

MAGNETIC PROPERTIES OF A NEARLY CLASSICAL ONE-COMPONENT PLASMA IN THREE OR TWO DIMENSIONS

II. STRONG FIELD

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The equilibrium statistical mechanics of a nearly classical one-component plasma, submitted to a strong magnetic field, is studied, in three or two dimensions, by a suitable expansion of the Wigner distribution function. A strong magnetic field quenches the quantum fluctuations transverse to the field. The situation is especially simple for a two-dimensional plasma, which has a classical behaviour in the strong-field limit; as a consequence, a classical Wigner crystallization can be induced by the magnetic field.

1. Introduction

The one-component plasma is a system of identical particles of charge e and mass m embedded in a uniform neutralizing background of opposite charge. The two-dimensional one-component plasma is a reasonable model for a system of electrons deposited on the surface of liquid helium¹⁾ (in that case the quantum effects are weak) and also for a MOS (metal–oxide–semiconductor) inversion layer¹⁾ (for which the quantum effects are usually dominant).

In a previous paper²⁾, we have studied the equilibrium statistical mechanics of a nearly classical one-component plasma in a magnetic field. In the present paper, we extend this study to high field values, for which the straight expansion with respect to Planck's constant \hbar , which was used in I, no longer works. Indeed, an expansion with respect to \hbar is useful only if the dimensionless parameters built with \hbar are small enough. The state of the plasma is characterized by the temperature T , and the number density ρ (or, alternatively, the average interparticle distance a defined as $a = (3/4\pi\rho)^{1/3}$ in the three-dimensional case and $a = (\pi\rho)^{-1/2}$ in the two-dimensional case). The corresponding dimensionless parameters can be chosen as $s = \hbar^2/ma^2k_B T$ and $t = k_B T\hbar^2/me^4$, where k_B is Boltzmann's constant. When a magnetic field B is

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present, a third dimensionless parameter appears, which can be chosen as $u = \hbar\omega/2k_B T$, where $\omega = eB/mc$ is the cyclotron frequency. These parameters can also be considered as ratios of characteristic lengths, since $s = (2\pi)^{-1}(\lambda/a)^2$, $t = (2\pi)^{-1}(\lambda/b)^2$, and $u = (8\pi)^{-1/2}\lambda/R$, where $\lambda = (2\pi\hbar^2/mk_B T)^{1/2}$ is the de Broglie wavelength, $b = e^2/k_B T$ is the two-body average classical distance of closest approach, and $R = (mc^2 k_B T/e^2 B^2)^{1/2}$ is an average gyration radius in the field. In I, all three parameters s , t , u , were assumed to be small, and therefore too strong magnetic fields were excluded by the small- u requirement. In the present paper, we relax any restriction about u , keeping, however, the assumption that s and t are sufficiently small; we devise a modified approach, in which we expand with respect to s and t only, for some arbitrary value of u . The strength of the interactions is characterized as usual by the parameter $\Gamma = (s/t)^{1/2} = e^2/k_B T a$; Γ may have any value, although we are more interested in the non-trivial strong-coupling case $\Gamma \gg 1$.

In the strong-field limit ($u \gg 1$), all particles are in the lowest Landau level; therefore this situation is sometimes called the extreme quantum limit. Nevertheless, this situation may as well be considered as a very classical one, especially in the two-dimensional case, for which a very simple picture emerges. The Landau orbits behave like heavy points, as if the electron mass were very large, and the validity of a classical description is improved as the field is increased. A lowest Landau orbit has a characteristic radius $l = (\hbar c/eB)^{1/2}$. If $l < \lambda$, the Landau radius l replaces the de Broglie wavelength λ as a characteristic quantum length scale, and the density range of the classical domain is increased towards higher densities.

The organization of this paper is as follows. In section 2, we discuss the Wigner distribution function for a system of charged particles submitted to a magnetic field. In section 3, a generalized semi-classical expansion, adapted to the strong-field case, is obtained for this Wigner distribution function. This expansion is used for computing the current density in section 4, the density in configuration space and the thermodynamics in section 5. The exchange effects are studied in section 6, for the two-dimensional case; they are found to be exponentially small in the strong-field limit. Finally, the solid-fluid phase transition in two dimensions is discussed in section 7.

2. Wigner distribution

The exchange effects will be considered in section 6; they will be shown to be small. Therefore, Boltzmann statistics can be used as a starting point. The spins of the particles are then decoupled from the orbital degrees of freedom, and the spin magnetic moments contribute to the total magnetization a trivial

paramagnetic term which has been described in I and will not be further considered. here. We first introduce the Wigner distribution function³). We consider an N -particle system, with a hamiltonian H . We call $r = (r_1, r_2, \dots, r_N)$ the position in configuration space, $p = (p_1, p_2, \dots, p_N)$ the position in momentum space, and set $\beta = 1/k_B T$. The Wigner distribution $f(r, p, \beta)$ is a representation of the density matrix, defined* as

$$f(r, p, \beta) = \int ds e^{i(\hbar)p \cdot s} \left\langle r - \frac{s}{2} \left| e^{-\beta H} \right| r + \frac{s}{2} \right\rangle. \tag{2.1}$$

The density matrix obeys Bloch's equation, which becomes in the Wigner representation

$$\frac{\partial f(r, p, \beta)}{\partial \beta} = -H(r, p) \left\{ \cos \left[\frac{\hbar}{2} (\nabla_p \cdot \nabla_r - \nabla_r \cdot \nabla_p) \right] \right\} f(r, p, \beta), \tag{2.2}$$

where $H(r, p)$ is the classical hamiltonian; the gradient operators operate either to the right or to the left, as indicated by the arrows. The initial condition is $f = 1$ at $\beta = 0$. Our first aim is to compute f by solving eq. (2.2).

Here, we consider a system of charged particles submitted to an external magnetic field B , and to a scalar potential V which includes their mutual interactions. The magnetic field is described by a vector potential $A(r)$, and we set $A = (A(r_1), A(r_2), \dots, A(r_N))$. The classical hamiltonian is

$$H = \frac{1}{2m} \left(p - \frac{e}{c} A \right)^2 + V(r). \tag{2.3}$$

It is especially convenient to use the "physical" momentum (mass times velocity) $\pi = p - (e/c)A$ rather than the canonical momentum p . Changing from the variables (r, p) to the variables (r, π) , and using $B = \text{curl } A$, we obtain from (2.2) and (2.3),

$$\frac{\partial f(r, \pi, \beta)}{\partial \beta} = - \left(\frac{\pi^2}{2m} + V \right) \left\{ \cos \left[\frac{\hbar}{2} \left[\nabla_\pi \cdot \left(\nabla_r + \frac{e}{c} B \wedge \nabla_\pi \right) - \nabla_r \cdot \nabla_\pi \right] \right] \right\} f(r, \pi, \beta), \tag{2.4}$$

where $B \wedge \nabla_\pi = (B \wedge \nabla_{\pi_1}, B \wedge \nabla_{\pi_2}, \dots, B \wedge \nabla_{\pi_N})$. After the derivations to the left have been performed, (2.4) becomes**

$$\frac{\partial f(r, \pi, \beta)}{\partial \beta} = - \left\{ \frac{\pi^2}{2m} + V(r) - \frac{1}{8m} \left(\hbar \nabla_r + \frac{e\hbar}{c} B \wedge \nabla_\pi \right)^2 - 2 \left[\sin^2 \left(\frac{\hbar}{4} \nabla_r \cdot \nabla_\pi \right) \right] V(r) \right\} f(r, \pi, \beta), \tag{2.5}$$

* A slightly different definition has been used in I.
 ** This equation is essentially equivalent to a previously derived one¹³).

where ∇_r in the argument of the sine operates only on $V(r)$; from now on, all gradients operate to the right as usual.

Eq. (2.5) provides a very convenient approach to the diamagnetism of a nearly classical one-component plasma. It has the nice feature of being explicitly gauge-invariant, since it contains the field \mathbf{B} itself rather than the vector potential \mathbf{A} . It can be used for computing microscopic quantities such as the current density⁴⁾ or macroscopic quantities such as the free energy.

The derivation of (2.5) requires no assumption about the spatial dependence of the field \mathbf{B} . From now on, we shall assume \mathbf{B} to be uniform and directed along the z -axis. In the two-dimensional case, \mathbf{B} is assumed to be normal to the plane (x, y) of the system.

3. Modified Wigner-Kirkwood expansion

When all three quantum parameters s, t, u , which have been defined in the introduction, are small, one can look for a solution of (2.5) as a series in powers of \hbar^2 . The leading term is the classical Maxwell-Boltzmann distribution

$$f_{cl} = \exp\left[-\beta\left(\frac{\pi^2}{2m} + V\right)\right]; \quad (3.1)$$

the field \mathbf{B} has no effect on this leading term, in agreement with the Bohr-Van Leeuwen theorem which states that magnetism does not exist in classical physics. Further terms of order \hbar^2, \hbar^4, \dots , can be computed and lead to the same results as in I with a little less labour.

Here, we are interested in the situation when s and t are small, but u may be large. It is then possible to devise a modified expansion method.* We shall look for a solution of (2.5) as a series in powers of $\hbar\nabla_r$, but no expansion will be made with respect to $\hbar\mathbf{B}$.

To zeroth order in $\hbar\nabla_r$, the solution f_0 of (2.5) obeys

$$\frac{\partial f_0}{\partial \beta} = \left(\frac{e^2 \hbar^2 \mathbf{B}^2}{8mc^2} \nabla_{\pi \perp}^2 - \frac{\pi^2}{2m} - V\right) f_0, \quad (3.2)$$

where $\nabla_{\pi \perp}$ means that only the components of ∇_{π} normal to \mathbf{B} must be kept (in two dimensions, $\nabla_{\pi \perp}$ is identical with ∇_{π}). One looks for a solution of (3.2) of the form

$$f_0 = \exp\left[-\beta\left(\frac{\pi_x^2}{2m} + V\right) - a(\beta)(\pi_x^2 + \pi_y^2) - b(\beta)\right], \quad (3.3)$$

where a notation such as π_x^2 means a sum on all particles $\pi_{1x}^2 + \pi_{2x}^2 + \dots + \pi_{Nx}^2$

* Another expansion method for the strong-field case has been recently devised¹⁴⁾.

(π_{ix} is the x -component of the “physical” momentum π_i of particle i); for a two-dimensional system, the z -components must be omitted. Carrying (3.3) into (3.2), one obtains two coupled differential equations for $a(\beta)$ and $b(\beta)$, which are easily solved (the initial conditions are such that $f_0 = 1$ at $\beta = 0$). One finds

$$f_0(r, \pi, \beta) = \frac{1}{\cosh u} \exp \left\{ -\frac{\beta}{2m} \left[\frac{\tanh u}{u} (\pi_x^2 + \pi_y^2) + \pi_z^2 \right] - \beta V(r) \right\}. \quad (3.4)$$

The essential feature of the strong-field case already appears in this zeroth-order expression: the effect of the field is to replace the mass m in front of $\pi_x^2 + \pi_y^2$ by $mu/\tanh u$, a quantity which increases with the field. Therefore, in the strong-field limit, the motion transverse to the field is governed by a large effective mass, and the transverse quantum fluctuations are expected to be quenched.

Further terms of order $\hbar \nabla_r$, $\hbar^2 \nabla_r^2$, \dots , are obtained by looking for a solution of (2.5) of the form

$$f = f_0(1 + \hbar \chi_1 + \hbar^2 \chi_2 + \dots). \quad (3.5)$$

Let us stress again that the \hbar which appear explicitly in (3.5) come from the expansion of (2.5) with respect to $\hbar \nabla_r$, only, while χ_1, χ_2, \dots , still depend on $\hbar B$. Using (3.5) in (2.5), one obtains equations for χ_1, χ_2, \dots , by successive iterations. It is convenient to change from the variable β to the variable u . The equation for χ_1 is

$$\begin{aligned} \frac{\partial \chi_1}{\partial u} + \tanh u \left(\pi_x \frac{\partial}{\partial \pi_x} + \pi_y \frac{\partial}{\partial \pi_y} \right) \chi_1 - \frac{m \hbar \omega}{4} \left(\frac{\partial^2}{\partial \pi_x^2} + \frac{\partial^2}{\partial \pi_y^2} \right) \chi_1 \\ = \frac{2}{m(\hbar \omega)^2} u \tanh u \left(\pi_x \frac{\partial V}{\partial y} - \pi_y \frac{\partial V}{\partial x} \right), \end{aligned} \quad (3.6)$$

where, again, a notation such as $\pi_x(\partial/\partial \pi_x)$ means a sum on all $\pi_{ix}(\partial/\partial \pi_{ix})$.

Guided by the form of the right-hand side of (3.6), one looks for a solution of the form

$$\chi_1 = \frac{2}{m(\hbar \omega)^2} y(u) \left(\pi_x \frac{\partial V}{\partial y} - \pi_y \frac{\partial V}{\partial x} \right). \quad (3.7)$$

Carrying (3.7) into (3.6), one obtains for $y(u)$ the differential equation

$$\frac{dy}{du} + y \tanh u = u \tanh u. \quad (3.8)$$

Multiplying both sides of (3.8) by $\cosh u$, we obtain

$$\frac{d}{du} (y \cosh u) = u \sinh u. \quad (3.9)$$

The initial condition is such that $\chi_1 = 0$ at $u = 0$. One finds immediately from (3.9)

$$y(u) = u - \tanh u, \quad (3.10)$$

and therefore

$$\chi_1 = \frac{\beta^2}{2mu^2} (u - \tanh u) \left(\pi_x \frac{\partial V}{\partial y} - \pi_y \frac{\partial V}{\partial x} \right). \quad (3.11)$$

The equation for χ_2 , although more complicated, can be solved by similar methods. One is led to differential equations of the form

$$\frac{dy}{du} + 2y \tanh u = g(u), \quad (3.12)$$

which are solved by multiplying both sides by $\cosh^2 u$. The result is

$$\begin{aligned} \chi_2 = & \frac{\beta^4}{8m^2u^4} (u - \tanh u)^2 \left(\pi_x \frac{\partial V}{\partial y} - \pi_y \frac{\partial V}{\partial x} \right)^2 \\ & + \frac{\beta^3}{16m^2u^3} (-3u + u \tanh^2 u + 3 \tanh u) \left(\pi_x \frac{\partial}{\partial y} - \pi_y \frac{\partial}{\partial x} \right)^2 V \\ & + \frac{\beta^3}{16m^2u^3} (-u + u \tanh^2 u + \tanh u) \left(\pi_x \frac{\partial}{\partial x} + \pi_y \frac{\partial}{\partial y} \right)^2 V \\ & + \frac{\beta^3}{4m^2u^3} (u - \tanh u) \pi_z \frac{\partial}{\partial z} \left(\pi_x \frac{\partial}{\partial x} + \pi_y \frac{\partial}{\partial y} \right) V \\ & + \frac{\beta^3}{24m^2} \pi_z^2 \frac{\partial^2 V}{\partial z^2} + \frac{\beta^3}{8mu^3} (u - \tanh u) \left[\left(\frac{\partial V}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial y} \right)^2 \right] \\ & - \frac{\beta^2}{8mu} \tanh u \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) + \frac{\beta^3}{24m} \left(\frac{\partial V}{\partial z} \right)^2 - \frac{\beta^2}{8m} \frac{\partial^2 V}{\partial z^2}. \end{aligned} \quad (3.13)$$

At this point, we have obtained an explicit expression for the Wigner distribution function f up to the second order in $\hbar \nabla_r$. For a two-dimensional system, the terms involving z must be omitted.

In the zero field case ($u = 0$), we recover a known \hbar -expansion³⁾ of f . In the strong-field limit ($u \gg 1$), the x and y parts of χ_1 and χ_2 go to zero, another indication that the quantum fluctuations transverse to the field are quenched.

4. Current density

The Wigner distribution function can be used for computing the electrical current density; the current density at r_1 is

$$\mathbf{j}(\mathbf{r}_1) = \frac{e}{m} \frac{N \int f \pi_1 d\pi dr_2 \dots dr_N}{\int f d\pi dr} \tag{4.1}$$

When the expansion (3.5), up to order \hbar^2 , is used for f , only the term of order \hbar , which is odd in π , contributes to the numerator of (4.1); to the same order, f can be replaced by f_0 in the denominator of (4.1). The integrals upon π are easily performed. One finds, to first order in $\hbar \nabla_r$,

$$\mathbf{j}(\mathbf{r}) = c \operatorname{curl} \mathbf{M}(\mathbf{r}), \tag{4.2}$$

where $\mathbf{M}(\mathbf{r})$ plays the role of a magnetization density. \mathbf{M} is related to the classical one-body density

$$n(\mathbf{r}_1) = \frac{N \int e^{-\beta V} dr_2 \dots dr_N}{\int e^{-\beta V} dr} \tag{4.3}$$

by

$$\mathbf{M}(\mathbf{r}) = -e_z \frac{e\hbar}{2mc} \left(\coth u - \frac{1}{u} \right) n(\mathbf{r}), \tag{4.4}$$

where e_z is the unit vector along the field \mathbf{B} .

Therefore, the current density is localized in the regions of variable density, i.e. near the surface for a uniform system, as it should. Within the first order in $\hbar \nabla_r$, which is used here, $\mathbf{M}(\mathbf{r})$ depends on the local density $n(\mathbf{r})$ and on the applied field by a simple Langevin law, the same as the one which describes the diamagnetism of a system of free particles. In the weak-field limit $u \ll 1$, one finds, to order \hbar^2 ,

$$\mathbf{M}(\mathbf{r}) = -\frac{\beta e^2 \hbar^2 n(\mathbf{r})}{12m^2 c^2} \mathbf{B}, \tag{4.5}$$

in agreement with a previous calculation⁴). In the strong-field limit $u \rightarrow \infty$, \mathbf{M} saturates to minus one Bohr magneton per particle:

$$\mathbf{M}(\mathbf{r}) = -e_z \frac{|e|\hbar}{2mc} n(\mathbf{r}). \tag{4.6}$$

5. Density in configuration space and thermodynamics

The N -body density in configuration space is obtained by an integration upon π :

$$\langle r | e^{-\beta H} | r \rangle = \frac{1}{(2\pi\hbar)^{dN}} \int f(r, \pi, \beta) d\pi, \quad (5.1)$$

where d is the number of dimensions (2 or 3). The expansion (3.5) is used for f ; there is no contribution from χ_1 which is odd in π . One finds, to second order in $\hbar\nabla_r$,

$$\begin{aligned} \langle r | e^{-\beta H} | r \rangle &= \left(\frac{u}{\lambda^d \sinh u} \right)^N e^{-\beta V(r)} \\ &\times \left\{ 1 + \frac{\hbar^2 \beta^2}{8m} \left(\frac{1}{u} \coth u - \frac{1}{u^2} \right) \left[\beta \left(\frac{\partial V}{\partial x} \right)^2 + \beta \left(\frac{\partial V}{\partial y} \right)^2 - 2 \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) \right] \right. \\ &\left. + \frac{\hbar^2 \beta^2}{24m} \left[\beta \left(\frac{\partial V}{\partial z} \right)^2 - 2 \frac{\partial^2 V}{\partial z^2} \right] \right\}. \end{aligned} \quad (5.2)$$

In the weak-field limit, the small- u expansion of (5.2) is in agreement with the results of I. In the strong-field limit ($u \gg 1$), only the z -part of the quantum fluctuations survives in (5.2); for a two-dimensional system (no z -part), up to a multiplicative factor, the spatial density is just the classical one, $\exp(-\beta V)$.

The free energy F is given by

$$F = -k_B T \ln \left[\frac{1}{N!} \int \langle r | e^{-\beta H} | r \rangle dr \right]. \quad (5.3)$$

Using (5.2), we find, after integrations by parts,

$$F = F_{cl} + F_1 + F_2, \quad (5.4)$$

where

$$F_{cl} = -k_B T \ln \left[\frac{1}{\lambda^{dN} N!} \int e^{-\beta V(r)} dr \right] \quad (5.5)$$

is the classical free energy,

$$F_1 = -Nk_B T \ln \frac{u}{\sinh u} \quad (5.6)$$

is the diamagnetic free energy of a system of free particles, and

$$F_2 = \frac{\hbar^2 \beta}{8m} \left[\left(\frac{1}{u} \coth u - \frac{1}{u^2} \right) \left\langle \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right\rangle + \frac{1}{3} \left\langle \frac{\partial^2 V}{\partial z^2} \right\rangle \right] \quad (5.7)$$

is the quantum correction of order $(\hbar\nabla_r)^2$; the notation $\langle A \rangle$ means a classical average

$$\langle A \rangle = \int e^{-\beta V} A dr / \int e^{-\beta V} dr. \quad (5.8)$$

It was shown in I that, in the three-dimensional case, for a fluid, or a solid of

cubic symmetry,

$$\left\langle \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right\rangle = \frac{8}{3} \pi N e^2 \rho, \quad \left\langle \frac{\partial^2 V}{\partial z^2} \right\rangle = \frac{4}{3} \pi N e^2 \rho, \tag{5.9}$$

while, in the two-dimensional case (for which there is no z -term)

$$\left\langle \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right\rangle = N e^2 \rho \int \frac{1}{r^3} g(\mathbf{r}) d\mathbf{r}, \tag{5.10}$$

where $g(\mathbf{r})$ is the pair distribution function; except for very small values of Γ , the integral in (5.10) is of the order of a^{-1} .

Let us discuss, in the two-dimensional case, the importance of the quantum correction (5.7) as the field is increased. In the zero-field limit, $\beta F_2/N$ is of the order of $\Gamma(\lambda/a)^2$ (except for very small values of Γ). In the strong-field limit, $\beta F_2/N$ becomes of the order of $\Gamma(l/a)^2$ (and ultimately goes to zero as the field becomes infinite). The system will behave classically if the quantum correction $\beta F_2/N$ is sufficiently small. Therefore, the criterion for classical behaviour, which was $\Gamma(\lambda/a)^2 < 1$ for weak fields becomes $\Gamma(l/a)^2 < 1$ for strong fields. As announced in the introduction, l replaces λ as a characteristic quantum length scale. More precisely, F_2 is multiplied by $3/u$ when the field goes from zero to large values, and λ must be replaced by $(12\pi)^{1/2}l$.

Finally, let us study the magnetization. The magnetization density is

$$M = -\frac{\rho}{N} \frac{\partial F}{\partial B} = M_1 + M_2, \tag{5.11}$$

where

$$M_1 = -\rho \frac{e\hbar}{2mc} \left(\coth u - \frac{1}{u} \right) \tag{5.12}$$

is the diamagnetic magnetization of a system of free particles, while the interactions appear only in the quantum correction

$$M_2 = \rho \frac{e\hbar}{2mc} \frac{\hbar^2 \beta^2}{8m} \left\langle \frac{1}{N} \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) \right\rangle \frac{d}{du} \left(\frac{1}{u^2} - \frac{1}{u} \coth u \right). \tag{5.13}$$

In the weak-field limit, through a small- u expansion of (5.12) and (5.13), one recovers the results of I, with the remarkable feature that, near the classical limit, the interaction-dependent term M_2 is of a higher order in \hbar and therefore is small compared to the free-particle term M_1 . Here, one obtains the additional result that, for stronger fields, the interaction-dependent term M_2 becomes even more negligible, since $|M_2/M_1|$ is easily shown to be a decreasing function of u ; in the strong-field limit, M_2 goes to zero while M_1 goes to its saturation value $-\rho|e|\hbar/2mc$.

6. Exchange effects (two dimensions)

In this section, we compute the exchange effects which we have neglected before. We are interested in the strong-field limit $u \rightarrow +\infty$. The exchange of two free particles would essentially occur when they approach one another at a distance of the order of the Landau radius l , which is small in the strong-field limit. The Coulomb repulsive potential strongly inhibits such configurations; we show that the exchange effects are then exponentially small.

Here, we consider only the case of a two-dimensional system. In three dimensions, the explicit computation of exchange effects is more difficult (however, these effects should also be small under nearly classical conditions). We omit the spins of the particles (electrons for instance) because all particles are in the same spin state in the strong-field limit; therefore, we only have to antisymmetrize the spatial wave functions (for fermions) in order to compute exchange effects. At the lowest order, it is enough to keep two-body exchange effects. Following the same method as in I, we write the partition function as

$$Z = \frac{1}{N!} \left[\int \langle \mathbf{r}_1 \mathbf{r}_2 \dots \mathbf{r}_N | e^{-\beta H} | \mathbf{r}_1 \mathbf{r}_2 \dots \mathbf{r}_N \rangle d\mathbf{r} - \frac{N(N-1)}{2} \int \langle \mathbf{r}_2 \mathbf{r}_1 \dots \mathbf{r}_N | e^{-\beta H} | \mathbf{r}_1 \mathbf{r}_2 \dots \mathbf{r}_N \rangle d\mathbf{r} \right]. \quad (6.1)$$

In order to split the free energy $F = -k_B T \ln Z$ as $F = F_{\text{direct}} + F_{\text{exch}}$, where F_{direct} is the direct part and F_{exch} is the exchange part, we write

$$Z = \frac{1}{N!} \int \langle \mathbf{r}_1 \mathbf{r}_2 \dots \mathbf{r}_N | e^{-\beta H} | \mathbf{r}_1 \mathbf{r}_2 \dots \mathbf{r}_N \rangle d\mathbf{r} \times \left[1 - \frac{N(N-1)}{2} \frac{\int \langle \mathbf{r}_2 \mathbf{r}_1 \dots \mathbf{r}_N | e^{-\beta H} | \mathbf{r}_1 \mathbf{r}_2 \dots \mathbf{r}_N \rangle d\mathbf{r}}{\int \langle \mathbf{r}_1 \mathbf{r}_2 \dots \mathbf{r}_N | e^{-\beta H} | \mathbf{r}_1 \mathbf{r}_2 \dots \mathbf{r}_N \rangle d\mathbf{r}} \right], \quad (6.2)$$

and we expand $\ln Z$ with respect to the exchange term. We find

$$\beta F_{\text{direct}} = -\ln \left[\frac{1}{N!} \int \langle \mathbf{r}_1 \mathbf{r}_2 \dots \mathbf{r}_N | e^{-\beta H} | \mathbf{r}_1 \mathbf{r}_2 \dots \mathbf{r}_N \rangle d\mathbf{r} \right], \quad (6.3)$$

and

$$\beta F_{\text{exch}} = \frac{N(N-1)}{2} \frac{\int \langle \mathbf{r}_2 \mathbf{r}_1 \dots \mathbf{r}_N | e^{-\beta H} | \mathbf{r}_1 \mathbf{r}_2 \dots \mathbf{r}_N \rangle d\mathbf{r}}{\int \langle \mathbf{r}_1 \mathbf{r}_2 \dots \mathbf{r}_N | e^{-\beta H} | \mathbf{r}_1 \mathbf{r}_2 \dots \mathbf{r}_N \rangle d\mathbf{r}}. \quad (6.4)$$

The direct term has been studied in section 5. We now turn to the exchange term.

In the strong-field limit, we can replace the denominator of (6.4) by $[Q/(2\pi l^2)^N] \exp(-\beta N \hbar \omega / 2)$, where Q is the classical configuration integral $Q = \int e^{-\beta V} dr$.

In the numerator of (6.4), it is sufficient to treat "classically" the motion of particles 3 to N and of the center of mass of particles 1 and 2; we obtain

$$\begin{aligned} & \langle r_2 r_1 \dots r_N | e^{-\beta H} | r_1 r_2 \dots r_N \rangle \\ &= \frac{2}{(2\pi l^2)^{N-1}} e^{-\beta(N-1)\hbar\omega/2} e^{-\beta W(r_1, r_1, \dots, r_N)} \langle -r_{12} | e^{-\beta H_{12}} | r_{12} \rangle, \end{aligned} \tag{6.5}$$

where $r_{12} = r_2 - r_1$. We have split the potential energy as

$$V = \frac{e^2}{r_{12}} + W(r_1, r_2, \dots, r_N); \tag{6.6}$$

H_{12} describes the relative motion of particles 1 and 2,

$$H_{12} = \frac{1}{m} \left\{ -i\hbar \nabla_{12} - \frac{e}{4c} \mathbf{B} \wedge r_{12} \right\} + \frac{e^2}{r_{12}}. \tag{6.7}$$

Since we are in nearly classical conditions, exchange effects would occur for values of r_{12} small compared to the mean interparticle distance a . Therefore we have replaced $W(r_1, r_2, \dots, r_N)$ by its value at $r_2 = r_1$ in (6.5). Using (6.5) in (6.4), we obtain

$$\frac{\beta F_{\text{exch}}}{N} = 2\pi\rho l^2 e^{\beta\hbar\omega/2} e^C \int dr_{12} \langle -r_{12} | e^{-\beta H_{12}} | r_{12} \rangle, \tag{6.8}$$

where C is a classical quantity defined by

$$\rho e^C = \frac{N-1}{Q} \int \exp[-\beta W(r_1, r_1, r_3, \dots, r_N)] dr_1 dr_3 \dots dr_N. \tag{6.9}$$

C is related to the short-range behaviour of the classical pair distribution function

$$g(r) \underset{r \rightarrow 0}{\sim} \exp\left[-\frac{\beta e^2}{r} + C + \dots\right]. \tag{6.10}$$

C is a function of the coupling parameter Γ . It was shown in I that C can be simply estimated within the two-dimensional ion-sphere model as $C = 0.9537\Gamma$.

At this stage, we have reduced the computation of exchange effects to a quantum one-body problem by a "classical" treatment of the many-body effects. Many-body effects are contained in the screening factor e^C in (6.8); we now turn to study the one-body exchange integral

$$I_{\text{exch}} = \int d\mathbf{r}_{12} \langle -\mathbf{r}_{12} | e^{-\beta H_{12}} | \mathbf{r}_{12} \rangle. \quad (6.11)$$

Up to now, we have only assumed that the parameter u is sufficiently large. In order to compute asymptotic analytical expressions for I_{exch} , we have to specify the values of the two dimensionless parameters $\eta = \beta e^2/l$ and $\zeta = \hbar^2/me^2l$ which entirely determine the behaviour of I_{exch} . These parameters are simply related to the previous ones defined in the introduction; for instance $u = \eta\zeta/2$; however, here, they are more convenient. In view of practical applications, we study the case where ζ is small and fixed, η going to infinity; it is then ensured that u goes to infinity (this means a finite field and a sufficiently low temperature). Under these conditions, we calculate I_{exch} in the appendix with the result

$$I_{\text{exch}} = \frac{4}{(4\pi)^{1/3}\sqrt{3}} \eta^{1/3} \sin \left[\frac{3}{4} \left(\frac{\pi}{2} \eta^2 \right)^{1/3} - \frac{\pi}{12} \right] \\ \times \exp \left[-\frac{3\sqrt{3}}{4} \left(\frac{\pi}{2} \eta^2 \right)^{1/3} \right] \exp \left(-\frac{\beta\hbar\omega}{2} \right). \quad (6.12)$$

Using the value (6.12) of I_{exch} in (6.8), we finally find in terms of the dimensionless parameters s , t , u ,

$$\frac{\beta F_{\text{exch}}}{N} = \frac{4}{6^{1/2}\pi^{1/3}} \frac{s}{t^{1/6}u^{5/6}} \sin \left[\frac{3}{4} \left(\frac{\pi u}{t} \right)^{1/3} - \frac{\pi}{12} \right] \\ \times \exp \left[C - \frac{3\sqrt{3}}{4} \left(\frac{\pi u}{t} \right)^{1/3} \right]. \quad (6.13)$$

The strong-field limit $u \rightarrow \infty$ can be reached of course for other ranges of values of the parameters η and ζ . We show in the appendix that the expression (6.13) can be extended to the case (less realistic than the previous one) of an infinite magnetic field, which means l small compared to both βe^2 and \hbar^2/me^2 . Thus, it seems reasonable to believe that the exchange effects are always exponentially small in the strong-field limit $u \rightarrow \infty$.

In I, we have studied the exchange effects for a weak field. They were found to be of the order of $\exp[C - (3\pi/2)(\beta e^2/l)^{2/3}]$. Here, in a strong field, the dominant term in (6.13) is $\exp[C - (3\sqrt{3}/4)(\pi/2)^{1/3}(\beta e^2/l)^{2/3}]$. Again, it is seen that l replaces λ as the characteristic quantum length scale. There is a competition between the many-body effects which enhance the exchange through the factor C and the Coulomb repulsion between the particles under consideration which quenches the exchange. The Coulomb repulsion wins, and the exchange effects are exponentially small, if $C < (\beta e^2/l)^{2/3}$; since C is of the order of Γ (except for very small values of Γ), this criterion for small exchange effects can be reexpressed as $\Gamma(l/a)^2 < 1$. It is the same criterion as

for the smallness of the direct quantum corrections. However, for small l , the exchange term vanishes faster, since it is exponentially small.

Conversely, the classical picture breaks down when $\Gamma(l/a)^2 > 1$. A quantum description is then necessary, and several authors⁵⁻⁷⁾ indeed have used the Hartree-Fock approximation. One should however be careful, because it is likely that strong correlations will survive in a large range of the parameters.

7. Fluid-solid phase transition (two dimensions)

The phase diagram of a one-component plasma has been drawn on theoretical grounds, for both three-dimensional⁸⁾ and two-dimensional⁹⁾ cases. In the density-temperature plane, the coexistence curve encloses a finite solid-phase domain. The fluid-to-crystal transition has been experimentally observed¹⁰⁾ in the two-dimensional case, for electrons on liquid helium. It has been argued by several authors^{5,11,12)} that the application of a magnetic field will increase the size of the solid-phase domain, since the field reduces the quantum fluctuations. Here, we come to the same conclusion.

We shall concentrate on the two-dimensional case, which is simpler, because *all* quantum fluctuations are then transverse to the field and therefore reduced by it. Classically, the fluid-solid transition curve would be a straight line in the plane $(T, \rho^{1/2})$, starting from the origin with a slope determined by some fixed value of $\Gamma = \beta e^2/a$ of about 125. This classical picture is valid at low densities and temperatures. For high densities, the quantum effects become important, with the consequence that the transition line departs from its classical straight line shape, bends over, and ultimately terminates on the $\rho^{1/2}$ axis.

When no magnetic field is present, the transition line shows little departure from the classical straight line, in the $(T, \rho^{1/2})$ plane, in a low-density region approximately defined⁹⁾ by $\hbar^2/me^2a < 10^{-2}$; since $\hbar^2/me^2a = s/\Gamma$, this criterion can be reformulated as $s < 1$, or $\lambda < (2\pi)^{1/2}a$. This condition is less drastic than the criterion for classical behaviour, $\Gamma(\lambda/a)^2 < 1$, which emerged in sections 5 and 6. Actually, it is not surprising that the quantum displacement of the transition line be less important than the quantum corrections to the free energies of each phase, since the quantum corrections for both phases are almost equal.

In section 5, we have seen that, when a strong magnetic field is applied, λ is replaced by $(12\pi)^{1/2}l$ in the expression for the direct quantum correction to the free energy; in section 6, we have seen that this replacement is also appropriate for the exchange quantum correction to within a numerical factor of the order of one. Therefore, we expect that the phase diagram in the

($T, \rho^{1/2}$) plane will keep the same general shape when a strong magnetic field is present, but now the transition line will remain close to the classical straight line in a larger domain defined by $6^{1/2}l < a$, i.e. $s < u/3$ (note that, for a system of free particles, the complete filling of the lowest Landau level occurs for $2^{1/2}l = a$).

A transparent description of the strong-field case can be given in terms of Landau orbits. Every particle is confined by the field in a lowest Landau orbit of size l , and these small orbits behave like classical point-particles, provided that al/l be large enough. As the density is increased, the correlation created by the Coulomb forces become more and more important, until the orbits organize themselves in a crystal, beyond the classical transition density corresponding to $\Gamma \approx 125$. This classical description, however, which is valid only if the Landau orbits are well separated, breaks down at higher densities and temperatures.

For a system of electrons on liquid helium, the liquid–solid transition is not affected by a magnetic field, as already shown in I. The system is already very classical when no field is present, and a magnetic field makes it even more classical. The transition remains in the neighbourhood of $\Gamma = 125$.

The situation might be more interesting for a MOS inversion layer. When there is no magnetic field, such a system is far from being nearly classical! However, it may become so in a strong field. Let us describe the inversion layer as a two-dimensional one-component plasma of particles of mass m^* interacting through a Coulomb law $e^2/\epsilon r$ (where ϵ is a characteristic dielectric constant), and coupled to the applied magnetic field B with the bare charge e ; of course, this is certainly a rather oversimplified model. Assuming typical values $m^* = 0.2m$ and $\epsilon = 10$, one finds⁹), when there is no magnetic field, that the maximum temperature at which the solid exists is only 0.1 K, below the usual experimental conditions. However, if a magnetic field $B = 2.5 \times 10^5$ G is applied, at the more reasonable temperature $T = 1$ K, for a typical density $\rho = 2 \times 10^{11}$ electrons per cm^2 , one finds $s = 28$, $u = 84$, $\Gamma = 132$. The criterion for a classical phase transition, $s < u/3$, is just satisfied, and since Γ is near the classical transition value 125, the system should be on the verge of the crystalline state. This is a typical case where a strong magnetic field will induce a Wigner crystallisation which could not otherwise occur at such a “high” temperature.

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Appendix

In this appendix, we calculate the asymptotical form of the one-body exchange integral (6.11), when ξ is small and fixed, η going to infinity

It is convenient to introduce a complete set of eigenfunctions of the relative hamiltonian

$$H_{12} = \frac{1}{m} \left(-i\hbar\nabla - \frac{e}{4c} \mathbf{B} \wedge \mathbf{r} \right)^2 + \frac{e^2}{r}, \tag{A.1}$$

where $\mathbf{r} = \mathbf{r}_{12}$. Since H_{12} commutes with the angular momentum, we can choose eigenfunctions of the form

$$\psi_{n,L}(\mathbf{r}) = \frac{e^{iL\phi}}{(2\pi)^{1/2}} \frac{F_{n,L}(r)}{r^{1/2}}; \tag{A.2}$$

L is a positive or negative integer, r and ϕ are the polar coordinates of \mathbf{r} . $F_{n,L}(r)$ is a solution of the one-dimensional Schrödinger equation

$$\left[-\frac{\hbar^2}{m} \frac{d^2}{dr^2} + U_{\text{eff}}(r) \right] F_{n,L}(r) = E_{n,L} F_{n,L}(r), \tag{A.3}$$

where $E_{n,L}$ is the energy and

$$U_{\text{eff}}(r) = \frac{e^2}{r} + \frac{\hbar^2}{4mr^2} (4L^2 - 1) + \frac{e^2 B^2}{16mc^2} r^2 - \frac{L\hbar\omega}{2}. \tag{A.4}$$

Choosing $\psi_{n,L}(r)$ normalized to unity, we obtain

$$I_{\text{exch}} = \int_0^\infty dr \sum_{n,L} (-1)^L e^{-\beta E_{n,L}} |F_{n,L}(r)|^2. \tag{A.5}$$

In the range of parameters ζ and η considered here, the temperature is relatively low; so the dominant contribution to I_{exch} is given by the states of lowest energy, which turn out to be those of large positive angular momentum L . For such states, we can treat the Coulomb potential as a perturbation in the effective potential $U_{\text{eff}}(r)$. We have checked that we can replace $E_{n,L}$ by its first-order expansion with respect to e^2 and keep the unperturbed eigenfunctions $F_{n,L}^0$ for calculating the dominant term of (A.5). The energy levels of a free particle in the magnetic field are $E_{n,L}^0 = (n + \frac{1}{2})\hbar\omega$ for $L \geq 0$, n being a positive integer. Since we are studying the strong-field limit $u \rightarrow +\infty$, the ground states $n = 0$ give the dominant contribution to (A.5). For such states, $E_{0,L}$ and $F_{0,L}^0$ have the following asymptotical behaviours for large values of L :

$$E_{0,L} = \frac{1}{2} \hbar \omega + \frac{e^2}{2L^{1/2}l} \tag{A.6}$$

and

$$F_{0,L}^0 = \frac{1}{(2\pi l^2)^{1/4}} \exp\left[-\frac{(r - 2L^{1/2}l)^2}{4l^2}\right]. \tag{A.7}$$

In the large- η limit, we can restrict the sum in (A.5) to the strictly positive values of L , and replace each term by (A.6) and (A.7). We then obtain

$$I_{\text{exch}} = \frac{e^{-\beta \hbar \omega l^2}}{(2\pi)^{1/2}} \int_0^\infty d\xi \sum_{L=1}^\infty (-1)^L \exp\left[-\frac{\eta}{2L^{1/2}} - \frac{1}{2}(\xi - 2L^{1/2})^2\right], \tag{A.8}$$

where $\xi = r/l$.

In order to compute the sum in (A.8), we introduce the function $f(\xi, z)$ defined by

$$f(\xi, z) = \frac{\pi}{\sin \pi z} \exp\left[-\frac{\eta}{2z^{1/2}} - \frac{1}{2}(\xi - 2z^{1/2})^2\right]. \tag{A.9}$$

ξ being fixed, $f(\xi, z)$ is an analytical function of z with a branch line along the negative real axis and an essential point at the origin; we choose for \sqrt{z} the determination with a positive real part. The poles of $f(\xi, z)$ are the non-zero positive integers, and their residues are the terms of the sum in (A.8). Therefore,

$$\sum_{L=1}^p (-1)^L \exp\left[-\frac{\eta}{2L^{1/2}} - \frac{1}{2}(\xi - 2L^{1/2})^2\right] = \frac{1}{2\pi i} \oint_{\mathcal{C}_p} f(\xi, z) dz, \tag{A.10}$$

where \mathcal{C}_p is the integration contour shown in fig. 1.

In the limit $p \rightarrow +\infty$, the contour integral reduces to the integral along the imaginary axis. After a simple change of the integration variable, we find

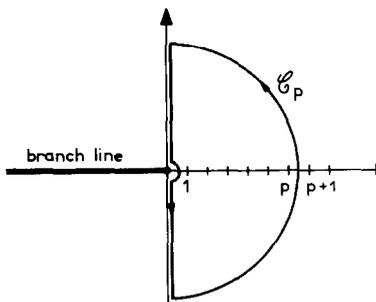


Fig. 1. The integration contour \mathcal{C}_p .

$$\begin{aligned} & \sum_{L=1}^{\infty} (-1)^L \exp\left[-\frac{\eta}{2L^{1/2}} - \frac{1}{2}(\xi - 2L^{1/2})^2\right] \\ &= 2 \int_0^{\infty} \frac{dx}{\sinh \pi x^2} \sin\left(2x^2 - \frac{\eta}{2\sqrt{2}x} - \xi x \sqrt{2}\right) \exp\left(-\frac{\xi^2}{2} + \xi x \sqrt{2} - \frac{\eta}{2\sqrt{2}x}\right). \end{aligned} \tag{A.11}$$

Using (A.11) in (A.8), we obtain a double integral representation of I_{exch} . It is convenient to perform first the integration upon the ξ variable. Since only the large values of x give a non-negligible contribution to I_{exch} , we can extend the ξ integration domain to $(-\infty, +\infty)$. We find

$$I_{\text{exch}} = -2 e^{-\beta \hbar \omega / 2} \int_0^{\infty} \frac{dx}{\sinh \pi x^2} \sin \frac{\eta}{2\sqrt{2}x} \exp\left(-\frac{\eta}{2\sqrt{2}x}\right). \tag{A.12}$$

Finally, we perform the integration upon x by the saddle-point method which provides the asymptotical behaviour of I_{exch} in the large- η limit, given by eq. (6.12).

Let us point out that I_{exch} has the same asymptotical behaviour (6.12) in the case of an infinite magnetic field, namely l small compared to both βe^2 and \hbar^2/me^2 . Indeed, in the limit $\zeta \rightarrow +\infty$, we can treat the Coulomb potential as a perturbation in $U_{\text{eff}}(r)$ for all values of the angular momentum L ; on the other hand, as u goes to infinity, only the ground states of positive angular momentum give a non negligible contribution to I_{exch} . Then, (A.6) and (A.7) again suffice for deriving the asymptotical behaviour of I_{exch} , and we find again the same expression (6.12) for I_{exch} .

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