos\theta' &= \frac{\cos\theta - \beta}{1 - \beta\cos\theta}
\end{aligned}$$

and

$$\tan\theta' = \frac{1}{\gamma}\frac{\sin\theta}{\cos\theta - \beta}\,.$$

2. Boost of the photon momentum

(a) Write the way a photon of 4-momentum $k = (k^0, \vec{k})$ is transformed into $k' = (k^{0'}, \vec{k}')$ under the boost from $F$ to $F'$, separating $\perp$ and $\|$ components. Introduce the angles $\theta$ and $\theta'$.

*Solution*

First of all, for a photon, $k^0 = \|\vec{k}\|$. Therefore,

$$\begin{aligned} k^{0'} &= \gamma\, k^0 - \beta\,\gamma k^3 = k^0\gamma(1-\beta\cos\theta) \\ \vec{k}'_\perp &= \vec{k}_\perp \\ k'_\| &= -\beta\,\gamma\, k^0 + \gamma k_\| = k^0\gamma(\cos\theta - \beta) = k^{0'}\cos\theta' \end{aligned}$$

with

$$\begin{aligned} \|\vec{k}_\perp\| &= \|\vec{k}\|\sin\theta = k^0\sin\theta \\ \|\vec{k}'_\perp\| &= \|\vec{k}'\|\sin\theta' = k^{0'}\sin\theta'\,. \end{aligned}$$

(b) Relate $\theta'$ and $\theta$ and check the consistency with the results obtained in the question 2.1.

*Solution*

From the above result we get immediately

$$\begin{aligned} \sin\theta' &= \frac{1}{\gamma}\frac{\sin\theta}{1-\beta\cos\theta} \\ \cos\theta' &= \frac{\cos\theta-\beta}{1-\beta\cos\theta\,.} \end{aligned}$$

which are hopefully identical with the results obtained from the transformation of the velocity.

(c) Show that the change of frequency of the photon under the above boost is given by

$$\nu' = \nu\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}}\,. \qquad (6)$$

*Solution*

Since $k^0 = h\nu$ we get

$$\nu' = \nu\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}}\,.$$

3. A few particular cases

(a) Discuss the cases $\theta = 0$, $\theta = \frac{\pi}{2}$ and $\theta = \pi$, both for the aberration of light and for the frequency. Comment in each case whether there is a blue shift or a red shift.

*Solution*

Case $\theta = 0$ :

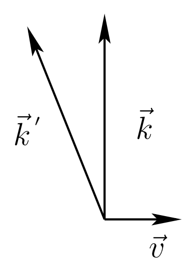
$$\nu' = \gamma(1-\beta)\nu = \nu\sqrt{\frac{1-\beta}{1+\beta}}$$

and $\theta' = \theta = 0$. The photon does not face aberration, and its frequency decreases, leading to a red-shift effect.

Case $\theta = \frac{\pi}{2}$ :

$$\nu' = \gamma\nu = \frac{\nu}{\sqrt{1-\beta^2}}$$

and $\sin\theta' = 1/\gamma$, $\cos\theta' = -\beta$ .




The photon faces aberration, and its frequency increases, leading to a blue-shift effect.

Case $\theta = \pi$ :

$$\nu' = \gamma\nu(1+\beta) = \nu\sqrt{\frac{1+\beta}{1-\beta}}$$

and $\theta' = \theta = \pi$. The photon does not face aberration, and its frequency increases, leading to a blue-shift effect.

(b) Show that there is a particular angle $\theta$ for which the photon frequency is unchanged. What is the aberration in this case?

*Solution*

Writing $\nu' = \nu$ leads to $\gamma(1-\beta\cos\theta) = 1$ so that $\sin\theta' = \sin\theta$. Besides, we have

$$\cos\theta = \frac{\gamma - 1}{\gamma\beta}\,.$$

We thus gets

$$\cos\theta' = \gamma(\cos\theta - \beta) = \frac{\gamma-1}{\beta} - \gamma\beta = \frac{\gamma(1-\beta^2)-1}{\beta} = \frac{1/\gamma - 1}{\beta} = \frac{1-\gamma}{\gamma\beta} = -\cos\theta$$

so that the light beam gets symmetrized with respect to the plane perpendicular to the $z-$axis. As expected, this angle $\theta$ is between 0 and $\pi/2$.

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4. Forward cone.

(a) Show that any light propagating outside a forward cone in the frame $F$ will appear to propagate backward in the frame $F'$. Express the half-opening angle of this cone in term of $\beta$.

*Solution*

We have $\cos\theta' < 0$ for $\theta$ satisfying $1 - \beta\cos\theta < 0$ i.e. $\theta > \theta_{\text{cone}} = \arccos\beta$. Thus, any photon propagating outside this forward cone will look like propagating backward.

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(b) Show that this half-opening angle in the ultra-relativistic limit $\beta \to 1$ is given by

$$\theta_{\text{cone}} \sim \frac{1}{\gamma}\,. \tag{7}$$

*Solution*

When $\beta \to 1$,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \sim \frac{1-\beta^2}{\beta} \sim \frac{1}{\gamma\beta} \sim \frac{1}{\gamma}$$

with $\tan\theta \sim \theta$ since $\theta \ll 1$, so that $\theta_{\text{cone}} \sim \frac{1}{\gamma}$.

---

(c) What would be the consequence for an hypothetical interstellar traveler moving at a speed close to the light speed?

*Solution*

Following the above discussion, in an hypothetical interstellar journey, the traveler would have the feeling to see the celestial sphere on his back to be just in front of him, because of the aberration of light, except for the part almost exactly on his back, so that he would have the feeling to move backward, toward his past landscape!

5. We consider the non-relativistic limit $\beta \ll 1$.

(a) Show that the aberration angle $\Delta\theta = \theta' - \theta$ is then given by

$$\Delta\theta = \beta \sin\theta\,. \tag{8}$$

*Solution*

In the limit $\beta \ll 1$, $\gamma \sim 1$, with a quadratic correction in $\beta$. Three possible reasonings:

(i) <u>from the expression of $\cos\theta'$ :</u>

On the one hand, from the differential of $\cos\theta$,

$$\Delta\cos\theta' \sim -\sin\theta\,\Delta\theta$$

and on the other hand

$$\cos\theta' \sim (\cos\theta - \beta)(1 + \beta\cos\theta) + o(\beta^2) \sim \cos\theta - \beta\,\sin^2\theta + o(\beta^2)$$

so that

$$\Delta\theta \sim \beta\,\sin\theta\,.$$

(ii) <u>from the expression of $\sin\theta'$ :</u>

On the one hand, from the differential of $\sin\theta$,

$$\Delta\sin\theta' \sim \cos\theta\,\Delta\theta$$

and on the other hand

$$\sin\theta' \sim \sin\theta(1 + \beta\cos\theta) + o(\beta)$$

and thus

$$\cos\theta\,\Delta\theta \sim \cos\theta\,\beta\sin\theta\,.$$

One should distinguish the particular situation $\theta = \frac{\pi}{2}$ (which as expected is a spurious singularity, see below the special discussion) from $\theta \neq \frac{\pi}{2}$:
if $\theta \neq \frac{\pi}{2}$,

$$\Delta\theta \sim \beta\,\sin\theta\,.$$

The case $\theta = \frac{\pi}{2}$ gives on the one hand

$$\Delta\sin\theta' \sim 1 - \frac{(\Delta\theta)^2}{2} + o\left((\Delta\theta)^2\right)$$

and on the other hand, keeping the quadratic term in $\beta$ coming from $\gamma$,

$$\sin\theta' \sim 1 - \frac{\beta^2}{2} + o(\beta^2)$$

i.e.

$$\Delta\theta \sim \beta$$

which enters the afore obtained general formula for $\Delta\theta$.

(iii) <u>from the expression of $\tan\theta'$:</u>

Again, a spurious singularity for $\theta = \frac{\pi}{2}$ appears. First, assuming $\theta$ different from $\pi/2$,

$$\tan\theta' \sim \frac{1}{\gamma}\frac{\sin\theta}{\cos\theta - \beta} \sim \tan\theta \frac{1}{1 - \frac{\beta}{\cos\theta}} \sim \tan\theta\left(1 + \frac{\beta}{\cos\theta}\right) \sim \tan\theta + \frac{\beta\,\sin\theta}{\cos^2\theta}\,.$$

Besides, from the differential of $\tan\theta$,

$$\Delta\tan\theta \sim \frac{\Delta\theta}{\cos^2\theta}$$

so that combining both expression reads

$$\Delta\theta \sim \beta\,\sin\theta\,.$$

The special case $\theta = \frac{\pi}{2}$ gives on the one hand

$$\tan\theta' \sim -\frac{1}{\Delta\theta}$$

and on the other hand

$$\tan\theta' \sim -\frac{1}{\beta}$$

so that

$$\Delta\theta \sim \beta$$

which enters the afore obtained general formula for $\Delta\theta$.

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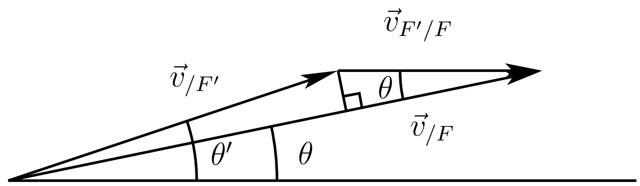
(b) Show that this result can be obtained when performing a classical galilean change of frame.

*Solution*

Starting from

$$\vec{v}_{/F} = \vec{v}_{/F'} + \vec{v}_{F'/F} \,,$$




we thus have, using the fact that in the non-relativistic limit $\beta c = v_{F'/F} = v \ll v_F = v_{F'} = c$,

$$v \sin\theta \sim |\theta' - \theta| c$$

so that

$$\Delta\theta \sim \frac{v}{c} \sin\theta \,.$$

Another way is to start from

$$\vec{v}_{/F} = \vec{v}_{/F'} + \vec{v}_{F'/F} \,,$$

which implies that

$$\vec{v}^{\,2}_{/F} = \vec{v}^{\,2}_{/F'} + 2\vec{v}_{/F'} \cdot \vec{v}_{F'/F} + \vec{v}^{\,2}_{F'/F}$$

and that

$$\|\vec{v}_{/F'} \wedge \vec{v}_{/F}\| = \|\vec{v}_{/F'}\| \|\vec{v}_{/F}\| |\sin(\theta' - \theta)| \,.$$

Besides, one has

$$\vec{v}_{/F} \wedge \vec{v}_{/F'} = \vec{v}_{F'/F} \wedge \vec{v}_{/F'} \,,$$

and thus

$$\|\vec{v}_{/F'} \wedge \vec{v}_{/F}\| = \|\vec{v}_{F'/F} \wedge \vec{v}_{/F'}\| = \|\vec{v}_{/F'}\| \|\vec{v}_{F'/F}\| \, |\sin\theta'| \,.$$

In the limit $\|\vec{v}_{/F}\| \gg \|\vec{v}_{F'/F}\|$ we therefore have $\|\vec{v}_{/F'}\| \sim \|\vec{v}_{/F}\|$ and equating the two obtained expressions for $\|\vec{v}_{/F'} \wedge \vec{v}_{/F}\|$, $|\sin(\theta' - \theta)| \sim |\theta' - \theta| \ll |\sin\theta'| \sim |\sin\theta|$ so that restoring the signs, we have algebraically ($v$ is the projection of $\vec{v}_{F'/F}$ on $\vec{u}_z$):

$$\Delta\theta \sim \frac{v}{c} \sin\theta' \sim \frac{v}{c} \sin\theta \,.$$

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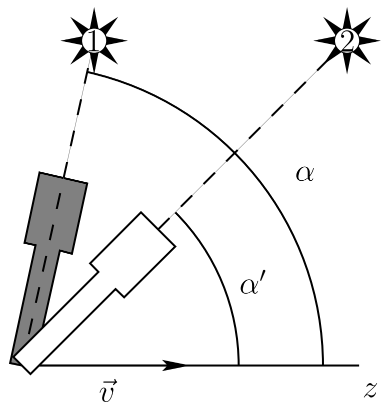
## 3 Aberration of light in astronomy (Bonus)

1. We assume that a star is at rest with respect to the Sun. A telescope points in the direction of this star. We denote the direction of the instant motion of Earth with respect to Sun as $z$.




In the rest frame of the Sun, the star is in the position (1), and $\alpha$ is the angle with respect to the axis $z$ toward which the telescope should be pointed if fixed with respect to the Sun. Due to the motion of Earth with respect to the Sun, the star seems to be in the position (2) and $\alpha'$ is the real angle (i.e. in the rest frame of the Earth at instant time) toward which the telescope should be pointed. Note: the average distance between Sun and Earth is $150 \cdot 10^6$ km (one astronomical unit).

(i) Discuss the importance of relativistic effect with respect to the classical one.

*Solution*

We have

$$v = R\omega = 150 \cdot 10^9 \frac{2\pi}{365 \times 24 \times 3600} \simeq 30 \cdot 10^3 \text{ m/s},$$

and thus $\beta = v/c = 30 \cdot 10^3/(3 \cdot 10^8) \simeq 1.0 \cdot 10^{-4}$, with

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \sim 1 + \frac{\beta^2}{2} \simeq 1 + \frac{10^{-8}}{2}$$

therefore the relativistic effects are completely negligible.

---

(ii) Show that the exact expression of $\tan\alpha'$ as observed by the telescope, as a function of $\alpha$ and $v_{\text{Earth}}$, is given by

$$\tan\alpha' = \frac{1}{\gamma} \frac{\sin\alpha}{\cos\alpha + \frac{v_{\text{Earth}}}{c}}. \tag{9}$$

*Solution*

Since light is now traveling from the star toward the telescope, one can simply perform the transformation

$$\begin{aligned} \theta &\to \pi - \alpha \\ \theta' &\to \pi - \alpha' \end{aligned}$$

in above obtained formula (5) for $\tan\theta'$ to get

$$\tan\alpha' = \frac{1}{\gamma}\frac{\sin\alpha}{\cos\alpha + \frac{v_{\mathrm{Earth}}}{c}}$$

---

(iii) Give a suitable approximation for $\alpha' - \alpha$.

*Solution*

From the approximated relation (8) and again using

$$\begin{aligned} \theta &\to \pi - \alpha \\ \theta' &\to \pi - \alpha' \end{aligned}$$

we get

$$\alpha - \alpha' \sim \frac{v_{\mathrm{Earth}}}{c}\sin\alpha \sim \frac{v_{\mathrm{Earth}}}{c}\sin\alpha'\,.$$

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2. We assume that the star is asymptotically far from the Sun. We recall that the *ecliptic* is the plane of Earth's orbit around the Sun.

(a) For which angle with respect to the direction of motion of Earth with respect to Sun is the aberration angle maximal? Compute numerically (as a precise fraction of degrees) this maximum aberration angle $\Delta\theta$.

*Solution*

The aberration is maximal for $\theta = \frac{\pi}{2}$. The aberration angle is then $\Delta\theta = \frac{v_{\mathrm{Earth}}}{c}$. One gets

$$\Delta\theta = 1.0 \cdot 10^{-4}\,\mathrm{rad} \simeq 0.0057^\circ \simeq 20,6''\,.$$

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(b) What is the numerical difference of the two observed angles for a star of maximal aberration when performing two measurements separated by half a year?

*Solution*

Half a year after a first measurement, $v_{\mathrm{Earth}}$ gets reversed, therefore $\Delta\theta$ is opposite, and thus there is roughly $41''$ between the two measured angles.

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(c) Describe the phenomena seen over a year in the case of a star:

(i) in the ecliptic, therefore with $0°$ ecliptic latitude.

*Solution*

In this case, the observed trajectory of the star is a segment of opening angle $41''$.

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(ii) at the pole of the ecliptic, therefore with $90°$ (north ecliptic pole) or $-90°$ ecliptic latitude (south ecliptic pole).

*Solution*

In this case, the observed trajectory of the star is a circle of opening angle $41''$.

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(iii) for an arbitrary star.

*Solution*

In the general case, both ecliptic latitude and ecliptic longitude oscillate with a period of one year, and an amplitude depending on the medium ecliptic latitude and longitude of the star, with a maximal amplitude of $20.5''$: therefore the observed variation of the position of the star is an ellipse.

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(c) The phenomenon of parallax is well known: a rather close object seems to move when you observe it either from your right eye or from you left eye. More generally, it is the difference in the apparent position of an object viewed along two different lines of sight. Compare the aberration effect discussed previously with the parallax effect due to the fact that the observation angle of a nearby star (typically of the order of a few light-year (ly) from us; Proxima Centaurus is at 4.24 ly) varies when observed at two times separated by half a year.

*Solution*

Approximating the tangent by the angle for small angles, the parallax angle $\alpha_p$ is given by

$$\alpha_p = \frac{d}{d_{Sun-Earth}} \simeq \frac{150 \cdot 10^9}{365 \times 24 \times 3600 \times 4.24 \times 3 \cdot 10^8} \simeq 3.7 \cdot 10^{-6}\,\mathrm{rad} \simeq 0.00021° \simeq 0.77''\,,$$

which is very small with respect to the aberration angle.

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(d) Explain why this allowed to perform one of the very first measurement of the speed of light (James Bradley, 1729), although limited by the precision on the Sun-Earth distance, by an astronomical measurement of $\gamma$-Draconis ($\sim 148$ light-year from us), which is almost at the north ecliptic pole.

*Solution*

The parallax effect was expected and looked for, as a clear proof of heliocentrism, already at the end of XVIth century. Several measurements were done. The first very first precise and convincing measurement was done by James Bradley, and looked incomprehensible. Let us see why.
With our modern understanding, it turns out that in each of the two directions (say semi-major and semi-minor axes), parallax is maximal when aberration vanishes. Indeed in the approximation of a circular orbit, which is almost the case for the movement of Earth around Sun, the speed of Earth (responsible for aberration) and Sun-Earth radial direction (responsible for parallax) are orthogonal to each other. As we have seen, the parallax effect never exceeds $1''$, which is negligible with respect to the aberration angle, of $20''$. For far distant stars, like $\gamma$-Draconis, there is even no parallax effect, but aberration remains, and are thus odd with parallax. This was exactly the outcome of the measurement done by James Bradley. It took him several years before he could understand and interpret correctly his observations. After he correctly interpreted his measurements, he could then extract the speed of light, as function of the Sun-Earth distance.