Spin-orbit Coupling and Tensor Forces (*).

B. JANCOVICI

Laboratoire de Physique de l'École Normale Supérieure - Université de Paris

(ricevuto il 2 Luglio 1957)

Résumé. — Nous avons cherché si on peut expliquer les forces spin-orbite dans les noyaux complexes comme un effet du 2e ordre des forces tensorielles. Suivant les méthodes de Brueckner, nous calculons d'abord une amplitude de réaction modifiée, pour les collisions de deux nucléons au sein de la matière nucléaire. Nous utilisons la deuxième approximation de Born, et nous tenons compte du principe d'exclusion dans les états intermédiaires. Nous obtenons ensuite le potentiel spin-orbite moyen auquel est soumis un nucléon en sommant sur toutes ses collisions l'amplitude de réaction. Le potentiel spin-orbite ainsi obtenu est d'un ordre de grandeur trop faible et peut même avoir le mauvais signe. Nous discutons quelques implications de ces résultats.

1. – Introduction.

The only non-central forces which are known with certainty to exist in the two-body nuclear problem (e.g. from the quadrupole moment of the deuteron) are the tensor forces. For the sake of simplicity, it may be asked if these forces are sufficient to account for the spin-orbit coupling which explains the shell model and the high energy nucleon-nucleus polarization experiments.

In the case of light nuclei, some evidence has been given that this is indeed the case [1]. We here treat the problem for heavy nuclei, using an approach

(*) Supported in part by the United States Air Force through the European Office, Air Research and Development Command.

[1] E. P. WIGNER and A. M. FEINGOLD: *Phys. Rev.*, **79**, 221 (1950); A. M. FEINGOLD: *Phys. Rev.*, **101**, 258 (1956); **105**, 944 (1957); D. H. LYONS: *Phys. Rev.*, **105**, 936 (1957).

in the spirit of Brueckner's model [2]. A preliminary account of this work has already been given [3]. Independent results obtained by L. S. KISSLINGER are in disagreement with ours [4].

We derive the spin-orbit effective potential from the reaction matrix for two-body collisions [5,6]. This reaction matrix will actually be a « modified » reaction matrix, which means that it is defined for the two-body collisions inside the nuclear matter and that it takes into account the presence of other nucleons. through the use of the exclusion principle in the intermediate states (statistical correlations). The modified reaction matrix has a spin-linear part

$$[\alpha + \beta(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)](\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_j)\cdot(\boldsymbol{k}_i'\boldsymbol{k}_j'\,|\,\boldsymbol{A}\,|\,\boldsymbol{k}_i\boldsymbol{k}_j) \tag{1}$$

for the collision of two particles i and j having initial and final momenta $\boldsymbol{k}_i$, $\boldsymbol{k}_j$ and $\boldsymbol{k}_i'$, $\boldsymbol{k}_j'$, respectively. $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ are the isotopic spin and spin operators.

We here assume that the effective potential in which the i-th nucleon moves is obtained by summing (1) on all two-body collisions with other nucleons j. This assumption will be discussed in Sect. **7**. A spin-orbit potential may emerge (Sect. **2**) from such a calculation only in a nucleus of non-uniform density: the spin-orbit potential is a surface effect [7]. However, it is convenient to compute (1) from the two-body forces in a medium of uniform density (Sect. **3**). Therefore, we use the Thomas-Fermi approximation: the density $\varrho(r)$ is varying, but, in the neighbourhood of each point, the nuclear structure may be approximated by a Fermi-gas with a local Fermi momentum

$$f(r) = \left[\frac{3\pi^2}{2}\,\varrho(r)\right]^{\frac{1}{3}}. \tag{2}$$

2. – From the modified reaction amplitude to the spin-orbit potential.

The average spin-dependent potential $\boldsymbol{\sigma}_i\cdot\boldsymbol{V}_i$ in which the i-th nucleon moves is obtained by adding the reaction amplitudes (1) for the collisions with all the other nucleons j. Thus, in momentum space, a matrix element of $\boldsymbol{V}_i$

(2) H. A. BETHE: *Phys. Rev.*, **103**, 1353 (1956).
(3) B. JANCOVICI: *Phys. Rev.*, **107**, 631 (1957); *Thèse* (Paris, 1957).
(4) L. S. KISSLINGER: *Phys. Rev.*, **104**, 1077 (1956). See note (9) in ref. (3).
(5) S. FERNBACH, W. HECKROTTE and J. V. LEPORE: *Phys. Rev.*, **97**, 1059 (1955).
(6) J. S. BELL and T. H. R. SKYRME: *Phil. Mag.*, **1**, 1055 (1956).
(7) E. FERMI: quoted by R. M. STERNHEIMER.

reads:

$$(3)\qquad \begin{cases} (\boldsymbol{k}'_i|\boldsymbol{V}_i|\boldsymbol{k}_i) = 4\sum_{\Psi_j}(2\pi)^{-6}\int{}^3k'_j\int \mathrm{d}^3k_j\,, \\ \Psi_j^*(\boldsymbol{k}'_j)\left[\alpha(\boldsymbol{k}'_i\boldsymbol{k}'_j|\boldsymbol{A}|\boldsymbol{k}_i\boldsymbol{k}_j) - (\tfrac{1}{2})(\alpha+3\beta)(\boldsymbol{k}'_j\boldsymbol{k}'_i|\boldsymbol{A}|\boldsymbol{k}_i\boldsymbol{k}_j)\right]\Psi_j(\boldsymbol{k}_j)\,, \end{cases}$$

where the $\Psi_j(\boldsymbol{k}_j)$ are the Fourier transforms of the one-particle orbital states $\Psi_j(\boldsymbol{r}_j)$:

$$(4)\qquad \Psi(\boldsymbol{k}) = \int \mathrm{d}^3r \exp[-i\boldsymbol{k}\cdot\boldsymbol{r}]\Psi(\boldsymbol{r})\,.$$

Eq. (3) can be brought, with some approximations, to the form of a usual spin-orbit potential ([5,6]). In our model, we assume a varying spherically symmetrical nuclear density.

$$(5)\qquad 4\sum_{\Psi_j}\Psi_i^*(\boldsymbol{r}_j)\,\Psi_j(\boldsymbol{r}_j) = \varrho(r_j)\,,$$

while around each point we describe the correlations by those of a «local» Fermi gas of «local» Fermi momentum f:

$$(6)\qquad 4\sum_{\Psi_j}\Psi_j^*(\boldsymbol{r}'_j)\Psi_j(\boldsymbol{r}_j) = \varrho(r'_j)[3/4\pi\ {}^3(r'_j)]\int_{\varkappa<f}\exp[-i\boldsymbol{k}\cdot(\boldsymbol{r}'_j-\boldsymbol{r}_j)]\,\mathrm{d}^3\varkappa\,.$$

In such a model, using (6) and the momentum conservation:

$$(7)\qquad (\boldsymbol{k}'_i\boldsymbol{k}'_j|\boldsymbol{A}|\boldsymbol{k}_i\boldsymbol{k}_j) = (2\pi)^3\,\delta(\boldsymbol{k}'_i+\boldsymbol{k}'_j-\boldsymbol{k}_i-\boldsymbol{k}_j)(\boldsymbol{k}'_j|\tilde{\boldsymbol{A}}|\boldsymbol{k}_i\boldsymbol{k}_j)\,,$$

we may bring (3) to the form:

$$(8)\qquad {}'|\boldsymbol{V}_i|\boldsymbol{k}_i) = \int\mathrm{d}^3g\int\mathrm{d}^3r'_j\exp[i\boldsymbol{g}\cdot\boldsymbol{r}'_j]\varrho(r'_j)\delta(\boldsymbol{k}'_i-\boldsymbol{k}_i+\boldsymbol{g})[3/4\pi f^3(r'_j\ \cdot$$
$$\cdot\int_{k_j<f}\mathrm{d}^3k_j[\alpha(\boldsymbol{k}_j+\boldsymbol{g}|\tilde{\boldsymbol{A}}|\boldsymbol{k}_i,\,\boldsymbol{k}_j) - (\tfrac{1}{2})(\alpha+3\beta)(\boldsymbol{k}_i-\boldsymbol{g}|\tilde{\boldsymbol{A}}|\boldsymbol{k}_i\boldsymbol{k}_j)]\,.$$

The last integral in the expression (8) is a pseudovector depending only on $\boldsymbol{k}_i$ and $\boldsymbol{g}$ (after this last integration has been performed). We may expand this expression in successive powers of $\boldsymbol{g}$, which would make successive derivatives of the density ϱ appear in (8). We keep only the linear term in $\boldsymbol{g}$. This means that we assume the spin-orbit effects to depend mostly on the first derivative of the density. Or, from another point of view, this approximation means that we only consider small angle (direct and exchange) scatterings of the i-th nucleon in the nuclear matter. This first non-vanishing term of the expansion must be of the form:

$$(9) \qquad (3/4\pi f^3)\int\limits_{k_j<f} \mathrm{d}^3k_j(\boldsymbol{k}_j+\boldsymbol{g}|\tilde{A}|\boldsymbol{k}_i,\boldsymbol{k}_j) = \boldsymbol{g}\times\boldsymbol{k}_i E_0(k_i,f)+\dots,$$

$$(10) \qquad (3/4\pi f^3)\int\limits_{k_j<f} \mathrm{d}^3k_j(\boldsymbol{k}_i-\boldsymbol{g}|\tilde{A}|\boldsymbol{k}_i,\boldsymbol{k}_j) = \boldsymbol{g}\times\boldsymbol{k}_i E_1(k_i,f)+\dots.$$

Using (9) and (10) in (8), and changing the integration variable $\boldsymbol{r}_j'$ into $\boldsymbol{r}_i$, we obtain the familiar expression

$$(11) \qquad (\boldsymbol{k}_i'|\boldsymbol{V}_i|\boldsymbol{k}_i) = \int \mathrm{d}^3r_i \exp[-i\boldsymbol{k}_i'\,\boldsymbol{r}_i + i\boldsymbol{k}_i\cdot\boldsymbol{r}_i]\,a(k_i,\;(r_i))\,\frac{1}{r_i}\frac{\partial\varrho}{\partial r_i}(\boldsymbol{r}_i\times\boldsymbol{k}_i)\,,$$

where

$$(12) \qquad a(k_i,f(r_i)) = a_0+a_1 = i\left[\alpha E_0(k_i,f)-(\tfrac{1}{2})(\alpha+3\beta)E_1(k_i,f)\right].$$

Thus the energy of the i-th nucleon is the same as if it were submitted to an effective spin-orbit force [8]

$$(13) \qquad a(k_i,f(r_i))\,\frac{1}{r_i}\frac{\partial\varrho}{\partial r_i}(\boldsymbol{l}_i\cdot\boldsymbol{\sigma}_i)\,.$$

In addition to $\boldsymbol{l}_i$, there is a velocity dependence in a because of k_i. However, if i is the « last » bound nucleon, k_i is peaked around f and we will just take $k_i=f(r_i)$. If i is a scattered high energy nucleon, $\hbar^2k_i^2/2M$ must be taken as its kinetic energy inside the potential well.

3. – The calculation of the modified reaction ampl ude.

Our task is now to obtain the modified reaction amplitudes (9) and (10) from the interaction between particles i and j, assuming that these particles are part of a locally uniform Fermi gas of Fermi momentum f. The reaction

(8) The a coefficient depends on both k_i and (through f) r_i. This could lead to some difficulties, since these two variables do not commute. Actually, we can consider them now as classical variables in the same approximation as the Thomas-Fermi approximation; in the latter, a momentum (e.g. the Fermi momentum) is defined at each point, which is also only a classical concept.

matrix θ_{ij} at energy E is given by the integral equation

$$\theta_{ij} = v_{ij} + v_{ij}\frac{Q}{E-H}\theta_{ij}\,, \tag{14}$$

where Q is the projector outside the states which are already occupied by other nucleons and H is the « model » hamiltonian for the average potential.

Here, we just use for (14) the second Born approximation, which gives the first non-vanishing spin-linear result in the case where the spin-dependent part of v_{ij} is a tensor force. In this second order Born approximation, the reaction amplitude is

$$(2\pi)^{-6}(2M/\hbar^2)\int\limits_{q_i>f}\mathrm{d}^3q_i\int\limits_{q_j>f}\mathrm{d}^3q_j(\boldsymbol{k}_i'\boldsymbol{k}_j'|v_{ij}|\boldsymbol{q}_i\boldsymbol{q}_j)\frac{1}{k_i^2+k_j^2-q_i^2-q_j^2}(\boldsymbol{q}_i\boldsymbol{q}_j|v_{ij}|\boldsymbol{k}_i\boldsymbol{k}_j). \tag{15}$$

The momenta $\boldsymbol{q}_i\boldsymbol{q}_j$ of the intermediate state must be outside the Fermi sphere. We assume that v_{ij} has a tensor part

$$v_{ij} = [1-\chi+\chi(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)]\,u(r_{ij})S_{ij}(\boldsymbol{r}_{ij}) + \dots\,, \tag{16}$$

where

$$S_{ij}(\boldsymbol{r}_{ij}) = \frac{3}{r_{ij}^2}(\boldsymbol{\sigma}_i\cdot\boldsymbol{r}_{ij})(\boldsymbol{\sigma}_j\cdot\boldsymbol{r}_{ij}) - (\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)\,, \tag{17}$$

and we are only interested in the part of (15) which is linear in the spins. By inserting (16) into (15), it is readily found that this part is (1) where

$$\alpha = 1-2\chi+4\chi^2\,, \qquad \beta = 2\chi-4\chi^2\,, \tag{18}$$

and

$$(\boldsymbol{k}_j'|\tilde{\boldsymbol{A}}|\boldsymbol{k}_i\boldsymbol{k}_j) = i(9M/16\pi^2\hbar^2)\int\limits_{\Omega(\boldsymbol{K})}\mathrm{d}^3q\,(\boldsymbol{k}'-\boldsymbol{q})\times(\boldsymbol{k}-\boldsymbol{q})\cdot$$
$$\cdot\frac{(\boldsymbol{k}'-\boldsymbol{q})\cdot(\boldsymbol{k}-\boldsymbol{q})}{k^2-q^2}\,\frac{w(|\boldsymbol{k}'-\boldsymbol{q}|)w(|\boldsymbol{k}-\boldsymbol{q}|)}{|\boldsymbol{k}'-\boldsymbol{q}|^2|\boldsymbol{k}-\boldsymbol{q}|^2}\,, \tag{19}$$

where we have inserted the relative momenta:

$$\boldsymbol{k}=\boldsymbol{k}_i-\boldsymbol{k}_j\,,\quad \boldsymbol{k}'=\boldsymbol{k}_i'-\boldsymbol{k}_j'\,,\quad \boldsymbol{q}=\boldsymbol{q}_i-\boldsymbol{q}_j\,,\quad \boldsymbol{K}=\boldsymbol{k}_i+\boldsymbol{k}_j\,, \tag{20}$$

and

$$w(k) = \int\limits_0^\infty j_2(kr)\,u(r)r^2\,\mathrm{d}r\,, \tag{21}$$

(j_2 is a spherical Bessel function ([9])). The domain of integration over $\boldsymbol{q}$, Ω, is defined by the conditions

$$(22)\qquad q_i,\, q_j > f\,, \quad \text{i.e.} \quad |\boldsymbol{q} \pm (\tfrac{1}{2})\boldsymbol{K}| > f\,.$$

We only need (19) for $\boldsymbol{k}'_j = \boldsymbol{k}_j + \boldsymbol{g}$ and $\boldsymbol{k}'_j = \boldsymbol{k}_i - \boldsymbol{g}$. We just keep the first term in an expansion in powers of $\boldsymbol{g}$:

$$(23)\qquad (\boldsymbol{k}_j + \boldsymbol{g}|\tilde{\boldsymbol{A}}|\boldsymbol{k}_i,\, \boldsymbol{k}_j) = i(9M/16\pi^2\hbar^2)\int_\Omega \mathrm{d}^3\boldsymbol{q}\, \frac{\boldsymbol{g}\times(\boldsymbol{g}-\boldsymbol{k})}{k^2-q^2}\left[\frac{w(|\boldsymbol{q}-\boldsymbol{k}|)}{|\boldsymbol{q}-\boldsymbol{k}|}\right]^2 + \dots\,,$$

and

$$(24)\qquad (\boldsymbol{k}_i - \boldsymbol{g}|\boldsymbol{A}|\boldsymbol{k}_i,\, \boldsymbol{k}_j) = i(9M/16\pi^2\hbar^2)\int_\Omega \mathrm{d}^3q\, \frac{w(\boldsymbol{q}-\boldsymbol{k})}{|\boldsymbol{q}-\boldsymbol{k}|^2}\cdot$$

$$\cdot\Bigg\{\boldsymbol{g}\times(\boldsymbol{q}-\boldsymbol{k})\frac{w(|\boldsymbol{q}+\boldsymbol{k}|)}{|\boldsymbol{q}+\boldsymbol{k}|^2} - \frac{2(\boldsymbol{k}\times\boldsymbol{q})[\boldsymbol{g}\cdot(\boldsymbol{q}-\boldsymbol{k})]}{k^2-q^2}\frac{w(|\boldsymbol{q}+\boldsymbol{k}|)}{|\boldsymbol{q}+\boldsymbol{k}|^2} +$$

$$+ 2(\boldsymbol{k}\times\boldsymbol{q})\left[\boldsymbol{g}\cdot(\boldsymbol{q}+\boldsymbol{k})\frac{1}{|\boldsymbol{q}+\boldsymbol{k}|}\frac{\partial}{\partial|\boldsymbol{q}+\boldsymbol{k}|}\frac{w(|\boldsymbol{q}+\boldsymbol{k}|)}{|\boldsymbol{q}+\boldsymbol{k}|^2}\right]\Bigg\} + \dots\,.$$

We need only the integrated values (9) and (10), the tensorial nature of which we know in advance, so that one may readily show that:

$$(25)\qquad E_0(k_i, f) = i\,\frac{9M}{16\pi^2\hbar^2}\,\frac{3}{4\pi f^3}\,\frac{1}{k_i^2}\int_{k_j<f}\mathrm{d}^3k_j\int_\Omega \mathrm{d}^3q\,\frac{\boldsymbol{k}_i\cdot(\boldsymbol{q}-\boldsymbol{k})}{k^2-q^2}\left[\frac{w(|\boldsymbol{q}-\boldsymbol{k}|)}{|\boldsymbol{q}-\boldsymbol{k}|}\right]^2,$$

and

$$(26)\qquad E_1(k_i, f) = i\,\frac{9M}{16\pi^2\hbar^2}\,\frac{3}{4\pi f^3}\,\frac{1}{k_i^2}\int_{k_j<f}\mathrm{d}^3k_j\int_\Omega \mathrm{d}^3q\,\frac{w(|\boldsymbol{q}-\boldsymbol{k}|)}{|\boldsymbol{q}-\boldsymbol{k}|^2}\cdot$$

$$\cdot\Bigg\{\left[-\boldsymbol{k}_i\cdot\boldsymbol{k} + \frac{(\boldsymbol{k}_i\cdot\boldsymbol{k})q^2 - (\boldsymbol{k}_i\cdot\boldsymbol{q})(\boldsymbol{k}\cdot\boldsymbol{q})}{k^2-q^2}\right]\frac{w(|\boldsymbol{q}+\boldsymbol{k}|)}{|\boldsymbol{q}+\boldsymbol{k}|^2} +$$

$$+ [(\boldsymbol{k}_i\cdot\boldsymbol{q})(\boldsymbol{k}\cdot\boldsymbol{q}) - (\boldsymbol{k}_i\cdot\boldsymbol{k})q^2 + k^2(\boldsymbol{k}_i\cdot\boldsymbol{q}) - (\boldsymbol{k}_i\cdot\boldsymbol{k})(\boldsymbol{k}\cdot\boldsymbol{q})]\cdot$$

$$\cdot\frac{1}{|\boldsymbol{q}+\boldsymbol{k}|}\frac{\partial}{\partial|\boldsymbol{q}+\boldsymbol{k}|}\frac{w(|\boldsymbol{q}+\boldsymbol{k}|)}{|\boldsymbol{q}+\boldsymbol{k}|^2}\Bigg\}\,.$$

In Appendix I, we give a method for computing by a one-variable numerical integration the integral (25), for any tensor force, the Bessel transform (21)

([9]) P. M. Morse and H. Feshbach: *Methods of Theoretical Physics* (New York, 1953).

of which is known. (25) is given by the integral

$$E_0(k_i, f) = -i36(M/\hbar^2)V_0^2 f^{-1}\int_0^\infty \Phi_{k_i}(x)\left[w\left(\frac{2f}{\mu}x\right)\right]^2 dx\,, \tag{27}$$

where $\Phi_{k_i}(x)$ is a universal function $1/\mu$ is the range of the tensor force. For the last bound nucleon, $k_i = f$, and

$$\left\{\begin{aligned}\Phi_f(x) &= \frac{3}{8}(1-x^2)^2 \operatorname{Log}\frac{x+1}{x-1} + \\ &+ 2x(x+1)^2(x-2)\operatorname{Log}2 + \frac{1}{4}x(-2x^3-3x^2+12x+13)\,, \quad \text{if } x<1,\\ \Phi_f(x) &= 2x(x-1)^2(x+2)\operatorname{Log}(x-1) + 2x(x+1)^2(x-2)\operatorname{Log}(x+1) + \\ &+ 4x^2(-x^2+3)\operatorname{Log}x + 2x^2 + 3\,, \quad \text{if } x>1.\end{aligned}\right. \tag{28}$$

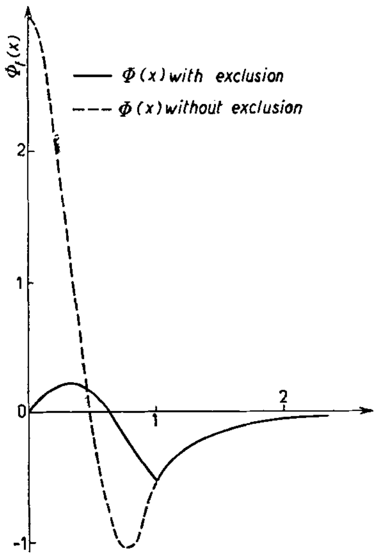

Fig. 1. – The functions $\Phi_f(x)$ for a bound nucleon.

$\Phi_f(x)$ is plotted on Fig. 1. One sees that it changes sign when x increases, so that the magnitude and sign of (1) depend in a sensitive way on the region where (21) is peaked, that is, on the shape of the tensor force and on the ratio f/μ.

In Appendix II, we give a method for computing the integral (26) for $k_i = f$ by a one-variable integration, in the special case of a tensor force

$$u(r) = V_0\,\mu^2 r^2 \exp\left[-\tfrac{1}{2}\mu^2 r^2\right]. \tag{29}$$

It is found that

$$E_1(f, f) = -i54\pi\frac{MV_0^2 f^3}{\hbar^2\mu^{10}}\int_0^\infty N(y)\,dy\,, \tag{30}$$

where

$$\begin{aligned}N(y) = \exp[-(2f^2/\mu^2)y]\Big\{&(1-y)^{\frac{3}{2}}\left(-(466/315)y^2-(272/315)y+(176/105)\right) + \\ &+ y^4-(118/105)y^3-(272/315)y^2+(1136/315)y-(176/105) + \\ &+ \left((16/35)y^4-(16/15)y^3\right)(1/\sqrt{y})\,\mathrm{tgh}^{-1}\sqrt{y} + \left(-(8/15)y+(8/35)\right)\cdot \\ &\cdot \operatorname{Log}(1-y) + (f^2/\mu^2)\big[(1-y)^{\frac{5}{2}}\left(-(944/945)y^2-(1541/945)y+(211/135)\right) + \\ &+ (4/45)y^6-(17/15)y^5+3y^4-(14/9)y^3-(62/15)y^2+(529/105)y - \\ &-(211/135)\big]\Big\}, \qquad \text{if } y<1.\end{aligned}$$

$$N(y) = \exp[-(2f^2/\mu^2)y]\{(512/735)K^7 + ((832/525)y - (1448/525))K^5 +$$
$$+ ((8/315)y^2 - (230/63)y + (334/105))K^3 + (y-1)(-(26/35)y^2 + (2/7)y -$$
$$- (22/105))K + (y-1)^2((13/35)y^2 + (4/21)y - (8/35))(2/\sqrt{y-1})\cdot$$
$$\cdot\ (\mathrm{tg}^{-1}\sqrt{2/\sqrt{y-1}-1} - \mathrm{tg}^{-1}(1/\sqrt{y-1})) - (512/735)Q^7 +$$
$$+ ((576/175)y + (8/25))Q^5 - Q^4 + (-(1352/315)y^2 + (4/45)y)Q^3 +$$
$$+ ((2/3)y + (8/15))Q^2 + (-(92/105)y^3 + (4/15)y^2)Q - (44/245)k^7 +$$
$$+ ((124/175)y - (64/75))k^5 + k^4 + (-(16/105)y^2 + (16/45)y)k^3 +$$
$$+ (-2y + (8/15))k^2 + (-(16/35)y^3 + (16/15)y^2)k + (16/21)q^7 +$$
$$+ (-4y + (16/15))q^5 + ((52/9)y^2 - (28/9)y)q^3 + ((4/3)y^3 - (4/3)y^2)q -$$
$$- (16/245)y^7 + (124/175)y^6 - (1508/525)y^5 + (1123/315)y^4 + (8/7)y^3 -$$
$$- (100/63)y^2 + (4034/1575)y - (1628/1225) - (2/\sqrt{y})[(-(8/35)y^4 +$$
$$+ (8/15)y^3)\,\mathrm{tgh}^{-1}(1/\sqrt{y}) + (-(46/105)y^4 + (2/15)y^3)\,\mathrm{tgh}^{-1}(\sqrt{y}/(1+\sqrt{y-1})) +$$
$$+ ((2/3)y^4 - (2/3)y^3)\,\mathrm{tgh}^{-1}\sqrt{y}/(2y-1) + (-(8/35)y^4 + (8/15)y^3)\cdot$$
$$\cdot\ \mathrm{tgh}^{-1}\sqrt{2-y}] + (-(8/15)y + (8/35))\,\mathrm{Log}\,(y(y-1)^{\frac{3}{2}}/2) + (f^2/\mu^2)\cdot$$
$$\cdot\ [-(16/63)K^9 + (-(16/15)y + (128/105))K^7 + (-(16/15)y^2 + (52/15)y -$$
$$- (28/15))K^5 + (y-1)((4/9)y^2 + (16/9)y - (8/9))K^3 - (4/3)(y-1)^3K +$$
$$+ (16/63)Q^9 + (-(128/105)y - (8/105))Q^7 + (4/9)Q^6 + (8/5)y^2Q^5 -$$
$$- (4/3)yQ^4 + ((4/3)y^2 - (16/15)y + (16/35))Q^2 - (8/189)k^9 +$$
$$+ ((32/105)y - (16/35))k^7 + (4/9)k^6 + (-(8/15)y^2 + (16/15)y)k^5 -$$
$$- (4/3)yk^4 + ((4/3)y^2 - (16/15)y + (16/35))k^2 - (8/27)q^9 +$$
$$+ ((32/31)y - (8/21))q^7 + (-(32/15)y^2 + (16/15)y)q^5 + (8/189)y^9 -$$
$$- (32/105)y^8 + (104/105)y^7 - (68/45)y^6 + (4/3)y^5 - (4/3)y^4 +$$
$$+ (88/45)y^3 - (496/105)y^2 + (64/15)y - (416/315)]\},$$

where

$$K = \sqrt{1 - y + 2\sqrt{y-1}}\,, \quad k = \sqrt{2y - y^2}\,, \quad q = \sqrt{2y-1}\,, \quad Q = 1 + \sqrt{y-1}$$

if $1 < y < 2$.

$$N(y) = \exp[-(2f^2/\mu^2)y]\{P(y, \sqrt{2y-1}) - P(y, \sqrt{2y})\},$$

where

$$P(y, q) = (16/21)q^7 + (-4y + (15/16))q^5 + ((52/9)y^2 - (28/9)y)q^3 + \\ + ((4/3)y^3 - (4/3)y^2)q + (-(4/3)y^4 + (4/3)y^3)(1/\sqrt{y})\,\mathrm{tgh}^{-1}(\sqrt{y}/q) + \\ + (f^2/\mu^2)[-(8/27)q^9 + ((32/21)y - (8/21))q^7 + \\ + (-(32/15)y^2 + (16/15)y)q^5], \qquad \text{if } y > 2. \tag{31}$$

4. – Experimental and calculated results.

For a bound nucleon, (13) is the well-known spin-orbit potential which is responsible for the shell-model. The analysis of low-lying energy levels may determine an average value $\bar{a}$ through the nuclear surface for a $[f(r_i), f(r_i)]$. This determination cannot be very precise, because of various uncertainties; however ^{15}O and ^{17}N are fairly accounted for by $\bar{a} \sim 70$ MeV $(10^{-13}\text{ cm})^5$ (6). The nuclei around ^{208}Pb indicate $\bar{a} \sim 57$ (10). $\bar{a}$ must be positive to account for the « inversed doublet » structure.

We have computed the direct term a_0 of (12), through (27), for some potentials, the Bessel transforms (21) of which are easy to obtain. These potentials are, together with their Bessel transforms:

$$\begin{cases} u(r) = V_0 \exp[-\mu_r]\,, \\ w(k) = V_0 \left[\dfrac{3}{k^3}\,\mathrm{tg}^{-1}\dfrac{k}{\mu} - \mu\,\dfrac{5k^2+3\mu^2}{k^2(k^2+\mu^2)^2}\right]. \end{cases} \tag{32a}$$

$$\begin{cases} u(r) = V_0 \dfrac{\exp[-\mu r]}{\mu r}\,, \\ w(k) = V_0 \left[-\dfrac{3}{k^3}\,\mathrm{tg}^{-1}\dfrac{k}{\mu} + \dfrac{2k^2+3\mu^2}{\mu k^2(k^2+\mu^2)}\right]. \end{cases} \tag{32b}$$

$$\begin{cases} u(r) = V_0 \left(\dfrac{1}{\mu r} + \dfrac{3}{\mu^2 r^2} + \dfrac{3}{\mu^3 r^3}\right) \exp[-\mu r], \ (^{11}) \\ w(k) = V_0 \dfrac{k^2}{\mu^3(k^2+\mu^2)}\,. \end{cases} \tag{32c}$$

$$\begin{cases} u(r) = V_0 \mu^2 r^2 \exp[-\tfrac{1}{2}\mu^2 r^2]\,, \\ w(k) = V_0 \sqrt{\dfrac{\pi}{2}}\,\dfrac{k^2}{\mu^5} \exp\left[-\dfrac{1}{2}\dfrac{k^2}{\mu^2}\right]. \end{cases} \tag{32d}$$

(10) R. J. Blin-Stoyle: *Phil. Mag.*, **46**, 973 (1955).

(11) Such a singular potential is fed into the second Born approximation only for indicative purposes.

The exchange dependence is as indicated in (16). The computed values of a_0 for these different potentials are listed in Table I. The parameters of the potentials were first taken from classical potentials ([12,13,14]) for (32 a, b, c), and for (32d) we tried approximately to reproduce Gartenhaus' potential ([15]). The results obtained in such a way are of the wrong sign. It is easy to see from (27) that a_0 can be positive only for not too singular forces with a long range, since only such forces will contain mostly low momenta, and thus appreciable contribution to the integral in (27) will come mostly from the region where Φ_f is positive. Accordingly, we tried to increase the range $1/\mu$, decreasing at the same time V_0 by a factor estimated from Feshbach's and Schwinger's calculations ([16]). The results are also listed in Table I. It is seen that a_0 may become positive, but remains too small by an order of magnitude.

TABLE I.

Potentials	$1/\mu$ (10^{-13} cm)	V_0 (MeV)	χ	Bound nucleon: a_0 in units of MeV$(10^{-13}$cm$)^5$, with exclusion	Bound nucleon: a_0 in units of MeV$(10^{-13}$cm$)^5$, without exclusion	Bound nucleon: a_1 in units of MeV $(10^{-13}$ cm$)^5$, with exclusion	Bound nucleon: a_1 in units of MeV $(10^{-13}$ cm$)^5$, without exclusion	300 MeV nucleon: a_0 in units of MeV $(10^{-13}$ cm$)^5$, with exclusion	300 MeV nucleon: a_0 in units of MeV $(10^{-13}$ cm$)^5$, without exclusion
(a) exponential	0.75	50.8	0.875	−4	−18	—	—	$19-i27$	$28-i30$
	0.984	22.6	»	5	19	—	—	$15-i16$	$28-i18$
(b) YUKAWA	1.529	8.11	1.29	< 1	−13	—	—	$7-i9$	$13-i10$
	2.362	2.39	»	3	29	—	—	$6-i5$	$14-i6$
	2.756	1.61	»	40	380	—	—	—	—
(d) Mesic	1.40	1.13	1	−9	−12	—	7	$-4-i7$	$-3-i8$
	$+\infty$	—	»	< 0	0	—	> 0	—	—
(e) GARTENHAUS	0.557	46.2	1	−26	−50	−5.7	20	$6-i14$	$8-i15$
	0.788	17	»	−13	−29	−92	17	—	—

a_0 is the calculated direct part of the spin-orbit interaction coefficient.
a_1 is the calculated exchange part.
The experimental value is $a=a_0+a_1=+70$ MeV $(10^{-13}$ cm$)^5$.

(12) R. JASTROW: *Phys. Rev.*, **81**, 165 (1951).
(13) H. H. HALL and J. L. POWELL: *Phys. Rev.*, **90**, 912 (1953).
(14) A. MARTIN and L. VERLET: *Nuovo Cimento*, **12**, 483 (1954).
(15) S. GARTENHAUS: *Phys. Rev.*, **100**, 900 (1955).
(16) H. FESHBACH and J. SCHWINGER: *Phys. Rev.*, **84**, 194 (1951).

In all calculations, the local Fermi momentum was taken as $f=1.27\cdot 10^{-13}$ cm through the whole nuclear surface, a value which would rather correspond to the inside of the nucleus. If the Thomas-Fermi approximation is valid, f should actually be smaller through the nuclear surface, and this would lead to still algebraically smaller values of a_0.

Other models for the nuclear surface do not appreciably change the results. A tentative description of the surface by the model of an infinite potential barrier was made ([17]). We also tried to take into account possible higher momenta in nuclear matter ([18]). All these attempts do not change the above result that the computed a_0 is algebraically too small.

It might be interesting to estimate the importance of the Pauli principle inasmuch as it is possible to consider exclusion effects as being less important on the nuclear surface. a_0 was also computed without taking into account the exclusion effects in the intermediate states, i.e. extending the integrations to all values of $\boldsymbol{q}_i$, $\boldsymbol{q}_j$, $\boldsymbol{q}$ (27) is still valid, but $\Phi_f(x)$ is now given by the second expression of (28) for all values of x. The results are listed in Table I It is seen that a_0 increases, but that a_0 cannot attain proper values for a reasonable range $1/\mu$.

The exchange term a_1 was computed only for the potential (32*d*), with and without exclusion. The results are listed on Table I. Again, proper values are not obtained.

5. – Explicit calculation of the non-modified reaction amplitude.

The failure of obtaining proper values for a may be understood from another point of view, in the special case where exclusion effects are not taken into account.

In this case, it is possible to have an explicit expression for the forward and backward reaction amplitudes, in second Born approximation, at least for the potentials (32*c*, *d*) and for

$$(32e) \qquad \begin{cases} u(r) = V_0\left(1+\dfrac{1}{\mu r}\right)\exp\left[-\mu r\right], \\[2mm] w(k) = V_0\dfrac{2k^2}{\mu(k^2+\mu^2)^2}\,. \end{cases}$$

Without exclusion effects, the reaction amplitude depends only on the

W. J. Swiatecki: *Proc. Phys. Soo.*, A **64**, 226 (1951).

([18]) G. F. Chew and M. L. Goldberger: *Phys. Rev.*, **77**, 470 (1950); K. A. Brueckner, J. Eden and N. C. Francis: *Phys. Rev.*, **98**, 1445 (1955).

relative momenta, and is of the form

$$(33)\qquad (\boldsymbol{k}'_i,\boldsymbol{k}'_j|A|\boldsymbol{k}_i,\boldsymbol{k}_j)=i(2\pi^3)\,\delta(\boldsymbol{k}'_i+\boldsymbol{k}'_j-\boldsymbol{k}_i-\boldsymbol{k}_j)(\boldsymbol{k}'\times\boldsymbol{k})D\,.$$

From (23) and (24), it is found that, for forward scattering

$$(34)\qquad D=D_0(k)=\frac{18}{\pi}\frac{M}{\hbar^2}\frac{1}{k^2}P\int\mathrm{d}^3q\,\frac{k^2-\boldsymbol{k}\cdot\boldsymbol{q}}{k^2-q^2}\left[\frac{w(|\boldsymbol{q}-\boldsymbol{k}|)}{|\boldsymbol{q}-\boldsymbol{k}|}\right]^2,$$

while, for backward scattering

$$(35)\qquad D=D_1(k)=\frac{18}{\pi}\frac{M}{\hbar^2}\frac{1}{k^2}P\int\mathrm{d}^3q\,\frac{w(|\boldsymbol{q}-\boldsymbol{k}|)}{|\boldsymbol{q}-\boldsymbol{k}|^2}\cdot$$
$$\cdot\left\{\left[-k^2+\frac{k^2q^2-(\boldsymbol{k}\cdot\boldsymbol{q})^2}{k^2-q^2}\right]\frac{w(|\boldsymbol{q}+\boldsymbol{k}|)}{|\boldsymbol{q}+\boldsymbol{k}|^2}+\frac{(\boldsymbol{k}\cdot\boldsymbol{q})^2-k^2q^2}{|\boldsymbol{q}+\boldsymbol{k}|}\,\frac{\partial}{\partial|\boldsymbol{q}+\boldsymbol{k}|}\,\frac{w(|\boldsymbol{q}+\boldsymbol{k}|)}{|\boldsymbol{q}+\boldsymbol{k}|^2}\right\}.$$

For potentials (32*c*, *d*, *e*), (34) and (35) provide the respective forward and backward amplitudes:

$$(36c)\qquad D_0(k)=12\pi\frac{MV_0^2}{\hbar^2\mu^7}\left[-\frac{3\mu^2(3k^2+\mu^2)}{k^2(4k^2+\mu^2)}+\frac{3\mu^3}{2k^3}\,\mathrm{tg}^{-1}\frac{2k}{\mu}\right],$$

$$(37c)\qquad D_1(k)=12\pi\frac{MV_0^2}{\hbar^2\mu^7}\left[-\frac{3\mu^2}{4k^2}+\frac{3\mu^3(4k^2+\mu^2)}{8k^3(2k^2+\mu^2)}\,\mathrm{tg}^{-1}\frac{2k}{\mu}\right].$$

$$(36d)\qquad D_0(k)=6\pi\sqrt{\pi}\,\frac{MV_0^2}{\hbar^2\mu^7}\cdot$$
$$\cdot\left[-\left(\frac{3\mu^3}{2k^3}+6\frac{\mu}{k}+12\frac{k}{\mu}\right)\frac{\sqrt{\pi}}{2i}\,\mathrm{Erf}\left(i\frac{2k}{\mu}\right)\exp\left[-\frac{4k^2}{\mu^2}\right]+\frac{3\mu^2}{k^2}+3\right],$$

$$(37d)\qquad D_1(k)=6\pi\sqrt{\pi}\,\frac{MV_0^2}{\hbar^2\mu^7}\cdot$$
$$\cdot\left[-1-2\frac{k^2}{\mu^2}+\frac{4k^3}{\mu^3}\exp\left[-\frac{k^2}{\mu^2}\right]\frac{\sqrt{\pi}}{2i}\,\mathrm{Erf}\left(i\frac{k}{\mu}\right)\right]\exp\left[-\frac{k^2}{\mu^2}\right].$$

$$(36e)\qquad D_0(k)=\frac{6\pi MV_0^2}{\hbar^2\mu^7}\,\frac{-1-24(k^2/\mu^2)+48(k^4/\mu^4)}{[1+4(k^2/\mu^2)]^3},$$

$$(37e)\qquad D_1(k)=\frac{6\pi MV_0^2}{\hbar^2\mu^7}\left[\frac{3\mu^8}{2k^2(k^2+\mu^2)(2k^2+\mu^2)^2}-\frac{3\mu^5(8k^4+4k^2\mu^2+\mu^4)}{2k^3(2k^2+\mu^2)^3}\cdot\right.$$
$$\left.\cdot\left(\mathrm{tg}^{-1}\frac{2k}{\mu}-\mathrm{tg}^{-1}\frac{k}{\mu}\right)\right].$$

These amplitudes are plotted on Fig. 2. a_0 and a_1 are respectively obtained by summing (36) and (37) on all collision, with the results:

$$a_0(f,f) = [(1-\chi)^2 + 3\chi^2]\frac{6}{f^3}\int_0^f \mathrm{d}k\, k^3(f^2-k^2)\,D_0(k)\,, \tag{38}$$

$$a_1(f,f) = \left(\frac{1}{2} + 2\chi - 4\chi^2\right)\frac{6}{f^3}\int_0^f \mathrm{d}k\, k^3(f^2-k^2)\,D_1(k)\,. \tag{39}$$

These expressions may be used to check the corresponding results, without exclusion, in Table I.

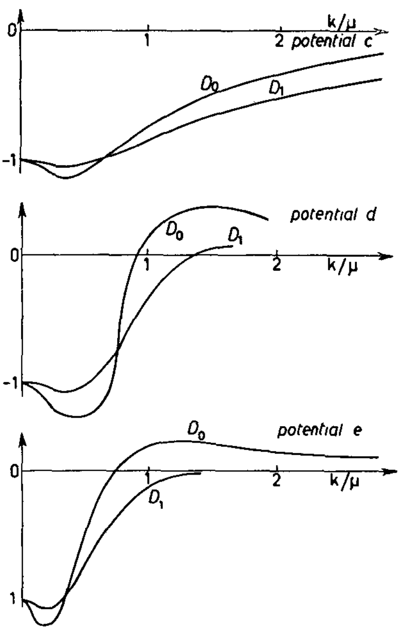

Fig. 2. – The reaction amplitudes D_0 and D_1 as functions of k/μ. c(D_0 and D_1 are in units of $12\pi M V_0^2/\hbar^2\mu^7$; d) D_0 and D_1 are in units of $6\pi\sqrt{\pi} M V_0^2/\hbar^2\mu^7$; e) D_0 and D_1 are in units of $6\pi M V_0^2/\hbar^2\mu^7$.

From a qualitative point of view, it is interesting to note that the contributions of the right sign (i.e. positive) to a_0 come only from the high values of k in (38), for which D_0 may become positive if the tensor potential $u(r)$ is not too singular (Fig. 2). The low values of k always make D_0 negative (this last result may be shown to be independent of the potential). The small net result comes from cancellation between the two regions.

The sign of a_1 is determined in (39) by the exchange character of the tensor potential. However, its magnitude was shown to be always too small, for reasonable potentials.

6. – High energies (300 MeV).

The strong polarization observed in the high scattering of nucleons by nuclei may be accounted for by a spin-orbit potential (13) ([19]) where a has an average value of the same order of magnitude as in shell-model problems. Values of a in the literature ([20]) are around 40 MeV $(10^{-13}\text{ cm})^5$.

([19]) E. FERMI: *Nuovo Cimento*, **11**, 407 (1954).

([20]) W. HECKROTTE and J. V. LEPORE: *Phys. Rev.*, **94**, 500 (1954); **95**, 1109 (1954); G. A. SNOW, R. M. STERNHEIMER and C. N. YANG: *Phys. Rev.*, **94**, 1073 (1954); B. J. MALENKA: *Phys. Rev.*, **95**, 522 (1954); R. M. STERNHEIMER: *Phys. Rev.*, **95**, 597 (1954); **97**, 1314 (1955); **100**, 886 (1955).

Our calculations of the direct term can be readily extended to higher energies. (The exchange term is not expected to be important). To describe a scattering problem, we use the scattering amplitude rather than the reaction amplitude, by inserting a small imaginary part into the energy denominator of (14). The actual calculations was performed for $k_i = 3f$; this corresponds to about 300 MeV for the incident nucleon. (27) is still valid but $\Phi(x)$ now reads:

$$(40)\quad \left\{\begin{array}{ll}
\Phi_{3f}(x) = \left(-4 + \frac{8}{9}\right)\mathrm{Log}\,2 + \left(-\frac{2}{27}x^4 + \frac{10}{9}x^2 - \frac{52}{27}x + \frac{8}{9}\right)\mathrm{Log}\,(1-x) + & \\
+ \left(\frac{2}{27}x^4 - \frac{10}{9}x^2 - \frac{56}{27}x - \frac{8}{9}\right)\mathrm{Log}\,(x+2) + & \\
+ \left(-\frac{1}{27}x^4 + x^2 + 2x\right)\mathrm{Log}\,(x+3) + \left(\frac{1}{27}x^4 - x^2 + 2x\right)\mathrm{Log}\,(3-x) + & \\
+ \frac{1}{9}x^2 + \frac{16}{9}x + i\pi\left(\frac{1}{27}x^4 - \frac{1}{9}x^2 - \frac{2}{27}x\right), & \text{if } 0 < x < 1. \\
 & \\
\Phi_{3f}(x) = \left(-\frac{2}{27}x^4 + \frac{10}{9}x^2 + 2x + \frac{8}{9}\right)\mathrm{Log}\,\frac{x+1}{x+2} + & \\
+ \left(\frac{5}{216}x^4 - \frac{7}{36}x^2 + \frac{5}{24} + \frac{7}{27x^2} - \frac{2}{9x^4}\right)\mathrm{Log}\,\frac{(2-x)(x+1)}{(x+2)(x-1)} + & \\
+ \frac{2}{27}x\,\mathrm{Log}\,\frac{(x-1)(2-x)}{4x^2} - \frac{1}{36}x^3 + \frac{1}{9}x^2 + \frac{79}{108}x + \frac{5}{6} + \frac{1}{54x} + \frac{2}{9x^3} + & \\
+ i\pi\left(-\frac{5}{216}x^4 + \frac{7}{36}x^2 - \frac{2}{27}x - \frac{5}{24} - \frac{7}{27x^2} + \frac{2}{9x^4}\right), & \text{if } 1 < x < 2. \\
 & \\
\Phi_{3f}(x) = \left(-\frac{2}{27}x^4 + \frac{10}{9}x^2 + 2x + \frac{8}{9}\right)\mathrm{Log}\,\frac{x+1}{x+2} + & \\
+ \left(-\frac{2}{27}x^4 + \frac{10}{9}x^2 - 2x + \frac{8}{9}\right)\mathrm{Log}\,\frac{x-1}{x-2} - & \\
- \frac{2}{27}x\,\mathrm{Log}\,\frac{(x+1)(x+2)}{(x-1)(x-2)} + \frac{2}{9}x^2 + \frac{5}{3}, & \text{if } x > 2.
\end{array}\right.$$

$\mathrm{Re}\,\Phi_{3f}(x)$ and $(1/\pi)\,\mathrm{Im}\,\Phi_{3f}(x)$ are plotted on Fig. 3.

We also compute a_0 neglecting the exclusion principle in the intermediate states. In that case, $\mathrm{Re}\,\Phi(x)$ is the third expression (40) for all values of x, and

$$(41)\quad \left\{\begin{array}{ll}
\mathrm{Im}\,\Phi_{3f}(x) = \pi\left(-\frac{4}{27}x\right), & \text{if } x < 1. \\
 & \\
\mathrm{Im}\,\Phi_{3f}(x) = \pi\left(-\frac{2}{27}x^4 + \frac{10}{9}x^2 - \frac{56}{27}x + \frac{8}{9}\right), & \text{if } 1 < x > 2. \\
 & \\
\mathrm{Im}\,\Phi_{3f}(x) = 0, & \text{if } x > 2.
\end{array}\right.$$

The results for a_0 are listed on Table I for the previously considered potentials.

It is interesting to see that the Pauli principle is still so important at 300 MeV, at least for the real part of a_0, except for the singular potential (32*c*). This looks quite general, and can be understood easily, because, for regular potentials, mostly small momentum transfers occur in (21), and the small transfers are still forbidden by the exclusion principle, whatever the incident nucleon energy is.

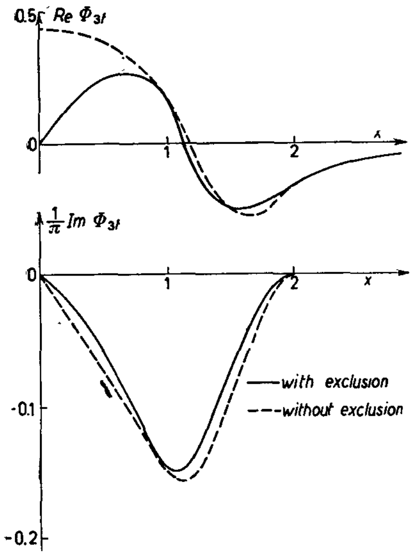

Fig. 3. – The function $\Phi_{3f}(x)$ for an incident 300 MeV nucleon.

The sign of a_0 is correct, for regular potentials, but its magnitude is too small. This is in agreement with the commonly accepted idea that regular tensor forces do not provide enough polarization at high energy. Insertion of singular forces (the Born approximation being no longer used) could probably lead to higher polarizations.

a_0 is complex. It may be noted that the real and imaginary parts are of opposite signs and of the same order of magnitude. Such a complex spin-orbit potential has been proposed ([21]) to explain the scattering from carbon, in the small-angle region where there is interference with the Coulomb scattering.

7. – Discussion and conclusions.

In principle, the failure of tensor forces to account for the shell-model spin-orbit coupling is a strong indication for the existence of elementary mutual spin-orbit forces in the two-body interaction. However, our computation involves several assumptions, some of which will be now discussed.

7·1. *The Born approximation.* – It was shown in Sect. 5 that, in the case where the exclusion effects are neglected, the negative value or smallness of a_0 come from the behaviour of $D_0(k)$ for small k. This behaviour was estimated

([21]) W. Heckrotte: *Phys. Rev.*, **101**, 1406 (1956).

by computations from the second Born approximation while the actual $D_0(k)$ could be different. Actually, $D_0(k)$ ought to have a pole at low energy, since there is a bound state for the deuteron (6), and our approximation does not reproduce this pole.

However, in the case where exclusion effects are taken into account the Born approximation was claimed to be much more justified (2), at least in the case of regular forces. It is still possible that singular tensor forces would alter the results. Of course, in the second Born approximation, we showed that for the singular tensor forces (32*c*), things are rather worse, but for such a force, the Born approximation is certainly bad. Thus, our results, at best, are restricted to smooth tensor forces, and, if singularities, hard cores, etc., could play a part even at low energy, this could change our results.

7·2. *The many-particle terms.* – In the spirit of Brueckner's theory, we have computed only these terms of a perturbation series, which are contained in the two-body reaction amplitude (14). Up to the second order in the interaction, these terms are the only ones which occur, only if momentum is conserved in two-body collisions. This is indeed the case in the approximation of infinite nuclear matter, and the assumption of momentum-conservation has some consistency with a Thomas-Fermi approximation.

However, in the nuclear surface, there could occur terms that do not conserve momentum. In the second order perturbation theory, these terms are for instance 3-particle groups, like $v_{12}v_{13}$. Such terms have been previously considered in light nuclei by FEINGOLD *et al.* (1). FEINGOLD could show that, in this case, the most important part of the vector forces are indeed such 3-body effective forces. The question therefore remains open to know if, in heavier nuclei, contrarily to the spirit of Brueckner's model, such 3-particle terms are important.

7·3. *Elementary spin-orbit forces.* – With the above restrictions, our calculations indicate that tensor forces cannot account for the spin-orbit coupling. Therefore, these results rather strengthen the idea of elementary mutual spin-orbit forces in the two-body interaction. From such forces, an average $\boldsymbol{l}\cdot\boldsymbol{s}$ effect can of course be easily derived (22). The point of not assuming elementary spin orbit forces from the beginning was for simplicity, since tensor forces are anyhow known to exist, and also because the field theory does not account for a spin-orbit two-body interaction. On another hand, there is

---

(22) J. HUGUES and K. J. LECOUTEUR: *Proc. Phys. Soc.*, A **63**, 1212 (1950); I. TALMI: *Helv. Phys. Acta*, **25**, 185 (1952); J. P. ELLIOTT and A. M. LANE: *Phys. Rev.*, **96**, 1160 (1954).

growing evidence from two-body scattering and polarization experiments that two-body spin-orbit forces do exist ([23]).

* * *

I am indebted to Dr. L. VERLET for helpful discussions.

APPENDIX I

The direct term.

To compute (25), we make the change of variable

$$\boldsymbol{z} = \boldsymbol{q} - \boldsymbol{k}\,. \tag{A.1}$$

Then

$$E_0(k_i, f) = -i\,\frac{9M}{16\pi^2\hbar^2}\,\frac{3}{4\pi f^3}\,\frac{1}{k_i^2}\int\limits_{|\boldsymbol{z}+\boldsymbol{k}_i|>f}\mathrm{d}^3 z \int\limits_{\substack{k_j<f\\|\boldsymbol{k}_j-\boldsymbol{z}|>f}}\mathrm{d}^3 k_j\,\frac{\boldsymbol{z}\cdot\boldsymbol{k}}{z^2+(\boldsymbol{k}_i-\boldsymbol{k}_j).\boldsymbol{z}}\left[\frac{w(z)}{z}\right]^2. \tag{A.2}$$

We first perform the integration with respect to k_j. The cases $z<2f$ and $z>2f$ must be distinguished. The integration on angles of $\boldsymbol{z}$ is then performed. The final integration on $\boldsymbol{z}$ is the numerical integral (27) where $x = z/2f$.

APPENDIX II

The exchange term.

The calculation is too long to be given here. We just indicate the method. To compute (25), in the special case (29), we make use of the same kinds of methods as Euler ([24]). We choose as variables $\boldsymbol{q}$, $\boldsymbol{k}$. The domain of integration Ω

([23]) P. S. SIGNELL and R. E. MARSHAK: *Phys. Rev.*, **106**, 832 (1957).
([24]) H. EULER: *Zeits. f. Phys.*, **105**, 553 (1937).

in $\boldsymbol{q}$ is of revolution around $\boldsymbol{K} = \boldsymbol{k}_i - \boldsymbol{k}$. We first integrate over polar angles of $\boldsymbol{q}$, $\boldsymbol{K}$ being the polar axis. Then, $\boldsymbol{k}$ is defined by k and K, and we perform the integration over the polar angles of $\boldsymbol{k}$, i. e. over K. Of course, many regions appear because of the limits. We are left with an integral over q and k. The further change of variable

$$k^2 - q^2 = 2\,x, \qquad k^2 + q^2 = 2y\,,$$

is done. The integration over x can still be performed analytically, and we are left with the numerical integration (30) over y.

RIASSUNTO (*)

Abbiamo indagato se si possano spiegare le forze spin-orbita nei nuclei complessi come un effetto del second'ordine delle forze tensoriali. Secondo i metodi di Brueckner calcoliamo dapprima un'ampiezza di reazione modificata per le collisioni di due nucleoni entro la materia nucleare, usando la seconda approssimazione di Born e tenendo conto del principio d'esclusione negli stati intermedi. Otteniamo in seguito il potenziale spin-orbita medio al quale è soggetto un nucleone sommando su tutte le sue collisioni l'ampiezza di reazione. Il potenziale spin-orbita così ottenuto risulta di ordine di grandezza troppo piccolo e può anche essere dotato del segno sbagliato. Discutiamo qualche conseguenza di tali risultati.

(*) *Traduzione a cura della Redazione.*