M1 General Physics **2020-2021**
Major PNU

Particles

Mid-term exam

Documents allowed

Notes:
- **The subject is deliberately long**. It is not requested to reach the end to get a good mark!
- For convenience, one may freely put $c = 1$ everywhere.
- Space coordinates maybe freely denoted as (x, y, z) or (x^1, x^2, x^3).
- The standard notation according to which a quantity with superscript * is measured in the center-of-mass frame will be used.

Any process is characterized by the scattering amplitude which gives the amplitude of probability to pass from a given initial state to another given final state. Knowing the phase space for the initial and final states, the modulus square of this amplitude, and the flux of initial particles, one can compute the cross-section of the process, which is an experimental observable. Our purpose here is obtain several generic properties of the flux and of the amplitude.

1 Flux

Considering the scattering of a beam on a target, the flux term accounts for the fact that a target has a given number density, and that the beam has a given number density, made of particles of type B with a velocity v_B.
More generally, for a head-on scattering (let us say along the z axis), the masses, velocities, energies of particles A and B being denoted as m_A, $\vec{p}_A$, E_A and m_B, $\vec{p}_B$, E_B, it can be shown that this flux factor reads

$$2K = |\vec{v}_A - \vec{v}_B|\, 2E_A\, 2E_B\,. \tag{1}$$

1. Consider a boost along the z axis, the new frame F' moving with respect to initial one F with a velocity $\vec{\beta}c$. As usual, the rapidity ϕ of this boost is defined through the relation $\beta = \tanh\phi$.

(a) Give the expression of γ and $\gamma\beta$ as hyperbolic trigonometric functions of the rapidity ϕ. *Hint:* $\cosh^2\phi - \sinh^2\phi = 1\,.$

(b) Write this boost using ϕ and then using β and γ.

2. Prove that under such a boost, the velocity of a particle transforms as

$$v^{1\prime} = \frac{1}{\gamma}\frac{v^1}{1-\beta\frac{v^3}{c}} \quad \text{and} \quad v^{2\prime} = \frac{1}{\gamma}\frac{v^2}{1-\beta\frac{v^3}{c}}\,, \tag{2}$$

$$v^{3\prime} = \frac{v^3 - \beta\, c}{1-\beta\frac{v^3}{c}}\,. \tag{3}$$

Hint: consider the differential of a boost.
Discuss the non-relativistic limit $v \ll c,\ \beta \ll 1$ and comment.

3. Show that the flux factor $2K$ given by Eq. (1) is invariant under boosts along the z axis.

4. Prove that the flux factor can be expressed as

$$2K = 4(E_B|\vec{p}_A| + E_A|\vec{p}_B|)\,. \tag{4}$$

5. Demonstrate that

$$2K = 4\sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2}\,. \tag{5}$$

6. In the center-of-mass frame, show that

$$2K = 4\,p_i^*\,W^*\,, \tag{6}$$

where $W^* = E_A^* + E_B^*$ is the total center of mass energy.

2 Mandelstam variables

Any 2 body $\to$ 2 body scattering between particles P_1, P_2, producing particles P_3 and P_4,

$$P_1(p_1)\,P_2(p_2) \to P_3(p_3)\,P_3(p_4) \tag{7}$$

is completely characterized, if one does not take into account spin effects, by the Mandelstam variables defined by

$$\boxed{\begin{array}{lllll} s & \equiv & (p_1+p_2)^2 & = & (p_3+p_4)^2\,, \\ t & \equiv & (p_1-p_3)^2 & = & (p_2-p_4)^2\,, \\ u & \equiv & (p_1-p_4)^2 & = & (p_2-p_3)^2\,, \end{array}} \tag{8}$$

the various equivalent expressions coming from energy-momentum conservation. These variables are illustrated on Fig. 1. Each of these variables can be considered as a "s"-variable for

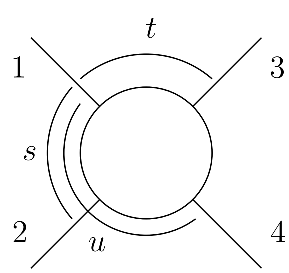

Figure 1: Mandelstam variables for a 2 body $\to$ 2 body process.

a crossed-channel process:

		"s"-variable	"t"-variable	"u"-variable
s-channel:	$1+2 \to 3+4$	s	t	u
t-channel:	$1+\bar{3} \to \bar{2}+4$	t	s	u
$u-$channel:	$1+\bar{4} \to 3+\bar{2}$	u	t	s

(9)

Indeed, a particle i of momentum p_i with $p^0 > 0$ should be considered as its antiparticle $\bar{i}$, of momentum $-p_i$ when $p^0 < 0$. The *same amplitude* $\mathcal{M}(s,\, t,\, u)$ thus describes these 3 reactions, as well as every desintegration process 1 body → 3 body (for example $1 \to \bar{2}+3+4$) and every back reaction (for example $3+4 \to 1+2$), by analytic continuation on variables s, t, u.

1. For further use, we denote, using the fact that $2p_1 \cdot p_2 = s - m_1^2 - m_2^2$,

$$\lambda(s, m_1^2, m_2^2) = 4[(p_1 \cdot p_2)^2 - m_1^2 m_2^2]\,. \tag{10}$$

We thus have

$$K(s) = \sqrt{\lambda(s, m_1^2, m_2^2)}\,. \tag{11}$$

(a) Show that

$$\lambda(s,\, m_1^2,\, m_2^2) = [s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]\,. \tag{12}$$

(b) Give the expression of $\lambda(s, 0, M)$ when one mass vanishes, and of $\lambda(s, 0, 0)$ when both vanish.

(c) Show that in the center-of-mass frame, $p_i^* = |\vec{p}_1| = |\vec{p}_2|$ and $p_f^* = |\vec{p}_3| = |\vec{p}_4|$ have very simple relativistic invariant forms:

$$p_i^* = \frac{\sqrt{\lambda(s,\, m_1^2,\, m_2^2)}}{2\sqrt{s}}\,, \tag{13}$$

and

$$p_f^* = \frac{\sqrt{\lambda(s,\, m_3^2,\, m_4^2)}}{2\sqrt{s}}\,. \tag{14}$$

2. Prove that

$$s + t + u = \sum_i m_i^2\,. \tag{15}$$

Hint: compute $2(s+t+u)$ is a "democratic way", using Eq. (8).

Consequently, the scattering amplitude only depends on two independent variables. One conventionally writes

$$\mathcal{M} = \mathcal{M}(s,\, t)\,.$$

3. (a) In the center-of-mass frame, show that

$$\begin{aligned} E_1^* &= \frac{s + m_1^2 - m_2^2}{2\sqrt{s}}\,, \qquad & E_3^* &= \frac{s + m_3^2 - m_4^2}{2\sqrt{s}}\,, \\ E_2^* &= \frac{s + m_2^2 - m_1^2}{2\sqrt{s}}\,, \qquad & E_4^* &= \frac{s + m_4^2 - m_3^2}{2\sqrt{s}}\,. \end{aligned} \tag{16}$$

Hint: use Eqs. (13) and (14).

(b) Prove that the following threshold conditions should be satisfied, in each indicated channel:

$$
\begin{aligned}
&s\text{-channel:} \quad 1+2 \to 3+4 : \quad s \geqslant (m_1+m_2)^2 \ \text{ and } \ (m_3+m_4)^2 \\
&t\text{-channel:} \quad 1+\bar{3} \to \bar{2}+4 : \quad t \geqslant (m_1+m_3)^2 \ \text{ and } \ (m_2+m_4)^2 \qquad (17) \\
&u\text{-channel:} \quad 1+\bar{4} \to 3+\bar{2} : \quad u \geqslant (m_1+m_4)^2 \ \text{ and } \ (m_2+m_3)^2
\end{aligned}
$$

4. The diffusion angle is by definition the scattering angle between particles 1 and 3, i.e. the scattering angle in the $s-$channel.

(a) Prove that

$$\cos\theta = \frac{t - m_1^2 - m_3^2 + 2\,E_1 E_3}{2\,|\vec{p}_1||\vec{p}_3|}\,. \qquad (18)$$

(b) In an arbitrary reference frame, for fixed s and E_1, explain why the discussion on the maximal/minimal values of $\cos\theta$ as a function of t is in general complicate.

(c) In the center-of-mass frame, one may write

$$\cos\theta^* = \frac{t - m_1^2 - m_3^2 + 2\,E_1^* E_3^*}{2\,p_1^*\,p_3^*}\,. \qquad (19)$$

(i) At fixed values of s and E_1, to which limit in t corresponds the forward reaction $\theta^* = 0$?

(ii) At fixed values of s and E_1, to which limit in u corresponds the backward reaction $\theta^* = \pi$?

(d) Show that

$$\cos\theta^* = \frac{s^2 + s(2t - m_1^2 - m_2^2 - m_3^2 - m_4^2) + (m_1^2 - m_2^2)(m_3^2 - m_4^2)}{\sqrt{\lambda(s, m_1^2, m_2^2)\,\lambda(s, m_3^2, m_4^2)}}\,. \qquad (20)$$

(e) In the large energy limit where $s \sim -u \gg -t,\ m_i^2$, called *Regge limit*, show that $\theta^* \to 0$.

5. Physical region for t.
(a) For a given s, show that the physical region for t looks like $t^- \leqslant t \leqslant t^+$ and give the values of t^- and t^+.

(b) In the case of equal masses ($m_i^2 = m^2$), give the physical region in t.

(c) Still in the case of equal masses ($m_i^2 = m^2$), express t and u as functions of s, m^2 and $\cos\theta^*$. Comment.