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On the Average Distance Between Particles in the Two-Dimensional Two-Component Plasma

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The asymptotic forms of the average distance of the closest particle to a fixed positive charge, and of the closest particle to the origin, are obtained for the two-dimensional two-component plasma in the low-density limit. The asymptotic forms of the average areas of the corresponding disks formed by the closest particle are also derived. These results are verified at a special coupling where exact results are available.

KEY WORDS: Two-component plasma; Kosterlitz–Thouless transition; exact solution.

1. INTRODUCTION

The two-dimensional two-component plasma (TCP) consists of an equal number of positive and negative two-dimensional Coulomb charges (magnitude q, say) interacting in a two-dimensional domain. Thus, the potential energy of a particle of charge q at $\mathbf{r}$ and charge q' at $\mathbf{r}'$ is

$$\phi(\mathbf{r}, \mathbf{r}') = -qq' \log(|\mathbf{r} - \mathbf{r}'|/L) \tag{1.1}$$

(L is an arbitrary length scale). In general, the system is characterized by the dimensionless parameters $\Gamma := q^2/kT$ and the product of the density ρ and the square of the hard-core radius σ.

A hard-core or similar short-distance regularization of the potential is necessary for $\Gamma \geqslant 2$ to prevent collapse of oppositely charged species. Even with this regularization, for a fixed small value of $\rho\sigma^2$ the system exhibits a transition from a conductive to a dipole phase. In the conductive phase the opposite species are free to screen perfectly an external charge density

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(in the long-wavelength limit), while in the dipole phase opposite charges are paired so that only a fraction of an external charge density is screened. In the limit $\rho\sigma^2 \to 0$ the transition occurs at $\Gamma = 4$[1], as distinct from $\Gamma = 2$, showing that this phenomenon is due to the large- rather than short-distance behavior of the logarithmic potential.

Our concern in this paper is to provide a quantitative description of the microscopic configurations at equilibrium. We are especially interested in investigating how the average distance (and the average squared distance) between a given particle and its closest neighbor depend on the coupling constant Γ. It will be argued that the behaviors of these averages undergo drastic changes as Γ increases, which enhances the pairing effects. A related problem has been addressed by Lavaud,[2] who proved that for $1 < \Gamma < 2$ the short-distance behavior of the pair distribution function between opposite charges is contrary to the prediction of a theorem of Widom.[3] This result was first observed by Hansen and Viot[4] through an analysis of Monte Carlo data.

In Section 2 we begin by studying the average distance from the origin to the closest particle, and the average area of the corresponding disk. Then we study the average distance between a positive charge and its nearest neighbor charge (which in the dipole phase, we expect to be a negative charge), together with the average area of the corresponding disk. These considerations are repeated for the two-component log-potential plasma confined to a one-dimensional domain. Although our reasoning throughout this section in heuristic, the results obtained are in the form of precise mathematical conjectures concerning the behavior of the averages in the low-density $\rho\sigma^2 \to 0$ limit.

In Section 3, exact results are presented, at the special coupling $\Gamma = 2$, for the TCP and its restriction to a one-dimensional domain. These results agree with the conjectures of Section 2.

The main results are summarized in Section 4.

Appendix A is a rederivation of an equation need in Section 3. In Appendix B, we give the large-R behavior of the distribution function for inserting a charge at the center of a hole of radius R in the TCP, when the TCP is in its conducting phase; this quantity first arises in Section 3, where it is calculated exactly at $\Gamma = 2$.

2. SPACING AVERAGES

2.1. Relationships Between Probabilities

Consider the TCP with a hard-core radius σ about the positive charges (for convenience the negative charges are taken as point particles). Denote by $h_n(R)$ the probability that exactly n charges are within a radius R of the origin [note that $\sum_{n=0}^{\infty} h_n(R) = 1$]. Let $S(R)$ denote the probability density that there is no particle within radius R, but there is a particle at a distance between R and $R + dR$ of the origin. Then

$$S(R)\,dR = h_0(R) - h_0(R + dR) \tag{2.1a}$$

so that

$$S(R) = -\frac{d}{dR} h_0(R) \tag{2.1b}$$

Consequently, the mean distance from the origin to the closest particle, $\bar{R}$ say, which is defined as

$$\bar{R} = \int_0^{\infty} RS(R)\,dR \tag{2.2a}$$

can be expressed, using (2.1b) and integration by parts, as

$$\bar{R} = \int_0^{\infty} h_0(R)\,dR \tag{2.2b}$$

Proceeding similarly, if $S(+, R)$ denotes the probability density that there is a positive particle fixed at the origin with no particle within radius R, but there is a particle within radius R and $R + dR$, then

$$\bar{R}_+ := \int_{\sigma}^{\infty} RS(+, R)\,dR \tag{2.3a}$$

can be rewritten as

$$\bar{R}_+ = \sigma + \int_{\sigma}^{\infty} h_0(+, R)\,dR \tag{2.3b}$$

The probability $h_0(+, R)$ is defined analogously to $h_0(R)$, except that we require that there is a positive particle fixed at the origin.

In an obvious notation, we also have

$$\pi\overline{R^2} = 2\pi \int_0^{\infty} Rh_0(R)\,dR \tag{2.4}$$

and

$$\pi\overline{R_+^2} = \pi\sigma^2 + 2\pi \int_{\sigma}^{\infty} Rh_0(+, R)\,dR \tag{2.5}$$

2.2. Behavior of $\bar{R}$ and $\pi\overline{R^2}$

From (2.2b) and (2.4) we see that $\bar{R}$ and $\pi\overline{R^2}$ are determined by $h_0(R)$. This quantity is defined so that $h_0(\sigma)=1$ and is expected to decrease monotonically to zero with the large-R behavior[5]

$$h_0(R) \sim e^{-\pi R^2 \beta P - 2\pi R \beta \gamma} \tag{2.6}$$

where P denotes the bulk pressure and γ denotes the surface tension.

To study the low-density behavior of $h_0(R)$ and consequently $\bar{R}$ and $\pi\overline{R^2}$ for the TCP, it is instructive to first consider a perfect gas of density ρ in two dimensions, for which

$$h_0(R) = e^{-\pi R^2 \rho} \tag{2.7}$$

We note that (2.7) is consistent with (2.6), since for a perfect gas $\beta P = \rho$ and $\gamma = 0$. Using (2.7) in (2.2b) and (2.4) gives

$$\bar{R} = \frac{1}{2}\frac{1}{\sqrt{\rho}} \quad \text{and} \quad \pi\overline{R^2} = \frac{1}{\rho} \tag{2.8}$$

From (2.7) we see that the characteristic length scale of $h_0(R)$ for a perfect gas in two dimensions is $1/\sqrt{\rho}$. We would expect this also to be true of the TCP in the limit $\rho \to 0$ with σ fixed. Quantatively, we conjecture that in the limit

$$\rho \to 0, \qquad R \to \infty, \qquad R^2\rho = x^2, \qquad \sigma \text{ fixed} \tag{2.9}$$

$h_0(R) \to H_0(x)$, where $H_0(x)$ is a well-defined integrable function.

Furthermore, we expect the low-density behavior of $\bar{R}$ and $\pi\overline{R^2}$ to be determined by the leading term in the virial expansion[1]

$$\beta P \sim a_0\rho + o(\rho), \qquad a_0 = \begin{cases} 1/2, & \Gamma \geqslant 2 \\ 1 - \Gamma/4, & \Gamma \leqslant 2 \end{cases} \tag{2.10}$$

where ρ denotes the total particle density. From a thermodynamic viewpoint, a_0 gives the number of free particles plus pairs as a fraction of the total number of particles in the system. The values of $\pi\overline{R^2}$ should therefore be $1/a_0$ times the perfect gas value while the value of $\bar{R}$ should be $(1/a_0)^{1/2}$ times that of the perfect gas. From (2.8) and (2.10) we thus conjecture that for the TCP

$$\bar{R} \sim \frac{1}{2}\frac{1}{(a_0\rho)^{1/2}} \quad \text{and} \quad \pi\overline{R^2} \sim \frac{1}{a_0\rho} \tag{2.11}$$

Using the conjecture stated in the sentence including Eq. (2.9), we have that (2.11) is equivalent to the conjecture that

$$\int_0^\infty H_0(x)\,dx = \frac{1}{2\sqrt{a_0}} \qquad \text{and} \qquad 2\pi \int_0^\infty x H_0(x)\,dx = \frac{1}{a_0} \tag{2.12}$$

We observe that the function

$$H_0(x) = e^{-\pi a_0 x^2} \tag{2.13}$$

which is an obvious generalization of the perfect gas result (2.7), satisfies (2.12). In Section 3, we will show that indeed (2.13) is exact at $\Gamma = 2$.

2.3. Behavior of $\bar{R}_+$ and $\pi\overline{R_+^2}$

From (2.3a) and (2.5) we see that $\bar{R}_+$ and $\pi\overline{R_+^2}$ are determined by $h_0(+, R)$. This quantity can be calculated from statistical mechanics according to the formula

$$h_0(+, R) = \Xi(+, R)/\Xi(+) \tag{2.14}$$

where $\Xi(+)$ denotes the grand partition function with a positive particle fixed at the origin, and $\Xi(+, R)$ denotes the grand partition function with a positive particle fixed at the center of an impenetrable disk of radius R. Simple manipulation of (2.14) and the formula

$$h_0(R) = \Xi(R)/\Xi \tag{2.15}$$

gives

$$h_0(+, R) = h_0(R)\,\frac{\Xi}{\Xi(+)}\,\frac{\Xi(+, R)}{\Xi(R)} \tag{2.16}$$

Whereas we have argued that for all values of Γ, $h_0(R)$ has the characteristic length scale $1/\sqrt{\rho}$, the final factor in (2.16) exhibits different length scales for its decay, depending on the value of Γ. Let us denote the last two factors in (2.16) by $p(+, R)$ and consider separately the cases $\Gamma < 2$ and $\Gamma > 2$.

Case 1. $\Gamma < 2$. As noted in the Introduction, for $\Gamma < 2$ all equilibrium quantities tend to well-defined limits as $\sigma \to 0$. If $\sigma = 0$, $p(+, R)$ is a function of $\rho^{1/2}$. Furthermore, we expect the equilibrium properties to behave uniformly in the limit $\sigma \to 0^+$, as they will be determined by the longrather than the short-range properties of the potential. The characteristic length scale of $p(+, R)$ should therefore be $1/\rho^{1/2}$ for nonzero as well as zero values, assuming $\sigma \ll 1/\sqrt{\rho}$.

Thus, in the limit (2.9) we conjecture that $p(+, R)$ tends to a well-defined limit $P(+, x)$ and consequently

$$\bar{R}_+ \sim \frac{1}{\sqrt{\rho}} \int_0^\infty H(x)\, P(+, x)\, dx \tag{2.17}$$

and

$$\pi \overline{R_+^2} \sim \frac{2\pi}{\rho} \int_0^\infty x H(x)\, P(+, x)\, dx \tag{2.18}$$

Case 2. $\Gamma > 2$. For $\Gamma > 2$, $p(+, R)$ is not defined at $\sigma = 0$, due to the short-distance collapse, so the reasoning for $\Gamma < 2$ is inapplicable. However, in contrast to the case $\Gamma < 2$, the first virial coefficient at least of a low-fugacity expansion is finite for $\Gamma > 2$. This is appropriate to probe the behavior of $p(+, R)$ for $R < 1/\sqrt{\rho}$, since $1/\sqrt{\rho}$ is the order of the average interparticle spacing, and the region within this range will most probably be sampled by one other particle only. The first term in the low-fugacity expansion gives the contribution to the grand partition function of such configurations.

To perform this expansion, consider the TCP defined with position-dependent fugacities

$$\zeta(|\mathbf{r}|) = \begin{cases} \zeta, & |\mathbf{r}| > R \\ \zeta_0, & |\mathbf{r}| < R \end{cases} \tag{2.19}$$

Then we have

$$\frac{\Xi(+, R)}{\Xi(R)} = \lim_{\zeta_0 \to 0} \frac{\rho_+(0)}{\zeta_0} \tag{2.20}$$

where $\rho_+(0)$ denotes the density of positive particles at the origin. For $\Gamma > 2$ we have the virial expansion

$$\Xi = 1 + \int_{D_{\sigma(\mathbf{r}_1)} \otimes D} \zeta(|\mathbf{r}_1|)\, \zeta(|\mathbf{r}_2|) \exp[-\Gamma \log(|\mathbf{r}_1 - \mathbf{r}_2|/L]\, d\mathbf{r}_1\, d\mathbf{r}_2 + \cdots \tag{2.21}$$

where $D_{\sigma(\mathbf{r}_1)}$ denotes the two-dimensional domain D (the container) with a disk of radius σ at $\mathbf{r}_1$ excluded. Since

$$\rho_+(0) = \zeta(0) \frac{\delta \log \Xi}{\delta \zeta(0)} \tag{2.22}$$

we obtain from (2.21) the expansion

$$\rho_+(0) = \zeta_0 \int_{D_{\sigma(0)}} \zeta(|\mathbf{r}|) \exp(-\Gamma \log |\mathbf{r}|/L)\, d^2\mathbf{r} + \cdots \tag{2.23}$$

and thus from (2.20)

$$\Xi(+, R)/\Xi(R) = \zeta \int_{D_{R(0)}} \exp(-\Gamma \log |\mathbf{r}|/L)\, d^2\mathbf{r} + \cdots \tag{2.24}$$

The same reasoning without a hole gives

$$\Xi/\Xi(+) = \zeta/\rho_+ \tag{2.25a}$$

$$= 1\Bigg/\left[\zeta \int_{D_{\sigma(0)}} \exp(-\Gamma \log |\mathbf{r}|/L)\, d^2\mathbf{r}\right] \tag{2.25b}$$

Combining (2.24) and (2.25b) gives that to leading order in ζ, and for $R < 1/\sqrt{\rho}$,

$$p(+, R) = \int_R^\infty r^{-\Gamma+1}\, dr \Bigg/ \int_\sigma^\infty r^{-\Gamma+1}\, dr$$
$$= (\sigma/R)^{\Gamma-2} \tag{2.26}$$

The characteristic length scale for the decay of $p(+, R)$ is thus σ, which by assumption is much, much smaller than $1/\sqrt{\rho}$, the characteristic length scale for the decay of $h(R)$. Thus the integrals (2.3b) and (2.5) defining $\bar{R}_+$ and $\pi\overline{R^2_+}$, in the low-density limit, are determined by the integrand in the region $R < 1/\sqrt{\rho}$. Terminating the upper range of integration at $O(1/\sqrt{\rho})$ and noting that within this region $1 > h(R) > \varepsilon$ for some $\varepsilon > 0$, we obtain the conjecture that for $\rho \to 0$ (with σ fixed)

$$\bar{R}_+ \sim \begin{cases} A(\Gamma)\, \sigma(\sigma\sqrt{\rho})^{\Gamma-3}, & 2 < \Gamma < 3 \\ \sigma A(3) \log(1/\sigma\sqrt{\rho}), & \Gamma = 3 \\ \sigma((\Gamma-2)/(\Gamma-3)) & \Gamma > 3 \end{cases} \tag{2.27}$$

and

$$\pi\overline{R^2_+} \sim \begin{cases} B(\Gamma)\, \sigma^2(\sigma\sqrt{\rho})^{\Gamma-4}, & 2 < \Gamma < 4 \\ \sigma^2 B(4) \log(1/\sigma\sqrt{\rho}), & \Gamma = 4 \\ \pi\sigma^2((\Gamma-2)/(\Gamma-4)) & \Gamma > 4 \end{cases} \tag{2.28}$$

where $A(\Gamma)$ and $B(\Gamma)$ are some undetermined functions of Γ.

We note from (2.28) that the transition of $\pi\overline{R_+^2}$ from a divergent to a finite quantity, in the low-density limit, coincides with the transition from a conductive to an insulator phase. (This result is implicit in the original paper of Kosterlitz and Thouless.[6])

2.4. Spacings for the TCP in a One-Dimensional Domain

The TCP confined to a one-dimensional domain is of interest in condensed matter physics, as it can be mapped into a model of quantum Brownian motion in a periodic potential.[7] Analogous to the situation in a two-dimensional domain, there is a transition from a conductive to an insulator (dipole) phase. However, in the low-density limit, the transition occurs a $\Gamma = 2$[8] (instead of $\Gamma = 4$).

Regarding the spacing averages, the area of a region is not relevant in one dimension, so we only need consider $\bar{R}$ and $\bar{R}_+$. The considerations of Section 2.3 are directly applicable. Repeating these considerations, we are led to conjecture that for σ fixed and $\rho \to 0$,

$$\bar{R} \sim \frac{1}{a_0 \rho} \qquad \text{where} \quad a_0 = \begin{cases} 1/2, & \Gamma \geqslant 1 \\ 1 - \Gamma/2, & \Gamma \leqslant 1 \end{cases} \tag{2.29}$$

and

$$\bar{R}_+ \sim \begin{cases} a(\Gamma)/\rho, & \Gamma < 1 \\ a(\Gamma)\, \sigma(\sigma\rho)^{\Gamma - 2}, & 1 < \Gamma < 2 \\ \sigma a(2) \log[1/(\sigma\rho)], & \Gamma = 2 \\ \sigma(\Gamma - 1)/(\Gamma - 2), & \Gamma > 2 \end{cases} \tag{2.30}$$

where $a(\Gamma)$ is an undertermined function of Γ.

3. EXACT RESULTS AT Γ = 2

3.1. Scaling Behavior of $h_0(R)$

At the coupling $\Gamma = 2$ we have the exact result[5] (see Appendix A for an alternative derivation)

$$\log h_0(R) = 2 \sum_{j=1}^{\Lambda} \log \left\{ \frac{2(mR/2)^j}{(j-1)!} K_j(mR) \right\} \tag{3.1}$$

where Λ is of the order of R/σ, $K_j(x)$ denotes the jth-order Bessel function of the second kind, and $m = 2\pi\zeta L$, where ζ denotes the fugacity. We want to investigate (3.1) in the limit (2.9). With σ fixed and $m \to 0$, we know[9]

$$\rho \sim \frac{m^2}{\pi} \log \frac{1}{m\sigma} \tag{3.2}$$

so that the limit (2.9) requires

$$m \to 0, \qquad R \to \infty, \qquad \frac{(mR)^2}{\pi} \log \frac{1}{m\sigma} = x^2 \text{ fixed} \tag{3.3}$$

Using (3.3) to eliminate R gives that we seek the value of

$$2 \lim_{m \to 0} \sum_{j=1}^{M} \log \left[\frac{2\{\pi^{1/2}x/[2\log(1/m\sigma)]^{1/2}\}^j}{(j-1)!} K_j\left(\frac{\pi^{1/2}x}{[\log(1/m\sigma)]^{1/2}} \right) \right] \tag{3.4}$$

where

$$M = \frac{\pi^{1/2}x}{\{\sigma m[\log(1/m\sigma)]^{1/2}\}} \tag{3.5}$$

The argument of the Bessel function in (3.4) is tending to zero. This suggests we use the small-z expansion[10]

$$K_n(z) \sim \frac{1}{2}\left(\frac{z}{2}\right)^{-n} (n-1)! \left[1 - \left(\frac{z}{2}\right)^2 \frac{1}{n-1} + O\left(\frac{z^4}{n^4}\right) \right] \tag{3.6}$$

valid for $n \geqslant 3$, with the leading-order asymptotic behavior given by the terms outside the square brackets for $n = 1$ and 2. The expression (3.4) then becomes

$$-2 \lim_{m \to 0} \frac{\pi x^2}{4 \log(1/m\sigma)} \sum_{j=2}^{M} \frac{1}{j-1} \tag{3.7}$$

which can immediately be evaluated to give

$$H_0(x) = e^{-\pi x^2/2} \tag{3.8}$$

This is in precise agreement with (2.13) since $a_0 = 1/2$ at $\Gamma = 2$.[1]

3.2. Exact Evaluation of $h_0(+, R)$

Since $h_0(R)$ is already known, from (2.16) we see that to calculate $h_0(+, R)$ exactly at $\Gamma = 2$, it suffices to calculate

$$\frac{\Xi}{\Xi(+)} \frac{\Xi(+, R)}{\Xi(R)} \tag{3.9}$$

Furthermore, from (2.25a) we have

$$\Xi/\Xi(+) = \zeta/\rho_+ \tag{3.10a}$$

$$\sim 1/[mL \log(1/m\sigma)] \tag{3.10b}$$

where we have used (3.2) to obtain (3.10b).

Thus it remains to calculate

$$f(R) := \Xi(+, R)/\Xi(R) \tag{3.11}$$

which we do by considering the TCP with position-dependent fugacities (2.19), using the method of Curnu and Jancovici[9] to calculate the density of the positive particles at the origin of such a system and then applying the formula (2.20). Let us define

$$m(r) = 2\pi L\zeta(r) \tag{3.12}$$

where $\zeta(r)$ is given by (2.19). The method of ref. 9 gives

$$f(R) = \lim_{m_0 \to 0} 2\pi L G(0) \tag{3.13}$$

where $m_0 = m(0)$ and $G(r)$ satisfies the differential equation

$$[m^2(r) - \nabla^2]\, G(r) = m(r)\, \delta(\mathbf{r}) \tag{3.14}$$

with the boundary conditions that at $r = R$ (the hole radius), G and its derivative

$$\frac{1}{m(r)}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) G$$

are continuous.

The general solution of (3.14) with $m(r)$ given by (3.12) and (2.19) is

$$G(r) = \begin{cases} (m_0/2\pi)[K_0(m_0 r) + aI_0(m_0 r)], & 0 < r < R \\ (m/2\pi)\, bK_0(mr), & r > R \end{cases} \tag{3.15}$$

The constants a and b are determined using the boundary conditions. We can then find the value of

$$\lim_{m_0 \to 0} G(r) \tag{3.16}$$

(which is independent of r for $0 < r < R$) and equating (3.16) to (3.13), we obtain

$$f(R) = \frac{L}{R}\frac{K_0(mR)}{K_1(mR)} \tag{3.17}$$

3.3. Low-Density Behavior of $\bar{R}_+$

From the formulas (2.3b), (2.16), and (2.25a) and the definition (3.10a), we have that in general

$$\bar{R}_+ = \sigma + \frac{\zeta}{\rho_+}\int_\sigma^\infty h_0(R)\, f(R)\, dR \tag{3.18}$$

Changing variables $x = \sqrt{\rho}\, R$ gives

$$\bar{R}_+ = \sigma + \frac{\zeta}{\rho_+} \frac{1}{\sqrt{\rho}} \int_{\sigma\sqrt{\rho}}^{\infty} h_0\left(\frac{x}{\sqrt{\rho}}\right) f\left(\frac{x}{\sqrt{\rho}}\right) dx \tag{3.19}$$

Now consider the limit $\rho \to 0$ (or equivalently, $m \to 0$) with σ fixed. From Section 3.1 we have

$$h_0(x/\sqrt{\rho}) \to e^{-\pi x^2/2} \tag{3.20}$$

while using (3.2) in (3.17) and the small-argument expansions of $K_0(x)$ and $K_1(x)$,[10] we obtain that for fixed x

$$(1/mL)\, f(x/\sqrt{\rho}) \sim \tfrac{1}{2} \log\log(1/m\sigma) \tag{3.21}$$

Using (3.20), (3.21), and (3.10b) in (3.19) gives

$$\bar{R}_+ \sim \frac{1}{\rho} \frac{1}{2\sqrt{2}} \frac{\log\log(1/m\sigma)}{\log(1/m\sigma)} \tag{3.22}$$

This behavior is intermediate between the conjectured forms (2.17) for $\Gamma < 2$ and (2.27) for $\Gamma > 2$, as would be expected.

3.4. Exact Calculation of $\bar{R}_+$ in a One-Dimensional Domain

As noted in the Introduction, in the dipole phase we expect the closest particle to a fixed positive charged particle to have a negative charge. Thus we can consider the system with a positive charge fixed at the origin and calculate the probability, $\hat{h}_0(+, R)$, say, that there are no negative charges within a distance R of the origin. Use of (2.3b) with $h_0(+, R)$ replaced by $\hat{h}_0(+, R)$ should then still give the correct asymptotic behavior of $\bar{R}_+$.

Here we seek to calculate $\hat{h}_0(+, R)$ for the TCP confined to a one-dimensional domain at the coupling $\Gamma = 2$. More explicitly, the domain is taken as two interlaced one-dimensional lattices, with the spacing within the sublattices τ and the closest spacing between lattices $\tau\phi$, $0 < \phi < 1/2$. The particle-free distance R is then measured in units of lattice spacings, so $R = l\tau$, say, where $l \in \mathbb{N}$.

The method of calculating $\hat{h}_0(+, R)$ for the TCP on a one-dimensional lattice is very different from that of calculating $h_0(+, R)$ for the TCP in a two-dimensional domain presented above. In the one-dimensional case we use the general formula[5]

$$\hat{h}_0(+, l\tau) = \frac{1}{\tau\rho_+} \sum_{j=0}^{2l} \frac{(-1)^j}{j!} \left(\sum_{n_1=-l}^{l-1} \cdots \sum_{n_j=-l}^{l-1} \right) \rho(0; n_1, n_2, \ldots, n_j) \tag{3.23}$$

where ρ_+ denotes the density of positive charges and $\rho(0, n_1,..., n_j)$ denotes the dimensionless distribution function for one positive charge fixed at the origin, on the sublattice for the positive charges, and j negative charges fixed at the lattice points labeled by $n_1,...,n_j$ on the sublattice for the negative charges.

At $\Gamma = 2$ we have the exact results[11,5]

$$\tau\rho_+ = \frac{\xi}{1+\xi} \tag{3.24}$$

and

$$\rho(0; n_1,..., n_j) = \det \begin{bmatrix} \dfrac{\xi}{1+\xi} & \left[\dfrac{\xi^{1/2}\sin\pi\phi}{\pi(1+\xi)(n_\beta+\phi)}\right]_{\beta=1,...,j} \\ \left[\dfrac{-\xi^{1/2}\sin\pi\phi}{\pi(1+\xi)(n_\alpha+\phi)}\right]_{\alpha=1,...,j} & \left[\dfrac{\xi}{1+\xi}\delta_{\alpha,\beta}\right]_{\alpha,\beta=1,...,j} \end{bmatrix} \tag{3.25}$$

where

$$\xi = \left(\frac{\pi\zeta}{\tau\sin\pi\phi}\right)^2 \tag{3.26}$$

Note that the matrix in (3.25) is diagonal apart from the first row and column. Substituting (3.24) and (3.25) in (3.23), we can recognize the resulting expression as an expanded form of the formula

$$\hat{h}_0(+, l\tau) = \frac{1+\xi}{\xi} \det \begin{bmatrix} \dfrac{\xi}{1+\xi} & \left[\dfrac{\xi^{1/2}\sin\pi\phi}{\pi(1+\xi)(k+\phi)}\right]_{k=-l,...,l-1} \\ \left[\dfrac{\xi^{1/2}\sin\pi\phi}{\pi(1+\xi)(j+\phi)}\right]_{j=-l,...,l-1} & \left[\left(1-\dfrac{\xi}{1+\xi}\right)\delta_{j,k}\right]_{j,k=-l,...,l-1} \end{bmatrix} \tag{3.27}$$

(see ref. 5 for similar identities). It is straightforward to evaluate this determinant and thus obtain the explicit evaluation

$$\hat{h}_0(+, l\tau) = \left(\frac{1}{1+\xi}\right)^{2l}\left(1 - \frac{\sin^2\pi\phi}{\pi^2}\sum_{j=-l}^{l-1}\frac{1}{(j+\phi)^2}\right) \tag{3.28}$$

Our aim in this section is to study $\bar{R}_+$. With the understanding that we interested in the distance to the closest negative charge, $\bar{R}_+$ and $\hat{h}_0(+, l\tau)$ are related by

$$\bar{R}_+ = \tau \sum_{l=0}^{\infty} l(\hat{h}_0(+, l\tau) - \hat{h}_0(+, (l+1)\tau)) \tag{3.29a}$$

$$= \tau \sum_{l=1}^{\infty} \hat{h}(+, l\tau) \tag{3.29b}$$

The low-density behavior of $\bar{R}_+$ is determined by the large-l form of $\hat{h}(+, l\tau)$. Using the summation formula

$$\sum_{j=-\infty}^{\infty} \frac{1}{(j+\phi)^2} = \frac{\pi^2}{\sin^2 \pi\phi} \tag{3.30}$$

we can readily deduce that for large l

$$\sum_{j=-l}^{l-1} \frac{1}{(j+\phi)^2} \sim \frac{\pi^2}{\sin^2 \pi\phi} - \frac{2}{l} \tag{3.31}$$

and thus

$$\hat{h}_0(+, l\tau) \sim \frac{2 \sin^2 \pi\phi}{\pi^2} \frac{1}{l} \left(\frac{1}{1+\xi}\right)^{2l} \tag{3.32}$$

Substituting (3.32) in (3.29b) and using (3.24) (with ρ_+ replaced by $\frac{1}{2}\rho$) shows that for low density

$$\bar{R}_+ \sim -2\tau \frac{\sin^2 \pi\phi}{\pi^2} \log \tau\rho \tag{3.33}$$

The symmetric case $\phi = 1/2$ is analogous to the continuum with $\tau = \sigma$. The exact asymptotic form (3.33) then agrees with (2.30) and furthermore gives $a(2) = 2/\pi^2$.

4. CONCLUSION

The main points of the present paper are the low-density behaviors of the average distance $\bar{R}_+$ between a particle and its nearest neighbor and of the average $\overline{R_+^2}$ of that distance squared. For $\Gamma < 2$, a hard core plays no role in the low-density limit; in this limit, the density ρ provides the only relevant length scale $\rho^{1/2}$, and $\bar{R}_+$ and $\overline{R_+^2}$ behave like $\rho^{-1/2}$ and ρ^{-1}, respectively [see (2.17) and (2.18)]. For $\Gamma > 2$, a hard core becomes necessary to prevent the collapse of oppositely charged particles owing to a short-range divergence of the potential. With a hard core included, the long-range property of the potential creates a stronger tendency of pairing between oppositely charged species for increasing Γ. This pairing effect is responsible for the behaviors (2.27) and (2.28): when Γ increases, $\bar{R}_+$ and $\overline{R_+^2}$ as functions of $\rho^{1/2}$ are less and less divergent at $\rho=0$. Their behaviors change discontinuously at special values of Γ. Finally, for Γ large enough, $\bar{R}_+$ and $\overline{R_+^2}$ become finite at $\rho=0$, as expected in a fluid made of tightly bound pairs.

APPENDIX A

Here we rederive (3.1) by showing that when a hole of radius R is made in the plasma, the number of particles is changed by

$$\Delta N = -2mR \sum_{l=1}^{\Lambda} \frac{K_{l-1}(mR)}{K_l(mR)} \tag{A1}$$

where Λ is a cutoff. At $\Gamma=2$, oppositely charged point particles would collapse; finite results have been obtained by introducing a small hard core of size σ in the interaction. An equivalent procedure is to truncate (A1) and similar sums of some value Λ; the known asymptotic form[9] of the density $\rho(m, \sigma)$ as $\sigma \to 0$ is recovered[5] by choosing Λ of the order of R/σ.

Assuming (A1), the result (3.1) can be reclaimed by integrating the statistical mechanical formula

$$\Delta N = m \frac{\partial}{\partial m} \log \frac{\Xi(R)}{\Xi} \tag{A2}$$

and using the condition

$$\Xi(0)/\Xi = 1 \tag{A3}$$

to determine the constant of integration.

We note that ΔN can naturally be decomposed into two parts,

$$\Delta N = \Delta N_{\text{in}} + \Delta N_{\text{out}} \tag{A4}$$

where $\Delta N_{\text{in(out)}}$ is the change in the number of particles inside (outside) the hole. Clearly,

$$\Delta N_{\text{in}} = -\pi R^2 \rho \tag{A5}$$

where ρ is the particle in the bulk. The term ΔN_{out} is given in terms of the densities $\rho_+(r) = \rho_-(r)$ outside the hole via the formula

$$\Delta N_{\text{out}} = 2 \int_R^\infty [\rho_+(r) - \rho_+] \, 2\pi r \, dr \tag{A6}$$

where $\rho_+ = \frac{1}{2}\rho$. We expect this term to be cutoff independent (i.e., finite in the limit $\Lambda \to \infty$).

The method of Cornu and Jancovici[9] allows the excess density in (A6) and thus ΔN_{out} to be calculated. We find

$$\Delta N_{\text{out}} = (mR)^2 \left[\frac{1}{2} + \sum_{l=1}^{\Lambda} f_l(mR) \right] - 2mR \sum_{l=1}^{\Lambda} \frac{K_{l-1}(mR)}{K_l(mR)} \tag{A7}$$

where

$$f_l(mR) := I_l(mR)\, K_l(mR) + I_{l-1}(mR)\, K_{l-1}(mR) \tag{A8}$$

A cutoff Λ in (A7) has been introduced since the sums considered separately diverge logarithmically; however, the divergences cancel and (A7) is in fact finite in the limit $\Lambda \to \infty$, as expected. The simplification which results from considering the sums separately is that for large Λ

$$\sum_{l=1}^{\Lambda} f_l(mR) \sim \log \frac{2\Lambda}{mR} + o(1) \tag{A9}$$

which can be deduced from (A8) and the formula

$$I_l(x)\, K_l(x) = \int_0^{\infty} J_0(2x \sinh t) e^{-2lt}\, dt \tag{A10}$$

Since with Λ as a cutoff[5]

$$\beta P = \frac{m^2}{2\pi} \left(\log \frac{2\Lambda}{mR} + 1 \right) \tag{A11}$$

and consequently

$$\rho = \frac{m^2}{\pi} \left(\log \frac{2\Lambda}{mR} + \frac{1}{2} \right) \tag{A12}$$

substituting (A9) in (A7) and adding to (A5) gives (A1), as required.

APPENDIX B

In this appendix we give the large-R behavior of $\Xi(+, R)/\Xi(R)$ for the TCP in its conducting phase. This follows from the assumption that at macroscopic distances from the plasma, it behaves like an ideal conductor. This implies that for large R

$$\Xi(+, R)/\Xi(R) \sim e^{-\beta U} \tag{B1}$$

where U denotes the electrostatic energy of the fixed charge–induced surface charge system. The electrostatic energy consists of two terms: the self-energy of the induced surface charge (U_1 say) and the energy of interaction between the induced surface charge and the charge at the origin (U_2 say). A simple calculation gives

$$U_1 = -\frac{q^2}{2} \log \frac{R}{L} \qquad \text{and} \qquad U_2 = q^2 \log \frac{R}{L} \tag{B2}$$

Substituting (B2) in (B1) gives the sum rule that for large R

$$\Xi(+, R)/\Xi(R) \sim (L/R)^{\Gamma/2} \tag{B3}$$

This sum rule can be tested at $\Gamma = 2$, since the exact result (3.17) gives

$$\Xi(+, R)/\Xi(R) \sim L/R \tag{B4}$$

which agrees with the prediction of (B3).

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