M1 General Physics
Major PNU

Particles

Classical theory of fields

We first consider, in the first two sections, a Lagrangian $\mathcal{L}(\phi, \partial_\mu \phi)$ for a scalar field, which does not depend explicitly on the space-time position.

1 Energy-momentum tensor

1. Using the transformation (see the notation used in the lectures)

$$\begin{array}{rcl} \delta x^\mu(x) & = & \text{constant} = \delta x^\mu, \\ \delta\phi & = & 0\,, \end{array} \tag{1}$$

show that this transformation allows one to construct a conserved current

$$\boxed{T^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} \mathcal{L}} \tag{2}$$

called energy-momentum tensor.

Solution

This result is obvious from the formula obtained for the Noether current: one gets that

$$j^\mu = \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} \mathcal{L} \right) \delta x_\nu \tag{3}$$

is conserved for arbitrary δx_ν, thus the conservation of $T^{\mu\nu}$ after factorizing out the arbitrary constant δx_ν. Note the fact that $T^{\mu\nu}$ thus depends on 2 indices.

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2. By analogy with the momentum in classical mechanics, defined as

$$p = \frac{\partial L}{\partial \dot{q}}\,, \tag{4}$$

define the field momentum.

Solution

By analogy, since the (infinite set of) degrees of freedom are encoded in $\phi(x)$, one naturally defines

$$\Pi = \frac{\delta \mathcal{L}}{\delta(\partial_0 \phi(x))} \,. \tag{5}$$

where the usual derivative with respect to $\dot{q}$ is replaced by the functional derivative with respect to $\partial_0 \phi(x)$.

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3. Express $T^{0\nu}$ in terms of Π, $\partial^\nu \phi$ and $\mathcal{L}$.

Solution

Combining (2) and the above defined field momentum, one has

$$T^{0\nu} = \Pi \, \partial^\nu \phi - g^{0\nu} \mathcal{L}. \tag{6}$$

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4. Consider in particular T^{00} and comment, in analogy with the hamiltonian of classical mechanics $H = p\dot{q} - L$.

Solution

$$T^{00} = \Pi \, \partial^0 \phi - \mathcal{L} \,, \tag{7}$$

which is just the Legendre transformation of the Lagrangian $\mathcal{L}$. This is thus the local density of energy.

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5. Provide an integral expression of the total energy of the field, and more generally of its total 4-momentum.

Solution

From the previous question, we obviously have

$$E = \int T^{00} d^3x \tag{8}$$

which is the time component of the total 4-momentum

$$P^\nu = \int T^{0\nu} d^3x \,. \tag{9}$$

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2 Angular-momentum tensor

1. Consider a Lorentz transformation Λ, close to identity, written as

$$\Lambda^{\mu\nu} = g^{\mu\nu} + \omega^{\mu\nu}\,, \tag{10}$$

with $\|\omega\| \ll 1$. Show that

$$\omega^{\mu\nu} + \omega^{\nu\mu} = 0\,. \tag{11}$$

Solution

The constraint satisfied by Λ reads

$$g_{\mu\nu}\Lambda^{\mu\rho}\Lambda^{\nu\sigma} = g^{\rho\sigma}\,,$$

which, after expansion around identity, gives

$$g_{\mu\nu}\left(g^{\mu\rho} + \omega^{\mu\rho}\right)\left(g^{\nu\sigma} + \omega^{\nu\sigma}\right) = g^{\rho\sigma},$$

i.e. at order 1 in ω

$$g^{\rho\sigma} + \omega^{\rho\sigma} + \omega^{\sigma\rho} = g^{\rho\sigma}$$

which thus leads to the result to be proven.

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2. Count the number of independent real parameters which are necessary to encode ω, and comment.

Solution

The 4×4 tensor $\omega^{\rho\sigma}$ is antisymmetric, therefore it depends on 6 parameters. This is in accordance to the fact that there exist 3 independent infinitesimal rotations along the axis x, y, z and 3 independent infinitesimal boots along the axis x, y, z.

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3. Using the transformation (see the notation used in the lectures)

$$\begin{array}{rcl} \delta x^\nu(x) & = & \omega^{\nu\mu} x_\mu\,, \\ \delta\phi & = & 0\,, \end{array} \tag{12}$$

show that

$$\boxed{J^{\mu,\nu\rho} = \left(x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu}\right)} \tag{13}$$

is conserved.

Solution

The Lagrangian being Lorentz invariant, the Noether theorem implies that the current

$$\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial^\nu\phi - g^{\mu\nu}\mathcal{L}\right)\omega_{\nu\rho}x^\rho = T^{\mu\nu}\omega_{\nu\rho}x^\rho$$

is conserved. Since $\omega_{\nu\rho}$ is antisymmetric, this implies that

$$\left(T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu\right)\omega_{\nu\rho}$$

is conserved. The antisymmetric tensor $\omega_{\nu\rho}$ being arbitrary, this leads to the expected result.

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4. Write the total angular momentum of the field.

Solution

The total angular momentum of the field is the conserved charge, i.e.

$$J^{\nu\rho} = \int d^3x\, J^{0,\nu\rho} = \int d^3x\, \left(x^\nu T^{0\rho} - x^\rho T^{0\nu}\right).$$

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5. From the conservation of the current (13), deduce that the energy-momentum tensor is symmetric.

Solution

From the expression of $J^{\mu,\nu\rho}$ we have

$$\partial_\mu J^{\mu,\nu\rho} = 0 = \partial_\mu\left(x^\nu T^{\mu\rho}\right) - \partial_\mu\left(x^\rho T^{\mu\nu}\right) = T^{\nu\rho} - T^{\rho\nu}$$

where we have used the conservation of $T^{\mu\nu}$. Thus, $T^{\nu\rho}$ is symmetric.
Remark (beyond the scope of the present lectures!):
This is expected since $T^{\mu\nu}$ is a measurable quantity, as coupled to gravitational field. Besides, starting from the Hilbert stress–energy tensor defined in General Relativity as

$$T^{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta[\sqrt{-g}\,\mathcal{L}_{matter}]}{\delta g^{\mu\nu}}, \tag{14}$$

its symmetry is obvious.

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3 Extension to fields with a spin

A scalar field transforms under Lorentz transformations as

$$\phi \longrightarrow \phi' \quad \text{with} \quad \phi'(x') = \phi(x) \quad \text{for} \quad x' = \Lambda x, \quad \text{where} \quad \Lambda \in \mathcal{L}. \tag{15}$$

In the more general case where the field is not scalar, one should specify the representation D the field belongs to.
We recall that, E being a $\mathbb{R}$ or $\mathbb{C}$-vector space of dimension n, a real or complex representation of a group G is defined has a morphism between the group and the bijective linear transformations of E, denoted has $GL(E)$, i.e.

$$\begin{aligned} &\forall g \in G, D(g) \in GL(E) \\ \text{with} \qquad &\forall g_1, g_2 \in G, D(g_1 g_2) = D(g_1) \circ D(g_2) \end{aligned} \tag{16}$$

or, if n is finite, using matrix notations for the operator $D(g)$,

$$\forall g_1, g_2 \in G, \quad \mathcal{D}(g_1 g_2) = \mathcal{D}(g_1)\mathcal{D}(g_2)\,. \tag{17}$$

In the case of the Lorentz group, considering a real representation S of finite dimension n, i.e.

$$\forall \Lambda_1, \Lambda_2 \in G, \quad \mathcal{S}(\Lambda_1 \Lambda_2) = \mathcal{S}(\Lambda_1)\mathcal{S}(\Lambda_2)\,. \tag{18}$$

the field, which has now n components labeled by a letter a, transforms as

$$\phi_a \longrightarrow \phi'_a \quad \text{with} \quad \phi_a(x') = S(\Lambda)_{ab}\phi_b(x) \quad \text{for} \quad x' = \Lambda x, \quad \text{where} \quad \Lambda \in \mathcal{L}\,. \tag{19}$$

Thus, the Lorentz transformation affects both the space-time coordinates x and the field components.
The matrix $S(\Lambda)$ of the representation can be expanded for Λ close to identity, see (10), as

$$S(\Lambda) = I + i\omega^{\mu\nu} L_{\mu\nu} \tag{20}$$

so that the transformation law (15) for a Lorentz transformation close to identity is now

$$\phi'_a(x') = \phi_a(x) + i\omega^{\mu\nu}(L_{\mu\nu})_{ab}\phi_b(x) \tag{21}$$

where $L_{\mu\nu}$ is antisymmetric in $\mu \leftrightarrow \nu$ because of the antisymmetry of $\omega^{\mu\nu}$, i.e.

$$\delta\phi_a(x) = i\omega^{\mu\nu}(L_{\mu\nu})_{ab}\phi_b(x)\,. \tag{22}$$

1. Write the conserved corresponding current in the form

$$J^{\mu,\nu\rho} = J^{\mu,\nu\rho}_{orbital} + J^{\mu,\nu\rho}_{spin} \tag{23}$$

where $J^{\mu,\nu\rho}_{orbital}$ has been obtained in Eq. (13), and give the expression of $J^{\mu,\nu\rho}_{spin}$.

Solution

From Eq. (22) inserted in the Noether current, one gets immediately that

$$(T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu)\,\omega_{\nu\rho} + i\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_a)}(L^{\nu\rho})_{ab}\phi_b(x)\,\omega_{\nu\rho}$$

is conserved. The antisymmetry of $L_{\nu\rho}$ then implies that

$$J^{\mu,\nu\rho} = J^{\mu,\nu\rho}_{orbital} + J^{\mu,\nu\rho}_{spin}$$

with

$$J^{\mu,\nu\rho}_{spin} = i\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_a)}(L^{\nu\rho})_{ab}\phi_b(x)$$

is conserved (while $J^{\mu,\nu\rho}_{orbital}$ and $J^{\mu,\nu\rho}_{spin}$ are not separately conserved). This applies in particular to the case of the electromagnetic field A^μ (spin 1 field) as well as to the case of a Dirac field (massive spin 1/2 field).

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2. Discuss the symmetry of the energy-momentum tensor.

Solution

Using the argument of question 5., one gets

$$\partial_\mu J^{\mu,\nu\rho} = 0 = \partial_\mu J^{\mu,\nu\rho}_{orbital} + \partial_\mu J^{\mu,\nu\rho}_{spin} = T^{\nu\rho} - T^{\rho\nu} + \partial_\mu J^{\mu,\nu\rho}_{spin}.$$

Since $\partial_\mu J^{\mu,\nu\rho}_{spin}$ does not a priori vanish, and further more $T^{\nu\rho} - T^{\rho\nu}$ and $\partial_\mu J^{\mu,\nu\rho}_{spin}$ are both antisymmetric with respect to $\nu\rho$, it is clear that there is now no reason why $T^{\nu\rho}$ would be symmetric.
Still, using a trick due to Belinfante, it is anyway possible to write a modified energy-momentum tensor with the same conserved charges, which turns out to be symmetric.
