M1 General Physics
Major PNU

Particles

Classical theory of fields

We first consider, in the first two sections, a Lagrangian $\mathcal{L}(\phi, \partial_\mu \phi)$ for a scalar field, which does not depend explicitly on the space-time position.

1 Energy-momentum tensor

1. Using the transformation (see the notation used in the lectures)

$$\begin{aligned} \delta x^\mu(x) &= \text{constant} = \delta x^\mu, \\ \delta \phi &= 0\,, \end{aligned} \tag{1}$$

show that this transformation allows one to construct a conserved current

$$\boxed{T^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} \mathcal{L}} \tag{2}$$

called energy-momentum tensor.

2. By analogy with the momentum in classical mechanics, defined as

$$p = \frac{\partial L}{\partial \dot{q}}\,, \tag{3}$$

define the field momentum.

3. Express $T^{0\nu}$ in terms of Π, $\partial^\nu \phi$ and $\mathcal{L}$.

4. Consider in particular T^{00} and comment, in analogy with the hamiltonian of classical mechanics $H = p\dot{q} - L$.

5. Provide an integral expression of the total energy of the field, and more generally of its total 4-momentum.

2 Angular-momentum tensor

1. Consider a Lorentz transformation Λ, close to identity, written as

$$\Lambda^{\mu\nu} = g^{\mu\nu} + \omega^{\mu\nu}\,, \tag{4}$$

with $\|\omega\| \ll 1$. Show that

$$\omega^{\mu\nu} + \omega^{\nu\mu} = 0\,. \tag{5}$$

2. Count the number of independent real parameters which are necessary to encode ω, and comment.

3. Using the transformation (see the notation used in the lectures)

$$\begin{array}{rcl} \delta x^\nu(x) & = & \omega^{\nu\mu} x_\mu\,, \\ \delta\phi & = & 0\,, \end{array} \tag{6}$$

show that

$$\boxed{J^{\mu,\nu\rho} = (x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu})} \tag{7}$$

is conserved.

4. Write the total angular momentum of the field.

5. From the conservation of the current (7), deduce that the energy-momentum tensor is symmetric.

3 Extension to fields with a spin

A scalar field transforms under Lorentz transformations as

$$\phi \longrightarrow \phi' \quad \text{with} \quad \phi'(x') = \phi(x) \quad \text{for} \quad x' = \Lambda x, \quad \text{where} \quad \Lambda \in \mathcal{L}. \tag{8}$$

In the more general case where the field is not scalar, one should specify the representation D the field belongs to.
We recall that, E being a $\mathbb{R}$ or $\mathbb{C}$-vector space of dimension n, a real or complex representation of a group G is defined has a morphism between the group and the bijective linear transformations of E, denoted has $GL(E)$, i.e.

$$\begin{array}{rl} & \forall g \in G, D(g) \in GL(E) \\ \text{with} & \forall g_1, g_2 \in G, D(g_1 g_2) = D(g_1) \circ D(g_2) \end{array} \tag{9}$$

or, if n is finite, using matrix notations for the operator $D(g)$,

$$\forall g_1, g_2 \in G, \quad \mathcal{D}(g_1 g_2) = \mathcal{D}(g_1)\mathcal{D}(g_2)\,. \tag{10}$$

In the case of the Lorentz group, considering a real representation S of finite dimension n, i.e.

$$\forall \Lambda_1, \Lambda_2 \in G, \quad \mathcal{S}(\Lambda_1 \Lambda_2) = \mathcal{S}(\Lambda_1)\mathcal{S}(\Lambda_2)\,. \tag{11}$$

the field, which has now n components labeled by a letter a, transforms as

$$\phi_a \longrightarrow \phi'_a \quad \text{with} \quad \phi_a(x') = S(\Lambda)_{ab}\phi_b(x) \quad \text{for} \quad x' = \Lambda x, \quad \text{where} \quad \Lambda \in \mathcal{L}\,. \tag{12}$$

Thus, the Lorentz transformation affects both the space-time coordinates x and the field components.
The matrix $S(\Lambda)$ of the representation can be expanded for Λ close to identity, see (4), as

$$S(\Lambda) = I + i\omega^{\mu\nu} L_{\mu\nu} \tag{13}$$

so that the transformation law (8) for a Lorentz transformation close to identity is now

$$\phi'_a(x') = \phi_a(x) + i\omega^{\mu\nu}(L_{\mu\nu})_{ab}\phi_b(x) \tag{14}$$

where $L_{\mu\nu}$ is antisymmetric in $\mu \leftrightarrow \nu$ because of the antisymmetry of $\omega^{\mu\nu}$, i.e.

$$\delta\phi_a(x) = i\omega^{\mu\nu}(L_{\mu\nu})_{ab}\phi_b(x)\,. \tag{15}$$

1. Write the conserved corresponding current in the form

$$J^{\mu,\nu\rho} = J^{\mu,\nu\rho}_{orbital} + J^{\mu,\nu\rho}_{spin} \tag{16}$$

where $J^{\mu,\nu\rho}_{orbital}$ has been obtained in Eq. (7), and give the expression of $J^{\mu,\nu\rho}_{spin}$.

2. Discuss the symmetry of the energy-momentum tensor.