M1 General Physics

Particles

Tensors, field strength and Lorentz transform of electromagnetic fields

1 Symmetric and antisymmetric tensors

1.1 Symmetrization, antisymmetrization and covariance

Consider a quadridimensional 2-tensor $M^{\mu\nu}$.

1. Provide a separation of $M^{\mu\nu}$ as a sum of two symmetric (S) and antisymmetric (A) tensors:

$$M^{\mu\nu} = S^{\mu\nu} + A^{\mu\nu} \,. \tag{1}$$

2. Recall the way $M^{\mu\nu}$ transform under a arbitrary Lorentz transformation Λ, encoded through its matrix elements $\Lambda^{\rho}{}_{\sigma}$.

3. Show that the decomposition (1) is covariant under Lorentz transformations.

4. Write the Lorentz transformation of $M^{\mu\nu}$ in a way which makes this separation explicit.

1.2 Transformation under boosts

We now consider a Lorentz boost of a frame F to a frame F' along the x axis, encoded by $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$.

5. Recall the explicit expression of Λ.

6. We focus on the case of a symmetric tensor $S^{\mu\nu}$.

Write the Lorentz transformation of the various components of S under the above boost.

7. We focus on the case of an antisymmetric tensor $A^{\mu\nu}$.

a. What can be said about $A^{00}, A^{11}, A^{22}, A^{33}$ and their transformations?

b. How does A^{23} transforms?

c. Compare the transformation of A^{12}, A^{13} and A^{02}, A^{03} with the transformation of x^1 and x^0. Deduce the transformation of these components.

d. Show that A^{01} is invariant under these boosts.

2 Lorentz transformations of electromagnetic fields

We now apply the previous results to the field strength

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix},$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields. We use a system of units such that $c = 1$.

8. Show that under the considered boost along x, these fields transform as

$$\begin{aligned} E'^1 &= E^1\,, & (2) \\ E'^2 &= \gamma(E^2 - \beta B^3)\,, & (3) \\ E'^3 &= \gamma(E^3 + \beta B^2) & (4) \end{aligned}$$

and

$$\begin{aligned} B'^1 &= B^1\,, & (5) \\ B'^2 &= \gamma(B^2 + \beta E^3)\,, & (6) \\ B'^3 &= \gamma(B^3 - \beta E^2)\,. & (7) \end{aligned}$$

9. Consider an arbitrary boost along the direction $\vec{n}$ ($\vec{n}^2 = 1$), i.e. with a velocity $\vec{v} = \beta\vec{n}$.

Show that under such a boost,

$$\begin{aligned} \vec{E}' &= (\vec{E}\cdot\vec{n})\vec{n} + \gamma\left[\vec{E} - (\vec{E}\cdot\vec{n})\vec{n}\right] + \gamma\,\vec{v}\wedge\vec{B}\,, & (8) \\ \vec{B}' &= (\vec{B}\cdot\vec{n})\vec{n} + \gamma\left[\vec{B} - (\vec{B}\cdot\vec{n})\vec{n}\right] - \gamma\,\vec{v}\wedge\vec{E}\,. & (9) \end{aligned}$$

10. Show that this can be rewritten in the form

$$\begin{aligned} \vec{E}' &= (\vec{E}\cdot\vec{n})\vec{n} + \gamma\left[\vec{n}\wedge(\vec{E}\wedge\vec{n}) + \vec{v}\wedge\vec{B}\right]\,, & (10) \\ \vec{B}' &= (\vec{B}\cdot\vec{n})\vec{n} + \gamma\left[\vec{n}\wedge(\vec{B}\wedge\vec{n}) - \vec{v}\wedge\vec{E}\right]\,. & (11) \end{aligned}$$

Hint: use $\vec{a}\wedge(\vec{b}\wedge\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$.

11. In the non-relativistic limit, simplify the transformations (8) and (9), keeping linear terms in β.