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The Two-Dimensional Coulomb Gas on a Sphere: Exact Results

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At the special value of the reduced inverse temperature $\Gamma = 2$, the equilibrium statistical mechanics of a two-dimensional Coulomb gas confined to the surface of a sphere is an exactly solvable problem, just as it was for the Coulomb gas in a plane. The thermodynamic quantities and all the correlation functions can be calculated. Use is made of an isomorphism between the classical Coulomb gas and the free Fermi field theory associated with the Dirac operator on the sphere.

KEY WORDS: Coulomb gas; solvable models; Dirac equation; curved space.

1. INTRODUCTION

Some time ago, exact results were found for the equilibrium statistical mechanics of a classical two-dimensional Coulomb gas (two-component plasma). At a special value of the temperature, the thermodynamic quantities and all the correlation functions can be calculated. Starting from a lattice version[1] of the model, it is possible to take the continuum limit.[2,3] Simplifications occur in this continuum limit because of an isomorphism between the classical Coulomb gas and a quantum relativistic free Fermi field described by the two-dimensional Dirac Lagrangian.

In the present paper, we consider the case of a two-dimensional Coulomb gas confined to the surface of a sphere at that same special temperature. The mathematical problem is amusing, because the isomorphism will now involve a two-dimensional Dirac equation on a curved space. Another motivation is that exact results on a sphere can be used as a guide

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for some numerical simulations for which it is convenient to use a spherical geometry.[4] The similar problem of the one-component plasma has already been investigated.[5,6]

In Section 2, we review the known results for the two-dimensional Coulomb gas in a plane. In Section 3, we show how the problem on a sphere is related to the problem in a plane by an appropriate stereographic projection. The thermodynamic properties and the correlations are obtained, including explicit curvature corrections for the former.

2. THE COULOMB GAS IN A PLANE

The model is a two-dimensional system of particles of charges $\pm e$. The Coulomb interaction between two particles of charges e and e' at a distance r from one another is $-ee' \ln(r/L)$, where L is some length scale. The dimensionless coupling constant is $\Gamma = \beta e^2$, where β is the inverse temperature.

The point-particle system is well behaved for $\Gamma < 2$. From scaling considerations,[7] one finds the very simple equation of state

$$\beta p = \rho\left(1 - \frac{\Gamma}{4}\right)$$

where p is the pressure and ρ the number density (total number of particles per unit area). More detailed results can be obtained at the special value $\Gamma = 2$ for the coupling constant (which we shall choose from now on), provided some short-distance cutoff is introduced to prevent the collapse of pairs of oppositely charged particles. We first review (with minor changes) the relevant parts of refs. 1–3.

It is convenient to represent the position of a particle either by the vector $\mathbf{r} = (x, y)$ or by the complex number $z = x + iy$. For a system of N positive and N negative particles, the complex coordinates of which are u_i and v_i respectively, the Boltzmann factor at $\Gamma = 2$ is

$$\exp\left\{2 \sum_{i<j} \left[\ln\left|\frac{u_i - u_j}{L}\right| + \ln\left|\frac{v_i - v_j}{L}\right|\right] - 2\sum_{i,j} \ln\left|\frac{u_i - v_j}{L}\right|\right\}$$

$$= L^{2N}\left|\frac{\prod_{i<j}(u_i - u_j)(v_i - v_j)}{\prod_{i,j}(u_i - v_j)}\right|^2$$

$$= L^{2N}\left|\det\left[\frac{1}{u_i - v_j}\right]_{i,j=1,\ldots,N}\right|^2$$

where the last equality stems from the Cauchy double-alternant identity. Let us consider first a lattice version of the model. Two interwoven lattices

U and V are introduced. The positive (negative) particles occupy the sublattice $U(V)$. Each lattice site contains at most one particle. Then the grand partition function (here defined as a sum including only neutral systems) is

$$Z = 1 + \lambda^2 \sum_{\substack{u \in U \\ v \in V}} \left| \frac{L}{u - v} \right|^2 + \lambda^4 \sum_{\substack{u_1, u_2 \in U \\ v_1, v_2 \in V}} \left| \det \left[\frac{L}{u_i - v_j} \right]_{i, j = 1, 2} \right|^2 + \cdots$$

where the sums are defined with the prescription that configurations which differ only by a permutation of identical particles are counted only once. It can be easily seen that this grand partition function is the expansion of a determinant built on *all* lattice sites:

$$Z = \det \begin{bmatrix} 1 & 0 & \cdots & \frac{\lambda L}{u_1 - v_1} & \frac{\lambda L}{u_1 - v_2} & \cdots \\ 0 & 1 & \cdots & \frac{\lambda L}{u_2 - v_1} & \frac{\lambda L}{u_2 - v_2} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ \frac{\lambda L}{\bar{v}_1 - \bar{u}_1} & \frac{\lambda L}{\bar{v}_1 - \bar{u}_2} & \cdots & 1 & 0 & \cdots \\ \frac{\lambda L}{\bar{v}_2 - \bar{u}_1} & \frac{\lambda L}{\bar{v}_2 - \bar{u}_2} & \cdots & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Let us now take formally the continuum limit, ignoring divergences. A more compact notation can be introduced using 2×2 Pauli matrices 1, σ_x, σ_y, σ_z, and their combinations

$$\sigma_+ = \frac{1}{2}(\sigma_x + i\sigma_y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and}$$

$$\sigma_- = \frac{1}{2}(\sigma_x - i\sigma_y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The grand partition function Z becomes

$$Z = \det[1 + \lambda L \langle \mathbf{r} | K | \mathbf{r}' \rangle] \tag{2.1}$$

where K is now a continuous matrix the elements of which in position space are defined as

$$\langle \mathbf{r} | K | \mathbf{r}' \rangle = \frac{\sigma_+}{z - z'} + \frac{\sigma_-}{\bar{z} - \bar{z}'} \tag{2.2}$$

These matrix elements are themselves 2×2 matrices. In the continuum limit sums become integrals provided the fugacity λ is redefined by writing Z as

$$Z = 1 + \lambda^2 \int \frac{L^2}{|z-z'|^2}\, d^2\mathbf{r}\, d^2\mathbf{r}' + \cdots$$

The integral kernel K has the remarkable property that its inverse is a very simple differential operator. Indeed, from the identities

$$\frac{\partial}{\partial \bar{z}} \frac{1}{z-z'} = \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} \ln(z-z')(\bar{z}-\bar{z}') = \frac{1}{2} \nabla^2 \ln|\mathbf{r}-\mathbf{r}'| = \pi\delta(\mathbf{r}-\mathbf{r}')$$

and

$$\frac{\partial}{\partial z} \frac{1}{\bar{z}-\bar{z}'} = \pi\delta(\mathbf{r}-\mathbf{r}')$$

it follows that

$$\left(\sigma_x \frac{\partial}{\partial x} + \sigma_y \frac{\partial}{\partial y}\right) \langle \mathbf{r}|\, K\, |\mathbf{r}'\rangle$$
$$= 2\left(\sigma_+ \frac{\partial}{\partial z} + \sigma_- \frac{\partial}{\partial \bar{z}}\right)\left(\frac{\sigma_+}{z-z'} + \frac{\sigma_-}{\bar{z}-\bar{z}'}\right) = 2\pi\delta(\mathbf{r}-\mathbf{r}')$$

and therefore the inverse of $(2\pi)^{-1} K$ is the Dirac operator

$$\partial\!\!\!/ = 2\left(\sigma_+ \frac{\partial}{\partial z} + \sigma_- \frac{\partial}{\partial \bar{z}}\right) = \sigma_x \frac{\partial}{\partial x} + \sigma_y \frac{\partial}{\partial y}$$

In terms of a rescaled fugacity $m = 2\pi L\lambda$, which has the dimensions of an inverse length, the grand partition function can be rewritten as

$$Z = \det(1 + m\partial\!\!\!/^{-1})$$

Therefore

$$\ln Z = \operatorname{Tr} \ln[1 + m\partial\!\!\!/^{-1}] = \operatorname{Tr} \ln[(\partial\!\!\!/ + m)\,\partial\!\!\!/^{-1}] \tag{2.3}$$

The trace is to be taken here both on space and Pauli matrices. Equation (2.3) expresses the well-known equivalence between the two-dimensional Coulomb gas at $\Gamma = 2$ and the free relativistic Fermi field associated to the Dirac operator $\partial\!\!\!/ + m$. The rescaled fugacity m becomes the mass in the Dirac operator.

One obtains the correlation functions by the usual trick of considering the fugacity m in (2.3) as position dependent, $m = m(\mathbf{r})$, and taking functional derivatives of $\ln Z$ with respect to $m(\mathbf{r})$. Marking the sign of the particle at $\mathbf{r}$ by the index $s = \pm 1$ and defining the Green function

$$G_{s_1 s_2}(\mathbf{r}_1, \mathbf{r}_2) = \langle \mathbf{r}_1 s_1 | \frac{1}{\not{\partial} + m} | \mathbf{r}_2 s_2 \rangle \tag{2.4}$$

one obtains the one-body density for each species

$$\rho_s = m G_{ss}(\mathbf{r}, \mathbf{r}) \tag{2.5}$$

the truncated two-body densities

$$\rho^{(2)T}_{s_1 s_2}(\mathbf{r}_1, \mathbf{r}_2) = -m^2 G_{s_1 s_2}(\mathbf{r}_1, \mathbf{r}_2)\, G_{s_2 s_1}(\mathbf{r}_2, \mathbf{r}_1) \tag{2.6}$$

and more generally the truncated n-body densities

$$\rho^{(n)T}_{s_1 s_2}(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_n) = (-1)^{n+1} m^n \sum_{(i_1 i_2 \cdots i_n)} G_{s_{i_1} s_{i_2}}(\mathbf{r}_{i_1}, \mathbf{r}_{i_2}) \cdots G_{s_{i_n} s_{i_1}}(\mathbf{r}_{i_n}, \mathbf{r}_{i_1}) \tag{2.7}$$

where the summation runs over all cycles $(i_1 i_2 \cdots i_n)$ built with $\{1, 2, ..., n\}$. It is useful to note the symmetry relations (the first of these was misprinted in ref. 3)

$$G_{ss}(\mathbf{r}_1, \mathbf{r}_2) = \overline{G_{ss}(\mathbf{r}_2, \mathbf{r}_1)}, \qquad G_{s\,-s}(\mathbf{r}_1, \mathbf{r}_2) = -\overline{G_{-s\,s}(\mathbf{r}_2, \mathbf{r}_1)} \tag{2.8}$$

An explicit calculation of (2.4) gives, for a constant fugacity m,

$$G_{++}(\mathbf{r}_1, \mathbf{r}_2) = \quad G_{--}(\mathbf{r}_1, \mathbf{r}_2) = \frac{m}{2\pi} K_0(m |\mathbf{r}_1 - \mathbf{r}_2|)$$

$$G_{-+}(\mathbf{r}_1, \mathbf{r}_2) = -\overline{G_{+-}(\mathbf{r}_2, \mathbf{r}_1)} = \frac{m}{2\pi} e^{i\varphi} K_1(m\, |\mathbf{r}_1 - \mathbf{r}_2|)$$

where φ is the polar angle of $\mathbf{r}_1 - \mathbf{r}_2$ and K_0 and K_1 are modified Bessel functions. When used in (2.6) and (2.7), this Green function gives finite *truncated* n-body densities ($n \geqslant 2$). However, the one-body density (2.5) diverges [$K_0(0)$ is infinite]. This divergence, expected for a point-particle system, can be removed by some regularization procedure. For example, if the particles are small, charged, hard disks of diameter σ ("small" means $m\sigma \ll 1$), K_0 must be replaced[2,3] by $K_0(m\sigma) \sim \ln(2/m\sigma) - \gamma$ (here $\gamma = 0.5772...$ is Euler's constant) and the expressions for the densities become[2,3]

$$\rho_+ = \rho_- = \frac{m^2}{2\pi} \left(\ln \frac{2}{m\sigma} - \gamma \right) \tag{2.9}$$

In the same way, the pressure obtained from (2.3)

$$\beta p = \int \frac{d^2\mathbf{k}}{(2\pi)^2} \ln\left(1 + \frac{m^2}{k^2}\right) \tag{2.10}$$

diverges and must be regularized to

$$\beta p = \frac{m^2}{2\pi}\left(\ln\frac{2}{m\sigma} - \gamma + \frac{1}{2}\right) \tag{2.11}$$

consistent with (2.9), which is a regularized form of

$$\rho_+ = \rho_- = \frac{1}{2} m \frac{\partial}{\partial m}(\beta p) = m^2 \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{m^2 + k^2} \tag{2.12}$$

3. THE COULOMB GAS ON A SPHERE: STEREOGRAPHIC PROJECTION

We now assume that the particles are on the surface of a sphere of radius R. We define the Coulomb interaction between two particles of charges e and e' as $-ee' \ln(r/L)$, where r is the length of the straight line in space which joins the particles, *not* as one might expect the geodesic distance on the sphere. With this choice, through an appropriate stereographic projection, the problem on the sphere can be mapped to a planar problem of the type considered in Section 2. The properties of the Coulomb gas on the sphere can be expressed in terms of the spectrum and Green functions of a Dirac operator on the sphere which is related to the usual Dirac operator in the plane by the stereographic projection.

3.1. Stereographic Projection

Let us consider a sphere of radius R and the plane tangent to its south pole (Fig. 1). Let P be the stereographic projection of a point M of the sphere onto the plane, from the north pole. In terms of the spherical coordinates (θ, φ) of M, the complex coordinate $z = x + iy$ of P is given by

$$z = 2Re^{i\varphi} \operatorname{cotan}\frac{\theta}{2} \tag{3.1}$$

This projection is a conformal transformation. An element of length dl at M and its projection $|dz|$ at P are related by the conformal weight

$$e^{\omega} = \sin^2\frac{\theta}{2} = \frac{1}{1 + (|z|^2/4R^2)}$$

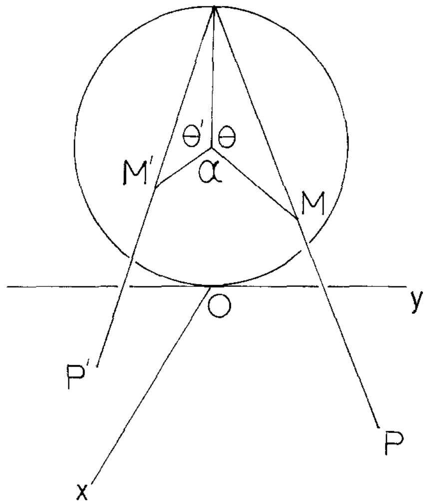

Fig. 1. The stereographic projection.

That is, $dl = e^{\omega}\,|dz|$, independent of the orientation of these elements. Therefore, the angles are conserved. The line joining two points M and M' has a Euclidean length $2R\sin(\alpha/2)$, where α is the angular distance from the center of the sphere. This length also has a simple relation with its projection $|z-z'|$:

$$2R\sin\frac{\alpha}{2} = e^{\omega/2}\,|z-z'|\,e^{\omega'/2} = \sin\frac{\theta}{2}\,|z-z'|\,\sin\frac{\theta'}{2} \tag{3.2}$$

The grand partition function of the Coulomb gas on the sphere is

$$Z = 1 + \int dS\,dS'\,\frac{\lambda^2 L^2}{[2R\sin(\alpha/2)]^2} + \cdots$$
$$= 1 + \int dS\,dS'\,\frac{\lambda^2 L^2}{|e^{\omega/2}(z-z')\,e^{\omega'/2}|^2} + \cdots$$

where dS and dS' are the area elements around points M and M', respectively. Following the same steps as in Section 2, with $z-z'$ replaced by $e^{\omega/2}(z-z')\,e^{\omega'/2}$, we now find for Z the expression

$$Z = \det[1 + \lambda L\langle M|\,K\,|M'\rangle] \tag{3.3}$$

where the matrix elements of K between two points M and M' are given by

$$\langle M|\,K\,|M'\rangle = \frac{\sigma_+}{e^{\omega/2}(z-z')\,e^{\omega'/2}} + \frac{\sigma_-}{e^{\omega/2}(\bar{z}-\bar{z}')\,e^{\omega'/2}} \tag{3.4}$$

Again, the integral kernel K has a diffrential operator as its inverse. The inverse of $(2\pi)^{-1} K$ is now

$$\not{D} = e^{-3\omega/2}\left(\sigma_x \frac{\partial}{\partial x} + \sigma_y \frac{\partial}{\partial y}\right) e^{\omega/2} = e^{-3\omega/2}\, \not{\partial} e^{\omega/2} \tag{3.5}$$

since

$$\begin{aligned}\not{D}\langle M|\, K\,|M'\rangle &= e^{-3\omega/2}\, \not{\partial} e^{\omega/2} e^{-\omega/2}\left(\frac{\sigma_+}{z-z'} + \frac{\sigma_-}{\bar{z}-\bar{z}'}\right) e^{-\omega'/2} \\ &= e^{-3\omega/2} 2\pi\delta(\mathbf{r}-\mathbf{r}')\, e^{-\omega'/2} = 2\pi e^{-2\omega'}\delta(\mathbf{r}-\mathbf{r}') = 2\pi\,\delta(M, M')\end{aligned}$$

Here $\delta(M, M')$ is the Dirac distribution on the sphere defined such that

$$\int \delta(M, M')\, dS' = \int \delta(M, M')\, R^2\, d(\cos\theta')\, d\varphi' = 1$$

and $dS' = e^{2\omega'}\, d^2\mathbf{r}'$.

Thus, the Dirac operator $\not{\partial}$ in the plane has to be replaced by $\not{D}$ defined by (3.5). It turns out that $\not{D}$ is the Dirac operator on the sphere. The Dirac operators in curved spaces have been investigated by many authors. A recent review of the Dirac operator on the sphere can be found in ref. 8. For more general references, see, e.g., ref. 9.

3.2. Thermodynamic Properties

If we define again a mass m in terms of the fugacity λ by $m = 2\pi L\lambda$, we have instead of (2.3)

$$\ln Z = \mathrm{Tr}\, \ln[1 + m\not{D}^{-1}]$$

The eigenvalues[8] of $\not{D}$ are $\pm in/R$, where n is any positive integer, with multiplicity $2n$. Thus, the pressure is given by

$$\begin{aligned}\beta p &= \frac{1}{4\pi R^2} \ln Z = \frac{1}{8\pi R^2}\, \mathrm{Tr}\, \ln[1 - m^2\not{D}^{-2}] \\ &= \frac{1}{2\pi R^2} \sum_{n=1}^{\infty} n \ln\left[1 + \frac{m^2 R^2}{n^2}\right]\end{aligned} \tag{3.6}$$

and the densities are

$$\begin{aligned}\rho_+ = \rho_- &= \frac{1}{2} m \frac{\partial}{\partial m}(\beta p) = \frac{m^2}{8\pi R^2}\, \mathrm{Tr}\, \frac{1}{m^2 - \not{D}^2} \\ &= \frac{m^2}{2\pi} \sum_{n=1}^{\infty} \frac{n}{m^2 R^2 + n^2}\end{aligned} \tag{3.7}$$

These pressure and densities are divergent quantities, unless they are regularized by a short-distance cutoff, as in the planar case.

In the limit $R \to \infty$, setting $k = n/R$, we retrieve the planar results (2.10) and (2.12). However, it is also possible to compute finite-size corrections, and these *corrections* turn out to be nondivergent. They can be computed as follows. From (3.6), the finite-size correction to the pressure is given by

$$\beta p - \beta p(R = \infty) = \frac{1}{2\pi R^2} \lim_{N \to \infty} \left[\sum_{n=1}^{N} n \ln(n^2 + m^2 R^2) - \int_0^N dn\, n \ln(n^2 + m^2 R^2) - 2 \sum_{n=1}^{N} n \ln n + 2 \int_0^N dn\, n \ln n \right]$$

From the Euler–MacLaurin summation formula, with $f(n) = n \ln(n^2 + m^2 R^2)$, one obtains

$$\begin{aligned} &\sum_{n=1}^{N} n \ln(n^2 + m^2 R^2) - \int_0^N dn\, n \ln(n^2 + m^2 R^2) \\ &\quad = \frac{1}{2} [f(N) - f(0)] + \frac{1}{12} [f'(N) - f'(0)] - \frac{1}{720} [f'''(N) \\ &\quad - f'''(0)] + \cdots = N \ln N + \frac{1}{6} (\ln N + 1) + O\left(\frac{1}{N^2}\right) \\ &\quad - \frac{1}{12} \ln(mR)^2 + \frac{1}{120(mR)^2} + \cdots \end{aligned}$$

Similarly, there exists an asymptotic expansion (see ref. 10, p. 42)

$$\sum_{n=1}^{N} n \ln n - \int_0^N dn\, n \ln n = \frac{1}{2} N \ln N + \frac{1}{12} (\ln N + 1) - \zeta'(-1) + O\left(\frac{1}{N^2}\right)$$

where ζ' is the derivative of Riemann's zeta function: $\zeta'(-1) = -0.1654\ldots$. Therefore

$$\beta p - \beta p(R = \infty) = \frac{1}{2\pi} \left[-\frac{\ln(mR)}{6R^2} + \frac{2\zeta'(-1)}{R^2} + \frac{1}{120 m^2 R^4} + \cdots \right] \tag{3.8}$$

Correspondingly, the finite-size correction to the densities (3.7) is

$$\rho_s - \rho_s(R = \infty) = \frac{1}{2\pi} \left[-\frac{1}{12R^2} - \frac{1}{120 m^2 R^4} + \cdots \right] \tag{3.9}$$

These corrections are respectively of order $(\ln R)/R^2$ and $1/R^2$. This is to be compared to the case of a disk of radius R, for which there is a boundary correction of order $1/R$.

3.3. Correlations

The correlations are still given by (2.6) and (2.7), where now, however, the Green function must be defined with $\not{D}$, the Dirac operator on the sphere:

$$G_{s_1 s_2}(\mathbf{r}_1, \mathbf{r}_2) = \langle \mathbf{r}_1 s_1 | \frac{1}{\not{D} + m} | \mathbf{r}_2 s_2 \rangle \tag{3.10}$$

We have defined here a point on the sphere by the coordinate $\mathbf{r}$ of its projection. This Green function has been studied in different forms. For our purpose, it is most convenient to adapt one of the earliest references.[11]

The definition (3.10) of the 2×2 matrix G is equivalent to the partial differential equations

$$(e^{-3\omega/2} \not{\partial} e^{\omega/2} + m)\, G(\mathbf{r}, \mathbf{r}') = \delta(M, M') = e^{-2\omega'} \delta(\mathbf{r} - \mathbf{r}') \tag{3.11}$$

In terms of

$$\tilde{G}(\mathbf{r}, \mathbf{r}') = e^{\omega/2} G(\mathbf{r}, \mathbf{r}')\, e^{\omega'/2} \tag{3.12}$$

(3.11) can be rewritten as

$$(\not{\partial} + m e^{\omega})\, \tilde{G}(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \tag{3.13}$$

This equation (3.13) has a remarkably simple interpretation: $\tilde{G}(\mathbf{r}, \mathbf{r}')$ is the Green function for the planar problem with a position-dependent fugacity $me^{\omega} = m[1 + (r^2/4R^2)]^{-1}$.

A Special Case. Since me^{ω} depends only on the distance r to the origin, we expect that this circular symmetry will make the problem simpler in the special case when the source point it at the origin $\mathbf{r}' = 0$, i.e., at the south pole of the sphere. Let us first deal with this case.

Equation (3.13) then can be written as

$$\tilde{G}(\mathbf{r}, 0) = (m - e^{-\omega} \not{\partial})[(\not{\partial} + m e^{\omega})(m - e^{-\omega} \not{\partial})]^{-1}$$

or equivalently as

$$\tilde{G}(\mathbf{r}, 0) = (m - e^{-\omega} \not{\partial})\, H(\mathbf{r}) \tag{3.14}$$

with $H(\mathbf{r})$ determined by

$$[m^2 e^{\omega} - \bar{\partial} e^{-\omega} \partial]\, H(\mathbf{r}) = \delta(\mathbf{r})$$

Because of the circular symmetry, $H = H(r)$ is a function of the distance r alone and the equation for H is the ordinary differential equation

$$\left(m^2 e^{\omega} - \frac{1}{r}\frac{d}{dr} r e^{-\omega} \frac{d}{dr}\right) H(r) = \delta(\mathbf{r}) \tag{3.15}$$

Near $\mathbf{r} = 0$, $\nabla^2 H(r) = -\delta(\mathbf{r})$, i.e., $H(r)$ is singular as $-(2\pi)^{-1} \ln r$. In terms of the variable $s = e^{\omega} = \sin^2(\theta/2)$, (3.15) becomes the hypergeometric equation

$$\left[s(1-s)\frac{d^2}{ds^2} - s\frac{d}{ds} - m^2 R^2\right] H = 0, \qquad s < 1 \tag{3.16}$$

The solution to (3.16) which is singular as $-(2\pi)^{-1} \ln r \sim -(4\pi)^{-1} \times \ln(1-s)$ at the south pole $s = 1$ and regular at the north pole $s = 0$ is of the form[12]

$$H = AsF(1 + imR, 1 - imR; 2; s)$$

where F is the hypergeometric function and A a constant. Indeed, near $s = 1$, the behavior of F is

$$F \sim -\frac{\sinh \pi m R}{\pi m R}[\ln(1-s) + \psi(1 + imR) + \psi(1 - imR) + 2\gamma] \tag{3.17}$$

where ψ is the logarithmic derivative of the gamma function and γ is Euler's constant. Therefore $A = mR/(4 \sinh \pi m R)$ and we obtain from (3.12) (with $e^{\omega'} = 1$) and (3.14)

$$\begin{aligned} G_{++}(\mathbf{r}, 0) &= G_{--}(\mathbf{r}, 0) \\ &= \frac{m^2 R}{4 \sinh \pi m R} \sin\frac{\theta}{2} F\left(1 + imR, 1 - imR; 2; \sin^2\frac{\theta}{2}\right) \end{aligned}$$

and

$$\begin{aligned} G_{+-}(\mathbf{r}, 0) &= \overline{G_{-+}(\mathbf{r}, 0)} \\ &= \frac{m}{4 \sinh \pi m R} e^{-i\varphi} \cos\frac{\theta}{2} F\left(1 + imR, 1 - imR; 1; \sin^2\frac{\theta}{2}\right) \end{aligned}$$

For future reference, it is preferable to write the solution in terms of the angular distance α between the two points involved in the Green function. Here $\alpha = \pi - \theta$, and

$$G_{++}(\mathbf{r}, 0) = G_{--}(\mathbf{r}, 0)$$
$$= \frac{m^2 R}{4 \sinh \pi m R} \cos \frac{\alpha}{2} F\left(1 + imR, 1 - imR; 2; \cos^2 \frac{\alpha}{2}\right) \quad (3.18a)$$
$$G_{+-}(\mathbf{r}, 0) = \overline{G_{-+}(\mathbf{r}, 0)}$$
$$= \frac{m}{4 \sinh \pi m R} e^{-i\varphi} \sin \frac{\alpha}{2} F\left(1 + imR, 1 - imR; 1; \cos^2 \frac{\alpha}{2}\right) \quad (3.18b)$$

We have written the Green function in terms of its components

$$G = \begin{pmatrix} G_{++} & G_{+-} \\ G_{-+} & G_{--} \end{pmatrix}$$

Because of the rotational symmetry of the sphere, the one-body density is a constant and the two-body densities for two points depend only on their angular distance α. Therefore, without any loss of generality, the one-body density can be retrieved by using (3.18a) in (2.5). The behavior (3.17) near $s = 1$ allows us to build an expansion in inverse powers of R and to retrieve (3.9). One obtains the truncated two-body densities by using (2.8) and (3.18) in (2.6).

General Case. The rotational invariance of the two-body truncated densities implies that the Green function for two points has a *modulus* which depends only on the angular distance α between these points. However, in general there is also a phase factor, depending on the coordinates of both points, and this phase is needed to compute the higher-order densities using (2.7). Thus, we have to look for the Green function in the general case of two arbitrary points.

To find $G(\mathbf{r}, \mathbf{r}')$ in general, following the method of ref. 11, we shall rotate the sphere around a horizontal diameter in such a way that $\mathbf{r}'$ comes to the south pole. This rotation transforms the complex coordinate z into

$$Z = \frac{z - z'}{1 + (z\bar{z}'/4R^2)} \quad (3.19)$$

This formula can be obtained, for instance, by noting that Z has to be a metromorphic function of z, since the transformation is a conformal one, and by considering first the special case when z and z' are real. If we make this change of variables in the Dirac operator of (3.11), we obtain, after a short calculation,

$$e^{-3\omega/2}2\left(\sigma_+\frac{\partial}{\partial z}+\sigma_-\frac{\partial}{\partial \bar{z}}\right)e^{\omega/2}+m$$

$$= S\left[e^{-3\Omega/2}2\left(\sigma_+\frac{\partial}{\partial Z}+\sigma_-\frac{\partial}{\partial \bar{Z}}\right)e^{\Omega/2}+m\right]S^{-1}$$

where $e^{\Omega}=[1+(|Z|^2/4R^2)]^{-1}$ is the conformal weight at point Z, and S is a 2×2 matrix

$$S=\begin{pmatrix} e^{i\zeta/2} & 0 \\ 0 & e^{-i\zeta/2}\end{pmatrix} \tag{3.20}$$

where

$$e^{i\zeta}=\frac{1-(Z\bar{z}'/4R^2)}{1-(\bar{Z}z'/4R^2)}=\frac{1+(\bar{z}z'/4R^2)}{1+(z\bar{z}'/4R^2)} \tag{3.21}$$

Multiplying both sides of (3.11) on the left by S^{-1} and noting that $S^{-1}\delta(M, M')=[S(Z=0)]^{-1}\,\delta(M, M')=\delta(M, M')$, we find

$$\left[e^{-3\Omega/2}2\left(\sigma_+\frac{\partial}{\partial Z}+\sigma_-\frac{\partial}{\partial \bar{Z}}\right)e^{\Omega/2}+m\right]S^{-1}G(\mathbf{r}, \mathbf{r}')=\delta(M, M')$$

Therefore $S^{-1}G(\mathbf{r}, \mathbf{r}')$ is the Green function of the operator which is obtained from the one in (3.11) by replacing z by Z but keeping the same Pauli matrices. The source point M' is located at $Z=0$. Consequently,

$$G(\mathbf{r}, \mathbf{r}')=SG(Z, 0)$$

where $G(Z,0)$ is given by (3.18) with $\varphi=\arg Z=\arg\{(z-z')/[1+(z\bar{z}'/4R^2)]\}$. Performing the multiplication by S gives the Green function, now in the general case:

$$G_{ss}(\mathbf{r}, \mathbf{r}')=e^{is\zeta/2}\cos\frac{\alpha}{2}A(\alpha) \tag{3.22a}$$

$$G_{+-}(\mathbf{r}, \mathbf{r}')=\overline{G_{-+}(\mathbf{r}, \mathbf{r}')}=e^{-i\xi/2}\sin\frac{\alpha}{2}B(\alpha) \tag{3.22b}$$

where

$$A(\alpha)=\frac{m^2R}{4\sinh \pi mR}F\left(1+imR, 1-imR; 2; \cos^2\frac{\alpha}{2}\right)$$

$$B(\alpha)=\frac{m}{4\sinh \pi mR}F\left(1+imR, 1-imR; 1; \cos^2\frac{\alpha}{2}\right)$$

The phase ζ is given by (3.21) and $\xi/2=\arg(z-z')$.

Using (3.1) and (3.2), we obtain alternative expressions for the phases:

$$\frac{\zeta}{2} = \arg\left[\sin\frac{\theta}{2}\sin\frac{\theta'}{2} + \cos\frac{\theta}{2}\cos\frac{\theta'}{2}e^{-i(\varphi-\varphi')}\right]$$

$$\frac{\xi}{2} = \arg\left[\cos\frac{\theta}{2}\sin\frac{\theta'}{2}e^{i\varphi} - \sin\frac{\theta}{2}\cos\frac{\theta'}{2}e^{i\varphi'}\right]$$

and a more compact expression for (3.22):

$$G(\mathbf{r}, \mathbf{r}') = e^{\omega/2}\left\{\left[1 + \frac{(\boldsymbol{\sigma}\cdot\mathbf{r})(\boldsymbol{\sigma}\cdot\mathbf{r}')}{4R^2}\right]A(\alpha) + \frac{\boldsymbol{\sigma}\cdot(\mathbf{r}-\mathbf{r}')}{2R}B(\alpha)\right\}e^{\omega'/2}$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y)$.

It may be remarked that the form (3.20) of S is not unexpected. The conformal transformation (3.19) rotates a line element by an angle

$$\arg\frac{\partial Z}{\partial z} = \arg\frac{1 + (|z'|^2/4R^2)}{[1 + (z\bar{z}'/4R^2)]^2} = -\zeta$$

Therefore, the local frame of reference at Z is obtained from the one at z by a rotation of angle $-\zeta$ around the vertical axis. It is well known that under such a rotation the Pauli matrices are transformed by (3.20).

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