

# Charge Fluctuations for a Coulomb Fluid in a Disk on a Pseudosphere

B. Jancovici<sup>1</sup> and G. Téllez<sup>2</sup>

*Received May 27, 2003; accepted July 25, 2003*

---

The classical (i.e., non-quantum) equilibrium statistical mechanics of a Coulomb fluid living on a pseudosphere (an infinite surface of constant negative curvature) is considered. The Coulomb fluid occupies a large disk communicating with a reservoir (grand-canonical ensemble). The total charge  $Q$  on the disk fluctuates. In a macroscopic description, the charge correlations near the boundary circle can be described as correlations of a surface charge density  $\sigma$ . In a macroscopic approach, the variance of  $Q$  and the correlation function of  $\sigma$  are computed; they are universal. These macroscopic results are shown to be valid for two solvable microscopic models, in the limit when the microscopic thickness of the surface charge density goes to zero.

---

**KEY WORDS:** Pseudosphere; negative curvature; two-dimensional Coulomb fluid; charge fluctuations; surface correlations; macroscopic electrostatics; microscopic solvable models.

## 1. INTRODUCTION

This paper is dedicated to Elliott Lieb on the occasion of his 70th birthday. It is a variation on a theme to which Elliott has brought major contributions.<sup>(1,2)</sup>

How statistical mechanics is affected by the curvature of space might be of some interest in general relativity, and also is an amusing problem *per se*. A simple case is a two-dimensional system living on a pseudosphere,

---

<sup>1</sup>Laboratoire de Physique Théorique, Bâtiment 210, Université de Paris-Sud, 91405 Orsay, France (Unité Mixte de Recherche No. 8627-CNRS); e-mail: Bernard.Jancovici@th.u-psud.fr

<sup>2</sup>Grupo de Física Teórica de la Materia Condensada, Departamento de Física, Universidad de Los Andes, A. A. 4976, Bogotá, Colombia; e-mail: gtellez@uniandes.edu.co

i.e., a surface of constant negative curvature. Unlike the sphere, the pseudosphere has an infinite area, and therefore, on a pseudosphere, one can consider the thermodynamic limit of some system while keeping a given curvature. A special feature is that, for a large domain, the neighborhood of the boundary has an area of the same order of magnitude as the whole area of this domain; this feature makes the approach to the thermodynamic limit rather different to what happens in a flat space.

More specifically, the present paper deals with a two-dimensional classical (i.e., non-quantum) Coulomb fluid living on a pseudosphere. This is a system of charged particles interacting by Coulomb's law, with this law defined on the pseudosphere, i.e., as the solution of the Poisson equation written with the pseudosphere metric. The Coulomb fluid is assumed to be in equilibrium and confined in a large disk drawn on the pseudosphere. The grand-canonical ensemble is used: the fluid can freely exchange particles with a reservoir. Thus, the total charge  $Q$  may fluctuate. Furthermore, there are charge correlations which, near the circle boundary of the disk, can be described as correlations of a surface charge density  $\sigma$ . The aims of the present paper are to compute the variance of  $Q$  and the two-point correlation function of  $\sigma$ . It will be shown that these quantities are universal (i.e., independent of the microscopic nature of the fluid). These universal results will be checked on two exactly solvable models: the two-component plasma, made of two species of particles of opposite signs, and the one-component plasma, made of one species of particles in a neutralizing background.

These problems have already been studied and solved in a flat space. For a finite two-dimensional Coulomb fluid in a plane, the total charge  $Q$  essentially does not fluctuate.<sup>(3)</sup> Furthermore, in the case of a large disk of radius  $R$  centered at the origin, the surface charge correlation function<sup>(4)</sup> is given by the universal expression

$$\beta \langle \sigma(\varphi) \sigma(0) \rangle^T = -\frac{1}{2\pi^2 [2R \sin(\varphi/2)]^2} \quad (1.1)$$

where  $\beta$  is the inverse temperature,  $\sigma(\varphi)$  the surface charge density on the boundary circle at the point of polar angle  $\varphi$ , and  $\langle \dots \rangle^T$  is a truncated statistical average. These results have been checked on exactly solvable models.<sup>(3,5)</sup>

Here, the same problems are considered, now on a pseudosphere. In Section 2, some basic properties of the pseudosphere and of Coulomb's law on it are recalled. In Section 3, macroscopic electrostatics on a pseudosphere is used for determining the variance of the total charge  $Q$  and the correlation function of the surface charge density  $\sigma$ . In Section 4, the

results are checked on a solvable model, the two-component plasma at a special temperature. In Section 5, the results are checked again on another solvable model, the one-component plasma at a special temperature.

## 2. PSEUDOSPHERE AND COULOMB'S LAW

Let us recall a few properties of the surface of constant negative curvature called a pseudosphere. Such a surface is a two-dimensional manifold, the entirety of which cannot be embedded in three-dimensional Euclidean space. Its properties are defined by its metric. Several sets of coordinates are commonly used.

The one which renders explicit the resemblance with the sphere is  $(\tau, \varphi)$  with  $\tau \in [0, \infty[$  and  $\varphi \in ]-\pi, \pi]$ , the metric being

$$ds^2 = a^2(d\tau^2 + \sinh^2 \tau d\varphi^2) \quad (2.1)$$

where  $-1/a^2$  is the Gaussian curvature (instead of  $1/R^2$  for a sphere of radius  $R$ ). The geodesic distance  $s$  between two points at  $(\tau, \varphi)$  and  $(\tau', \varphi')$  is given by

$$\cosh(s/a) = \cosh \tau \cosh \tau' - \sinh \tau \sinh \tau' \cos(\varphi - \varphi') \quad (2.2)$$

In particular, the geodesic distance of the point  $(\tau, \varphi)$  to the origin is  $a\tau$ . The Laplace–Beltrami operator is

$$\Delta = \frac{1}{a^2} \left( \frac{1}{\sinh \tau} \frac{\partial}{\partial \tau} \sinh \tau \frac{\partial}{\partial \tau} + \frac{1}{\sinh^2 \tau} \frac{\partial^2}{\partial \varphi^2} \right) \quad (2.3)$$

The set of points at a geodesic distance from the origin less than or equal to  $R = a\tau_0$  will be called a disk of radius  $R$ . Its boundary will be called a circle of radius  $R$ . Its circumference is

$$\mathcal{C} = 2\pi a \sinh \tau_0 \quad (2.4)$$

and its area is

$$\mathcal{A} = 4\pi a^2 \sinh^2(\tau_0/2) \quad (2.5)$$

It is remarkable that, for a large radius, both the circumference and the area are proportional to  $\exp \tau_0$ : the neighborhood of the boundary circle has an area of the same order of magnitude as the whole area!

Another often used set of coordinates is  $(r, \varphi)$  with  $r/(2a) = \tanh(\tau/2)$ . Then, the metric is

$$ds^2 = \frac{dr^2 + r^2 d\varphi^2}{\left(1 - \frac{r^2}{4a^2}\right)^2} \quad (2.6)$$

When these coordinates are used, the whole (infinite) pseudosphere maps on a disk of radius  $2a$ , the Poincaré disk.

Finally, here it will be convenient to use also the coordinates  $(\epsilon, \varphi)$  with

$$\tanh(\tau/2) = e^{-\epsilon} \quad (2.7)$$

Then the metric is

$$ds^2 = \frac{a^2}{\sinh^2 \epsilon} (d\epsilon^2 + d\varphi^2) \quad (2.8)$$

and the Laplace–Beltrami operator has the simple form

$$\Delta = \frac{\sinh^2 \epsilon}{a^2} \left( \frac{\partial^2}{\partial \epsilon^2} + \frac{\partial^2}{\partial \varphi^2} \right) \quad (2.9)$$

The Coulomb potential  $v(s)$  at a geodesic distance  $s$  from a unit point charge obeys the Poisson equation

$$\Delta v(s) = -2\pi \delta^{(2)}(s) \quad (2.10)$$

where  $\delta^{(2)}$  is the Dirac distribution on the pseudosphere. The solution of (2.10) which vanishes at infinity is

$$v(s) = -\ln \tanh \frac{s}{2a} \quad (2.11)$$

### 3. MACROSCOPIC ELECTROSTATICS, CHARGE FLUCTUATIONS, SURFACE CHARGE CORRELATIONS

#### 3.1. Two Problems in Macroscopic Electrostatics

Here are two problems, the solution of which will be needed in the following. On the pseudosphere, an ideal conductor fills the disk of radius  $R = a\tau_0$  centered at the origin.

**Capacitance.** The first problem, a very simple one, is: What is the capacitance of this disk? If the disk carries a charge  $Q$ , this charge uniformly spreads on its circumference and, by Newton's theorem (which can be easily shown to be valid on a pseudosphere), generates on the whole disk the constant electric potential  $Q v(R) = -Q \ln \tanh(\tau_0/2)$ . Therefore, the capacitance is

$$C = -\frac{1}{\ln \tanh \frac{\tau_0}{2}} \quad (3.1)$$

In the large-disk limit  $\tau_0 \rightarrow \infty$ ,  $C \sim \exp(\tau_0/2)$ .

**A Point Charge in the Presence of the Disk.** The second problem is: A unit point charge is located, outside the disk, at point  $(\tau', \varphi' = 0)$ . The disk is grounded (i.e., kept at zero potential). What is the electric potential  $\phi(\tau, \varphi; \tau')$  at some point  $(\tau, \varphi)$ , outside the disk? The method of images, which can be used for a flat disk, does not seem to work on a pseudosphere, and a Fourier expansion will be used.

The potential due to the unit point charge alone is (2.11). Expressing this potential in terms of  $\cosh(s/a)$ , using (2.2), and expanding as a Fourier series in  $\varphi$  (in the case  $\tau < \tau'$  which suffices for our purpose) gives

$$v(s) = \epsilon' + \sum_{\ell=1}^{\infty} \frac{2 \sinh \epsilon' \ell e^{-\ell}}{\ell} \cos \ell \varphi \quad (\tau_0 < \tau < \tau') \quad (3.2)$$

where we have gone from the variables  $\tau$  and  $\tau'$  to the variables  $\epsilon$  and  $\epsilon'$  defined by (2.7) and its analog for  $\epsilon'$ . Using (2.9), one easily checks that the Laplacian of each term of (3.2) vanishes.

The full potential in the presence of the disk is obtained by adding to (3.2) terms of zero Laplacian symmetrical in  $\epsilon$  and  $\epsilon'$ : a term of the form  $A_0 \epsilon' \epsilon$ , and terms of the form  $A_\ell \sinh \epsilon' \ell \sinh \epsilon \ell \cos \ell \varphi$ . These terms do vanish when  $\tau$  or  $\tau'$  goes to infinity. The coefficients  $A_\ell$  are determined by the condition that the potential vanishes on the disk, i.e., when  $\tau = \tau_0$ . The result is

$$\phi(\tau, \varphi; \tau') = \epsilon' - \frac{\epsilon \epsilon'}{\epsilon_0} + \sum_{\ell=1}^{\infty} 2 \sinh \epsilon' \ell \left( e^{-\epsilon \ell} - \frac{e^{-\epsilon_0 \ell}}{\sinh \epsilon_0 \ell} \sinh \epsilon \ell \right) \frac{\cos \ell \varphi}{\ell} \quad (3.3)$$

where  $\epsilon_0$  is related to  $\tau_0$  by the analog of (2.7).

### 3.2. Charge Fluctuations

The disk of radius  $R = a\tau_0$  centered at the origin is filled with a Coulomb fluid. It can freely exchange charges with a reservoir located at

infinity (grand canonical ensemble). If macroscopic electrostatics is applicable, what is the variance of the charge  $Q$  carried by the disk?

The same reasoning as in the flat space case,<sup>(3)</sup> using linear response theory, gives for the variance

$$\beta \langle Q^2 \rangle^T = C \quad (3.4)$$

where here the capacitance  $C$  is given by (3.1). This result (3.4) just says that the variation of the energy  $Q^2/2C$  has the usual thermal average  $(1/2) \beta^{-1}$ . In the large-disk limit  $\tau_0 \rightarrow \infty$ ,

$$\beta \langle Q^2 \rangle^T \sim \frac{e^{\tau_0}}{2} \sim \frac{\mathcal{C}}{2\pi a} \quad (3.5)$$

### 3.3. Surface Charge Correlations

In three dimensions, macroscopic electrostatics deals with volume charge densities and surface charge densities. In the present case of a two-dimensional system (a disk), the analog of the surface charge density actually is a charge per unit length on the boundary circle; we shall nevertheless still call it a surface charge density  $\sigma(\varphi)$ . If macroscopic electrostatics is applicable, what is the two-point correlation function of  $\sigma(\varphi)$ ?

The same reasoning as in the case of a flat space,<sup>(4)</sup> using linear response theory, gives for the two-point correlation function

$$\beta \langle \sigma(\varphi) \sigma(0) \rangle^T = -\frac{1}{(2\pi a)^2} \left( \frac{\partial^2 \phi(\tau, \varphi; \tau')}{\partial \tau \partial \tau'} \right)_{\tau = \tau' = \tau_0} \quad (3.6)$$

where  $\phi(\tau, \varphi; \tau')$  is the electric potential (3.3).

The second derivative in (3.6) can be expressed in closed form in terms of the Jacobi theta<sup>(6)</sup> function  $\theta_1$ . For the sake of dealing only with convergent series, from (3.3) one first computes the second derivative for  $\tau' > \tau_0$ , i.e.,  $\epsilon' < \epsilon_0$ :

$$\left( \frac{\partial^2 \phi(\tau, \varphi; \tau')}{\partial \tau \partial \tau'} \right)_{\tau = \tau_0} = -2 \sinh \epsilon_0 \sinh \epsilon' \left( \frac{1}{2\epsilon_0} + \sum_{\ell=1}^{\infty} \frac{\cosh \epsilon' \ell}{\sinh \epsilon_0 \ell} \ell \cos \ell \varphi \right) \quad (3.7)$$

For taking the limit of (3.7) when  $\tau' \rightarrow \tau_0$ , one subtracts from and adds to  $\cosh \epsilon' \ell / \sinh \epsilon_0 \ell$  a term  $e^{-(\epsilon_0 - \epsilon') \ell}$ . The limit of one of the two resulting series is computed after the summation has been performed:

$$\lim_{\epsilon' \rightarrow \epsilon_0} \sum_{\ell=1}^{\infty} e^{-(\epsilon_0 - \epsilon') \ell} \ell \cos \ell \varphi = -\frac{1}{4 \sin^2 \frac{\varphi}{2}} \quad (3.8)$$

The other series involves  $(\cosh \epsilon' \ell / \sinh \epsilon_0 \ell) - e^{-(\epsilon_0 - \epsilon') \ell}$ . It remains absolutely convergent when the limit  $\epsilon' = \epsilon_0$  is taken in each term. Thus

$$\left( \frac{\partial^2 \phi(\tau, \varphi; \tau')}{\partial \tau \partial \tau'} \right)_{\tau = \tau' = \tau_0} = -2 \sinh^2 \epsilon_0 \left( \frac{1}{2\epsilon_0} - \frac{1}{4 \sin^2 \frac{\varphi}{2}} + 2 \sum_{\ell=1}^{\infty} \frac{e^{-2\epsilon_0 \ell}}{1 - e^{-2\epsilon_0 \ell}} \ell \cos \ell \varphi \right) \quad (3.9)$$

The sum in (3.9) can be expressed in terms of the Jacobi  $\theta_1$  function, since<sup>(6)</sup>

$$\frac{\theta'_1(v, q)}{\theta_1(v, q)} = \pi \cot \pi v + 4\pi \sum_{\ell=1}^{\infty} \frac{q^{2\ell}}{1 - q^{2\ell}} \sin 2\ell \pi v \quad (3.10)$$

Setting  $v = \varphi / (2\pi)$  and  $q = e^{-\epsilon_0}$  in (3.10), and using its derivative with respect to  $\varphi$  in (3.9) gives for the correlation function (3.6) the closed form

$$\beta \langle \sigma(\varphi) \sigma(0) \rangle^T = \frac{1}{(2\pi a)^2} \sinh^2 \epsilon_0 \left[ \frac{1}{\epsilon_0} + \frac{1}{\pi} \frac{d}{d\varphi} \frac{\theta'_1\left(\frac{\varphi}{2\pi}, e^{-\epsilon_0}\right)}{\theta_1\left(\frac{\varphi}{2\pi}, e^{-\epsilon_0}\right)} \right] \quad (3.11)$$

In the flat-space limit  $a \rightarrow \infty$ ,  $\tau_0 \rightarrow 0$ , for a fixed value of  $R = a\tau_0$ , it can be checked that (1.1) is recovered. More interestingly, in the opposite limit of a radius  $R$  large compared to the ‘‘curvature radius’’  $a$ , i.e., when  $\tau_0 \rightarrow \infty$  and  $\epsilon_0 \rightarrow 0$ , (3.11) takes a simpler form. Indeed, after a Jacobi imaginary transformation,<sup>(6,7)</sup> the  $\theta_1$  function can be expressed as the series

$$\theta_1\left(\frac{\varphi}{2\pi}, e^{-\epsilon_0}\right) = \left(\frac{\pi}{\epsilon_0}\right)^{1/2} \sum_{n=-\infty}^{n=\infty} (-1)^n \exp\left[-\frac{\pi^2}{\epsilon_0} \left(\frac{\varphi}{2\pi} - \frac{1}{2} + n\right)^2\right] \quad (3.12)$$

If  $0 < |\varphi| < \pi$ , in the small- $\epsilon_0$  limit ( $\epsilon_0 \sim 2e^{-\tau_0}$ ), the first two leading terms of the series (3.12) are  $n=0$  and  $n=1$ . When only these terms are kept, to first order in their ratio  $\exp(-\pi |\varphi| / \epsilon_0)$ , (3.11) becomes

$$\beta \langle \sigma(\varphi) \sigma(0) \rangle^T \sim -\frac{1}{2a^2} \exp\left(-\frac{e^{\tau_0} \pi |\varphi|}{2}\right) \quad (0 < |\varphi| < \pi) \quad (3.13)$$

#### 4. TWO-COMPONENT PLASMA ON A PSEUDOSPHERE

The total charge fluctuation and the surface charge correlation have been obtained under the assumption that macroscopic electrostatics is valid. In the large-disk limit  $R \gg a$ , these results (3.5) and (3.13) will now be checked on two solvable microscopic models. Such checks are welcome,

because a two-dimensional case when macroscopic electrostatics is not valid, unexpectedly at first sight, is known: the charge fluctuations in a short-circuited circular condenser.<sup>(3)</sup>

Macroscopic electrostatics uses the concept of surface charge density. Actually, in a microscopic model, this “surface density” will have some microscopic thickness, and for macroscopic electrostatics to be valid, it is necessary that this thickness be negligible compared to the macroscopic lengths. The microscopic model which will be used in the present section is the two-component plasma at a special temperature. Its microscopic scale is characterized by a fugacity  $\zeta$ , with the dimension (length)<sup>-2</sup>. In a disk of radius  $R$  on a pseudosphere with the “radius of curvature”  $a$ , there are two dimensionless parameters involving  $\zeta$ :  $\zeta a^2$  and  $\zeta R^2$ . Necessary conditions for macroscopic electrostatics to be valid is that both these parameters be large compared to 1. Here, for simplicity, the disk is assumed to be large ( $R \gg a$ ).

#### 4.1. Review of the General Formalism

The two-component plasma is a system of two species of particles, of charges  $\pm 1$ . At the special inverse temperature  $\beta = 2$ , the model is exactly solvable in different geometries, in particular on a pseudosphere.<sup>(9)</sup> For the sake of completeness, the method of solution is briefly revisited, in a form simpler than in the original papers, by a generalization of what has been done in the case of a one-component plasma.<sup>(8)</sup>

In terms of the coordinates  $(r, \varphi)$  (the Poincaré disk representation), the Coulomb interaction (2.11) between two unit point charges at  $\mathbf{r}_i$  and  $\mathbf{r}_j$  is

$$v(s) = -\ln \left| \frac{(z_i - z_j)/(2a)}{1 - \frac{z_i \bar{z}_j}{4a^2}} \right| \quad (4.1)$$

where  $z_j$  is the complex coordinate of particle  $j$  (and  $\bar{z}_j$  its complex conjugate):  $z_j = r_j e^{i\varphi_j}$ . The interaction (4.1) happens to be the Coulomb interaction in a flat disk of radius  $2a$  with ideal conductor walls at zero potential. Therefore, one can use the techniques which have been developed<sup>(10, 11)</sup> for dealing with ideal conductor walls.

When  $\beta = 2$ , the Boltzmann factor for  $N_+$  positive particles with vector coordinates  $\mathbf{r}_i^+$  and corresponding complex coordinates  $z_i^+$ ,  $1 \leq i \leq N_+$ , and  $N_-$  negative particles with vector coordinates  $\mathbf{r}_i^-$  and corresponding complex coordinates  $z_i^-$ ,  $1 \leq i \leq N_-$ , can be written as (with, for the time being,  $2a$  taken as the unit of length:  $2a = 1$ )

$$B_{N_+, N_-} = \frac{\prod_{1 \leq i < j \leq N_+} (z_i^+ - z_j^+) (\bar{z}_i^+ - \bar{z}_j^+) \prod_{1 \leq k < l \leq N_-} (z_k^- - z_l^-) (\bar{z}_k^- - \bar{z}_l^-)}{\prod_{m=1}^{N_+} \prod_{n=1}^{N_-} (1 - z_m^+ \bar{z}_n^-) (1 - \bar{z}_m^+ z_n^-)} \frac{1}{\prod_{1 \leq i < j \leq N_+} (1 - z_i^+ \bar{z}_j^+) (1 - \bar{z}_i^+ z_j^+) \prod_{1 \leq k < l \leq N_-} (1 - z_k^- \bar{z}_l^-)} \times (1 - \bar{z}_k^- z_l^-) \prod_{m=1}^{N_+} \prod_{n=1}^{N_-} (z_m^+ - z_n^-) (\bar{z}_m^+ - \bar{z}_n^-) \quad (4.2)$$

(in the cases  $N_+ = 0$  and  $N_- = 0$ , the corresponding products in (4.2) should be replaced by 1; in particular  $B(0, 0) = 1$ ). It is convenient to define

$$B'_{N_+, N_-} = \frac{B_{N_+, N_-}}{\prod_{m=1}^{N_+} \prod_{n=1}^{N_-} (1 - z_m^+ \bar{z}_m^-) (1 - \bar{z}_n^- z_n^+)} \quad (4.3)$$

$B'$  is the Boltzmann factor in a disk with ideal conductor walls at zero potential, including now in its denominator the contribution from the interaction of each particle with its own image.  $B'$  has the advantage that it can be written as a  $N \times N$  determinant ( $N = N_+ + N_-$  is the total number of particles), by using the Cauchy identity

$$\frac{\prod_{1 \leq i < j \leq N} (u_i - u_j)(v_i - v_j)}{\prod_{i=1}^N \prod_{j=1}^N (u_i - v_j)} = (-1)^{N(N-1)/2} \det \left( \frac{1}{u_i - v_j} \right)_{i,j=1, \dots, N} \quad (4.4)$$

Indeed, choosing

$$\begin{aligned} u_i &= z_i^+, & v_i &= 1/\bar{z}_i^+, & 1 \leq i \leq N_+ \\ u_{i+N_+} &= 1/\bar{z}_i^-, & v_{i+N_+} &= z_i^-, & 1 \leq i \leq N_- \end{aligned} \quad (4.5)$$

in (4.4) gives, after some simple manipulations and the reestablishment of an arbitrary value of  $2a$ , a  $N \times N$  determinant

$$B'_{N_+, N_-} = \det A_{ij} \quad (4.6)$$

where

$$\begin{aligned} A_{ij} &= \frac{4a^2}{4a^2 - z_i^+ \bar{z}_j^+} & \text{if } 1 \leq i, j \leq N_+ \\ A_{ij} &= \frac{2a}{z_i^+ - z_{j-N_+}^-} & \text{if } 1 \leq i \leq N_+ \text{ and } N_+ < j \leq N \\ A_{ij} &= \frac{2a}{\bar{z}_{i-N_+}^- - \bar{z}_j^+} & \text{if } N_+ < i \leq N \text{ and } 1 \leq j \leq N_+ \\ A_{ij} &= \frac{4a^2}{4a^2 - \bar{z}_{i-N_+}^- z_{j-N_+}^-} & \text{if } N_+ < i, j \leq N \end{aligned} \quad (4.7)$$

If the (perhaps different) fugacities are  $\zeta_+$  and  $\zeta_-$  for the positive and negative particles, respectively, the grand partition function can be written as

$$\Xi = \sum_{N_+=0}^{\infty} \sum_{N_-=0}^{\infty} \frac{1}{N_+!N_-!} \int \prod_{m=0}^{N_+} d^2\mathbf{r}_m^+ \zeta_+(r_m^+) \prod_{n=0}^{N_-} d^2\mathbf{r}_n^- \zeta_-(r_n^-) B'_{N_+,N_-} \quad (4.8)$$

Indeed, one of the factors  $[1-(r^2/4a^2)]^{-1}$  in the area element on the pseudosphere  $dS = [1-(r^2/4a^2)]^{-2} d^2\mathbf{r}$  has been incorporated into the definition (4.3) of  $B'$ , while the other factor  $[1-(r^2/4a^2)]^{-1}$  has been incorporated in the definition of position-dependent fugacities

$$\zeta_{\pm}(r) = \frac{\zeta_{\pm}}{1-\frac{r^2}{4a^2}} \quad (4.9)$$

Although the integrals in the grand partition function (4.8) diverge (as the separation between a positive particle and a negative one goes to zero), this grand partition function can be formally manipulated for providing finite correlation functions.

It will now be shown that the grand canonical partition function can be expressed as one determinant of an infinite matrix, continuous in coordinate space. First, one considers the functional integral

$$Z_0 = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \int \exp \left[ \int \sum_{s,s'=\pm} \bar{\psi}_s(\mathbf{r})(M^{-1})_{ss'}(\mathbf{r},\mathbf{r}') \psi_{s'}(\mathbf{r}') d^2\mathbf{r} d^2\mathbf{r}' \right] \quad (4.10)$$

The fields  $\psi$  and  $\bar{\psi}$  are two-component Grassmann variables (anticommuting variables). The components of  $\psi$  are called  $\psi_+$  and  $\psi_-$ , and similarly for  $\bar{\psi}$ . The covariance of the Gaussian measure in (4.10) is the inverse of the kernel  $M^{-1}$ , which is chosen such that

$$\langle \bar{\psi}_s(\mathbf{r}) \psi_{s'}(\mathbf{r}') \rangle = M_{ss'}(\mathbf{r},\mathbf{r}') \quad (4.11)$$

where  $\langle \dots \rangle$  denotes an average taken with the Gaussian weight of (4.10) and the  $2 \times 2$  matrix  $M$  is

$$M(\mathbf{r},\mathbf{r}') = \begin{pmatrix} M_{++} & M_{+-} \\ M_{-+} & M_{--} \end{pmatrix} = \begin{pmatrix} \frac{4a^2}{4a^2-zz'} & \frac{2a}{z-z'} \\ \frac{2a}{z-z'} & \frac{4a^2}{4a^2-zz'} \end{pmatrix} \quad (4.12)$$

Second, one considers the functional integral

$$Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \int \exp \left[ \int \sum_{s,s'=\pm} \bar{\psi}_s(\mathbf{r})(M^{-1})_{ss'}(\mathbf{r},\mathbf{r}') \psi_{s'}(\mathbf{r}') d^2\mathbf{r} d^2\mathbf{r}' \right. \\ \left. + \int [\zeta_+(r) \bar{\psi}_+(\mathbf{r}) \psi_+(\mathbf{r}) + \zeta_-(r) \bar{\psi}_-(\mathbf{r}) \psi_-(\mathbf{r})] d^2\mathbf{r} \right] \quad (4.13)$$

and one expands  $Z/Z_0$  in powers of  $\zeta_+(r)$  and  $\zeta_-(r)$  as

$$\begin{aligned} \frac{Z}{Z_0} &= \sum_{N_+=0}^{\infty} \sum_{N_-=0}^{\infty} \frac{1}{N_+!N_-!} \int \prod_{m=0}^{N_+} d^2\mathbf{r}_m^+ \zeta_+(r_m^+) \prod_{n=0}^{N_-} d^2\mathbf{r}_n^- \zeta_-(r_n^-) \\ &\times \langle \bar{\psi}_+(\mathbf{r}_1^+) \psi_+(\mathbf{r}_1^+) \cdots \bar{\psi}_+(\mathbf{r}_{N_+}^+) \psi_+(\mathbf{r}_{N_+}^+) \\ &\times \bar{\psi}_-(\mathbf{r}_1^-) \psi_-(\mathbf{r}_1^-) \cdots \bar{\psi}_-(\mathbf{r}_{N_-}^-) \psi_-(\mathbf{r}_{N_-}^-) \rangle \end{aligned} \quad (4.14)$$

Third, from the Wick theorem for anticommuting variables<sup>(14)</sup> and the covariance (4.11), it results that the average in (4.14) is equal to the determinant of the matrix  $A_{ij}$  defined in (4.7), i.e., to  $B'_{N_+, N_-}$  as given by (4.6). Therefore (4.14) is identical to (4.8). The grand partition function of the Coulomb gas is

$$\mathcal{E} = \frac{Z}{Z_0} \quad (4.15)$$

Finally,  $Z_0 = \det(M^{-1})$  and  $Z = \det(M^{-1} + \zeta)$ . In these determinants of infinite order, the matrix elements of  $M$  are labeled both by the discrete charge indices  $s, s'$  and the continuous indices  $\mathbf{r}, \mathbf{r}'$ . The infinite diagonal matrix  $\zeta$  is defined as

$$\zeta = \begin{pmatrix} \zeta_+(r) & 0 \\ 0 & \zeta_-(r) \end{pmatrix} \quad (4.16)$$

Therefore (4.15) does give the grand partition function as the determinant of an infinite matrix, continuous in coordinate space:

$$\mathcal{E} = \det[M(M^{-1} + \zeta)] = \det(1 + M\zeta) \quad (4.17)$$

For computing the densities and many-body densities, some definitions are needed. Let us define

$$\tilde{G} = (1 + M\zeta)^{-1} M / (4\pi a) \quad (4.18)$$

(the factor  $(4\pi a)$  is there just for keeping the same notation as in previous papers). Thus,  $\tilde{G}$  is the solution of  $(1 + M\zeta) \tilde{G} = M / (4\pi a)$  or, more explicitly,  $\tilde{G}$  obeys the integral equation

$$\tilde{G}(\mathbf{r}, \mathbf{r}') + \int M(\mathbf{r}, \mathbf{r}'') \zeta(r'') \tilde{G}(\mathbf{r}'', \mathbf{r}') d\mathbf{r}'' = \frac{1}{4\pi a} M(\mathbf{r}, \mathbf{r}') \quad (4.19)$$

where it should be remembered that  $G, M, \zeta$  are  $2 \times 2$  matrices. We also define

$$G(\mathbf{r}, \mathbf{r}') = \left(1 - \frac{r^2}{4a^2}\right)^{1/2} \tilde{G}(\mathbf{r}, \mathbf{r}') \left(1 - \frac{r'^2}{4a^2}\right)^{1/2} \quad (4.20)$$

On (4.12), one sees the symmetries  $M_{ss'}(\mathbf{r}, \mathbf{r}') = ss' \bar{M}_{s's}(\mathbf{r}', \mathbf{r})$ . By formally expanding the definition  $\tilde{G} = (1 + M\zeta)^{-1} M / (4\pi a)$  in powers of  $M\zeta$  one finds that  $\tilde{G}$  has the same symmetries, which also hold for  $G$ :

$$G_{ss'}(\mathbf{r}, \mathbf{r}') = ss' \bar{G}_{s's}(\mathbf{r}', \mathbf{r}) \quad (4.21)$$

The density  $n_s(\mathbf{r})$  of particles of sign  $s$  is given from the grand partition function by a functional derivation:

$$n_s(\mathbf{r}) = \left(1 - \frac{r^2}{4a^2}\right)^2 \zeta_s(r) \frac{\delta \ln \Xi}{\delta \zeta_s(r)} \quad (4.22)$$

where the factor  $[1 - (r^2/4a^2)]^2$  insures that  $n_s(\mathbf{r}) dS$  is the average number of particles in the area element  $dS = [1 - (r^2/4a^2)]^{-2} d^2\mathbf{r}$ . Since, from (4.17),  $\ln \Xi = \text{Tr}(1 + M\zeta)$ , (4.22), (4.18), and (4.20) give

$$n_s(r) = 4\pi \zeta_s a G_{ss}(\mathbf{r}, \mathbf{r}) \quad (4.23)$$

(actually, for point particles, this density is infinite, but it can be made finite by the introduction of a small hard core). The two-body density Ursell functions are given by

$$U_{ss'}(\mathbf{r}, \mathbf{r}') = \left(1 - \frac{r^2}{4a^2}\right)^2 \left(1 - \frac{r'^2}{4a^2}\right)^2 \zeta_s(r) \zeta_{s'}(r') \frac{\delta^2 \ln \Xi}{\delta \zeta_s(r) \delta \zeta_{s'}(r')} \quad (4.24)$$

Taking into account the symmetry relations (4.21) gives

$$U_{ss'}(\mathbf{r}, \mathbf{r}') = -ss'(4\pi \zeta_s a)(4\pi \zeta_{s'} a) |G_{ss'}(\mathbf{r}, \mathbf{r}')|^2 \quad (4.25)$$

From now on, we restrict ourselves to the case of equal fugacities  $\zeta_+ = \zeta_- = \zeta$ . In the Poincaré disk representation, the Coulomb fluid fills a disk of radius  $r_0$ . Thus,  $\zeta(r) = 0$  when  $r > r_0$ . The radius  $r_0$  is related to the geodesic radius  $R = a\tau_0$  by  $r_0 = 2a \tanh(\tau_0/2)$ . Without loss of generality we can choose the polar angle of  $\mathbf{r}'$  as  $\varphi' = 0$ .

The integral equation (4.19) can be transformed into a differential one, by the application of the operator  $\not\partial = \sigma_x \partial_x + \sigma_y \partial_y$ , where  $\sigma_x$  and  $\sigma_y$  are Pauli matrices:

$$[\not\partial + 4\pi a \zeta(r)] \tilde{G}(\mathbf{r}, \mathbf{r}') = \delta_{\text{flat}}^{(2)}(\mathbf{r} - \mathbf{r}') \quad (4.26)$$

where  $\delta_{\text{flat}}^{(2)}$  is the Dirac distribution in the plane. This differential equation is to be supplemented by the condition that  $\tilde{G}(\mathbf{r}, \mathbf{r}')$  be continuous at the disk boundary  $r_0$  and by the boundary condition, seen on (4.19), that when  $r = 2a$ ,  $\tilde{G}_{-+} = e^{i\varphi} \tilde{G}_{++}$  (and a similar boundary relation between  $\tilde{G}_{+-}$  and  $\tilde{G}_{--}$ ).

In the case of an infinite system, eq.(4.26) could be solved,<sup>(9)</sup> for  $\mathbf{r}' = 0$ , in terms of hypergeometric functions. In the present case of a finite disk, i.e., when  $r_0 < 2a$ , an exact explicit solution of (4.26) for an arbitrary fugacity seems difficult to obtain. Fortunately, here we only need the large-fugacity limit, in which case there are important simplifications.

## 4.2. Large Fugacity

For a flat system, the Coulomb interaction (2.11) becomes  $-\ln(s/2a)$  where  $2a$  is an irrelevant length scale which only contributes an additive constant to the potential. In the flat case,<sup>(12)</sup> the rescaled fugacity  $m = 4\pi\zeta a$  (which has the dimension of an inverse length) was introduced, and the correlation length was found to be of the order of  $m^{-1}$ . In the present case of a system on a pseudosphere, it is convenient to keep the same definition of  $m$ .

On a pseudosphere, in the large fugacity limit  $4\pi\zeta a^2 = ma \gg 1$ , if we are interested in a solution of (4.26) only in a region of size  $m^{-1}$ , the curvature can be neglected and the flat system solutions can be used, with appropriately rescaled coordinates. In particular, if both  $r$  and  $r'$  are sufficiently close to  $r_0$ , the variation of  $\zeta(r)$  can be neglected: in (4.20) and (4.26),  $\zeta(r)$  can be replaced by the constant  $\zeta(r_0)$ . Here, we assume the disk to be large, and therefore  $\zeta(r_0) \sim \zeta e^{\tau_0}/4$ . Furthermore (4.20) becomes

$$\tilde{G}(\mathbf{r}, \mathbf{r}') = \frac{e^{\tau_0}}{4} G(\mathbf{r}, \mathbf{r}') \quad (4.27)$$

In terms of the rescaled variables  $(e^{\tau_0}/4) \mathbf{r} = \mathbf{t}$  and  $(e^{\tau_0}/4) \mathbf{r}' = \mathbf{t}'$ , (4.27) and (4.26) do give the flat system equation<sup>(12)</sup>

$$[\not\partial_{\mathbf{t}} + m] G(\mathbf{r}, \mathbf{r}') = \delta_{\text{flat}}^{(2)}(\mathbf{t} - \mathbf{t}') \quad (4.28)$$

In an infinite system, the  $(++)$  and  $(-+)$  elements of the solution of (4.28) would be  $G_{++}(\mathbf{r}, \mathbf{r}') = (m/2\pi) K_0(m|\mathbf{t}-\mathbf{t}'|)$  and  $G_{-+}(\mathbf{r}, \mathbf{r}') = (m/2\pi) e^{i\psi} K_1(m|\mathbf{t}-\mathbf{t}'|)$  where  $K_0$  and  $K_1$  are modified Bessel functions and  $\psi$  is the argument of  $te^{i\varphi}-t'$ . In the present case of a finite system in a disk, a "reflected wave" must be added. As a Fourier series in  $\varphi$ ,  $G_{++}$  is of the form

$$G_{++}(\mathbf{r}, \mathbf{r}') = \frac{m}{2\pi} \sum_{\ell=-\infty}^{\infty} [I_{\ell}(mt') K_{\ell}(mt) + a_{\ell} I_{\ell}(mt') I_{\ell}(mt)] e^{i\ell\varphi} \quad (t' < t < t_0) \quad (4.29)$$

where  $t_0 = (e^{\tau_0}/4) r_0$ . The first term in the sum corresponds to an expansion<sup>(6)</sup> of  $K_0(m|\mathbf{t}-\mathbf{t}'|)$ . The second term corresponds to the "reflected wave." The coefficients  $a_{\ell}$  are to be determined by the continuity and boundary conditions. Similarly,

$$G_{-+}(\mathbf{r}, \mathbf{r}') = \frac{m}{2\pi} \sum_{\ell=-\infty}^{\infty} [I_{\ell}(mt') K_{\ell+1}(mt) - a_{\ell} I_{\ell}(mt') I_{\ell+1}(mt)] e^{i(\ell+1)\varphi} \quad (t' < t < t_0) \quad (4.30)$$

The corresponding elements of  $\tilde{G}$  are given by (4.27). There are similar expansions in the case  $t < t' < t_0$ .

The coefficients  $a_{\ell}$  will now be determined. When  $t' < t_0 < t$ , (4.26) reduces to  $\not\partial \tilde{G}(\mathbf{r}, \mathbf{r}') = 0$  which means that  $\tilde{G}_{++}$  is an analytic function of  $z$  and  $\tilde{G}_{-+}$  an antianalytic function. Therefore, as a function of  $z = re^{i\varphi}$ ,  $\tilde{G}_{++}$  is of the form

$$\tilde{G}_{++} = \sum_{\ell=-\infty}^{\infty} b_{\ell} r^{\ell} e^{i\ell\varphi} \quad (t' < t_0 < t) \quad (4.31)$$

Taking into account the boundary condition  $\tilde{G}_{-+} = e^{i\varphi} \tilde{G}_{++}$  at  $r = 2a$  gives

$$\tilde{G}_{-+} = \sum_{\ell=-\infty}^{\infty} b_{\ell} (2a)^{2\ell+1} \frac{e^{i(\ell+1)\varphi}}{r^{\ell+1}} \quad (t' < t_0 < t) \quad (4.32)$$

For a large disk,  $r_0 = 2a \tanh(\tau_0/2) \sim 2a \exp(-2e^{-\tau_0})$ . The continuity of  $G_{++}$  and  $G_{-+}$  at  $r = r_0$  determines the coefficients  $a_{\ell}$  and  $b_{\ell}$ . One finds

$$a_{\ell} = \frac{\exp[-(2\ell+1)2e^{-\tau_0}] K_{\ell+1}(mt_0) - K_{\ell}(mt_0)}{\exp[-(2\ell+1)2e^{-\tau_0}] I_{\ell+1}(mt_0) + I_{\ell}(mt_0)} \quad (4.33)$$

In the present large-fugacity limit, the Bessel functions in (4.29), (4.30), and (4.33) can be replaced by their asymptotic forms  $I_\ell(x) \sim (2\pi x)^{-1/2} e^x$  and  $K_\ell(x) \sim (\pi/2x)^{1/2} e^{-x}$ . Then, whatever the relative order of  $t$  and  $t'$  might be,

$$G_{++}(\mathbf{r}, \mathbf{r}') \sim \frac{1}{4\pi t_0} \sum_{\ell=-\infty}^{\infty} \{e^{-m|t-t'|} - e^{-m(2t_0-t-t')}\} \times \tanh[(2\ell+1)e^{-\tau_0}] e^{i\ell\varphi} \quad (t, t' < r_0) \quad (4.34)$$

and

$$G_{-+}(\mathbf{r}, \mathbf{r}') \sim \frac{1}{4\pi t_0} \sum_{\ell=-\infty}^{\infty} \{e^{-m|t-t'|} + e^{-m(2t_0-t-t')}\} \times \tanh[(2\ell+1)e^{-\tau_0}] e^{i(\ell+1)\varphi} \quad (t, t' < r_0) \quad (4.35)$$

It should be recalled that these expressions are valid only near the boundary circle.

### 4.3. Charge Fluctuations

For the present model, by symmetry  $\langle Q \rangle = 0$  and the variance of the total charge is

$$\langle Q^2 \rangle = \int_{r, r' < r_0} \rho^{(2)}(\mathbf{r}, \mathbf{r}') dS dS' + \int_{r < r_0} n(r) dS \quad (4.36)$$

where  $\rho^{(2)}(\mathbf{r}, \mathbf{r}')$  is the two-body charge density,  $n(r)$  the total particle density, and  $dS$  an area element on the pseudosphere. In the bulk, perfect screening is expected, and furthermore  $\rho^{(2)}(\mathbf{r}, \mathbf{r}')$  has a range in the geodesic distance between  $\mathbf{r}$  and  $\mathbf{r}'$  of the order of  $m^{-1}$  only. Therefore, the only contributions to (4.36) come from  $r$  and  $r'$  close to  $r_0$ . When both  $r$  and  $r'$  are in the bulk (i.e., smaller enough than  $r_0$ ),  $\rho^{(2)}$  becomes a function  $\rho_{\text{bulk}}^{(2)}$  and it is convenient to define a surface part by  $\rho^{(2)}(\mathbf{r}, \mathbf{r}') = \rho_{\text{bulk}}^{(2)}(\mathbf{r}, \mathbf{r}') + \rho_{\text{surf}}^{(2)}(\mathbf{r}, \mathbf{r}')$ . Similarly, the density can be decomposed as  $n(r) = n_{\text{bulk}} + n_{\text{surf}}(r)$ . Assuming that perfect screening occurs in the bulk means

$$\int \rho_{\text{bulk}}^{(2)}(\mathbf{r}, \mathbf{r}') dS + n_{\text{bulk}} = 0 \quad (4.37)$$

where the integral extends on the whole pseudosphere. Using (4.37) allows to rewrite (4.36) as

$$\langle Q^2 \rangle = - \int_{r' < r_0 < r} \rho_{\text{bulk}}^{(2)}(\mathbf{r}, \mathbf{r}') dS dS' + \int_{r, r' < r_0} \rho_{\text{surf}}^{(2)}(\mathbf{r}, \mathbf{r}') + \int_{r < r_0} n_{\text{surf}}(r) dS \quad (4.38)$$

Because of the symmetry between positive and negative particles,  $\rho^{(2)}(\mathbf{r}, \mathbf{r}') = 2[U_{++}(\mathbf{r}, \mathbf{r}') - U_{--}(\mathbf{r}, \mathbf{r}')] + 2n_+(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')$  with the Ursell functions given by (4.25) and  $n(r) = 2n_+(r)$  with  $n_+$  given by (4.23). Thus, (4.36) becomes

$$\begin{aligned} \langle Q^2 \rangle = & -2m^2 \int_{r, r' < r_0} [ |G_{++}(\mathbf{r}, \mathbf{r}')|^2 + |G_{--}(\mathbf{r}, \mathbf{r}')|^2 ] dS dS' \\ & + 2m \int_{r < r_0} G_{++}(\mathbf{r}, \mathbf{r}) dS \end{aligned} \quad (4.39)$$

Using the Fourier series (4.34) and (4.35) in (4.39), and taking into account that in  $|G_{ss'}|^2$  only the term independent of  $\varphi$  survives the angular integration, gives

$$\begin{aligned} \langle Q^2 \rangle = & - \frac{4m^2}{(4\pi t_0)^2} \int_{t, t' < t_0} \sum_{\ell = -\infty}^{\infty} \{ e^{-2m|t-t'|} + e^{-2m(2t_0-t-t')} \\ & \times \tanh^2[(2\ell+1)e^{-\tau_0}] \} dS dS' \\ & + \frac{2m}{4\pi t_0} \int_{t < t_0} \sum_{\ell = -\infty}^{\infty} \{ 1 - e^{-2m(t_0-t)} \tanh[(2\ell+1)e^{-\tau_0}] \} dS \end{aligned} \quad (4.40)$$

The first term in each sum corresponds to  $\rho_{\text{bulk}}^{(2)}$  and  $n_{\text{bulk}}$ , respectively, and therefore the second term corresponds to  $\rho_{\text{surf}}^{(2)}$  and  $n_{\text{surf}}$ , respectively. Using (4.38) rather than (4.36) gives instead of (4.40)

$$\begin{aligned} \langle Q^2 \rangle = & - \frac{4m^2}{(4\pi t_0)^2} \sum_{\ell = -\infty}^{\infty} \left\{ - \int_{t' < t_0 < t} e^{-2m(t-t')} dS dS' \right. \\ & \left. + \int_{t, t' < t_0} e^{-2m(2t_0-t-t')} \tanh^2[(2\ell+1)e^{-\tau_0}] dS dS' \right\} \\ & - \frac{2m}{4\pi t_0} \sum_{\ell = -\infty}^{\infty} \int_{t < t_0} e^{-2m(t_0-t)} \tanh[(2\ell+1)e^{-\tau_0}] dS \end{aligned} \quad (4.41)$$

The integrands are indeed localized near the boundary circle. Using  $dS \sim t_0 dt d\varphi$  and performing the integrations gives

$$\langle Q^2 \rangle = \frac{1}{4} \sum_{\ell=-\infty}^{\infty} \{1 - \tanh^2[(2\ell+1)e^{-\tau_0}]\} - \frac{1}{2} \sum_{\ell=-\infty}^{\infty} \tanh[(2\ell+1)e^{-\tau_0}] \quad (4.42)$$

Finally, as  $\tau_0$  becomes large, the sums can be expressed as integrals on the variable  $x = (2\ell+1)e^{-\tau_0}$ . Since  $\tanh x$  is an odd function, the second sum can be considered as vanishing (actually, there are convergence factors at  $\ell \rightarrow \pm\infty$ , which have been omitted when the Bessel functions have been replaced by their asymptotic forms at fixed  $\ell$ ). One is left with

$$\langle Q^2 \rangle = \frac{e^{\tau_0}}{8} \int_{-\infty}^{\infty} (1 - \tanh^2 x) dx = \frac{e^{\tau_0}}{4} \quad (4.43)$$

in agreement with the macroscopic result (3.5), since here  $\beta = 2$ .

#### 4.4. Surface Charge Correlations

The first term in (4.34) or (4.35) corresponds to the bulk contribution  $(m/2\pi) K_0(m|t-t'|)$  or  $(m/2\pi) e^{i\psi} K_1(m|t-t'|)$ , respectively. The range  $m^{-1}$  of these bulk contributions goes to zero in the large-fugacity limit. Thus, for  $t \neq t'$ , only the second term survives. Let us assume that the relevant values of  $|\varphi|$  are small. Since  $e^{-\tau_0}$  is small for a large disk, after  $(2\ell+1)e^{-\tau_0}$  has been replaced by  $2\ell e^{-\tau_0} \sim \epsilon_0 \ell$ , the sum on  $\ell$  can be expressed in terms of an integral:

$$\sum_{\ell=-\infty}^{\infty} \tanh(\epsilon_0 \ell) e^{i\ell\varphi} \sim i \int_{-\infty}^{\infty} \tanh(\epsilon_0 \ell) \sin(\ell\varphi) d\ell \quad (4.44)$$

Here too, there are convergence factors as  $\ell \rightarrow \pm\infty$ , which have been omitted when the Bessel functions were replaced by their asymptotic forms at fixed  $\ell$ . These convergence factors can be taken into account by replacing  $\tanh(\epsilon_0 \ell)$  by  $\sinh(\epsilon_0 \ell)/\cosh(\epsilon \ell)$  (with  $\epsilon > \epsilon_0$ ), performing the integral which is a tabulated one,<sup>(13)</sup> and taking the limit  $\epsilon \rightarrow \epsilon_0$  afterwards. The result defines the integral as

$$i \int_{-\infty}^{\infty} \tanh(\epsilon_0 \ell) \sin(\ell\varphi) d\ell = \frac{i\pi}{\epsilon_0 \sinh \frac{\pi\varphi}{2\epsilon_0}} \quad (4.45)$$

The range in  $\varphi$  of this function is indeed of the order of  $\epsilon_0$ , an a posteriori justification of the above assumption that  $|\varphi|$  is small. Using (4.44) and (4.45), with  $|\sinh(\pi\varphi/2\epsilon_0)|$  replaced by  $(1/2) \exp \pi |\varphi|/2\epsilon_0$ , in (4.34) and (4.35) gives for the two-body charge density near the disk boundary

$$\begin{aligned} \rho^{(2)}(\mathbf{r}, \mathbf{r}') &= -2m^2[|G_{++}(\mathbf{r}, \mathbf{r}')|^2 + |G_{-+}(\mathbf{r}, \mathbf{r}')|^2] \\ &= -\frac{m^2}{a^2} e^{-2m(2t_0-t-t')} \exp\left(-\frac{e^{\tau_0}\pi|\varphi|}{2}\right) \end{aligned} \quad (4.46)$$

where  $\epsilon_0 \sim 2e^{-\tau_0}$  and  $t_0 \sim ae^{\tau_0}/2$  have been used. This two-body charge density is indeed localized near the disk boundary. The surface charge correlation is defined as

$$\langle \sigma(\varphi) \sigma(0) \rangle = \int_{-\infty}^{t_0} dt \int_{-\infty}^{t_0} dt' \rho^{(2)}(\mathbf{r}, \mathbf{r}') \quad (4.47)$$

Using (4.46) in (4.47) and performing the integrals reproduces the macroscopic result (3.13), since here  $\beta = 2$  and  $\langle \sigma(\varphi) \rangle = 0$ .

## 5. ONE-COMPONENT PLASMA ON A PSEUDOSPHERE

The macroscopic results (3.5) and (3.13) will now be checked on another solvable model, the one-component plasma. This is a system of one species of particles, of charges  $+1$ , embedded in a uniform background carrying the negative charge density  $-n_b$ . At the inverse temperature  $\beta = 2$ , the system is exactly solvable in a variety of geometries, in particular for a large disk of radius  $R = a\tau_0$  on a pseudosphere.<sup>(8)</sup> A grand canonical ensemble is used. For the grand partition function to be convergent, it is necessary to define it with a fixed value<sup>(2)</sup> of the background charge density  $-n_b$ ; the fugacity  $\zeta$  controls the number of particles. Thus, in general, the system is not globally neutral, except for a particular choice of the fugacity.

In the bulk the properties of the system are controlled by the background: the particle number density away from the boundary is  $n_b$ . However, near the boundary, the particle density differs from  $n_b$ , and, since on a pseudosphere the neighborhood of the boundary has an area of the same order of magnitude as the whole area, this neighborhood gives an important contribution to the total number of particles and thus to the total charge of the system.

The macroscopic results (3.5) and (3.13) are expected to be valid only when the microscopic thickness of the surface charge density goes to zero. How to reach this regime in the most general way by varying both parameters  $n_b$  and  $\zeta$  has not been clear to us. Here we content ourselves by

considering the limit  $\zeta \rightarrow \infty$  for a fixed value of  $n_b$ . In this limit, the total charge of the system is expected to become infinite and to be carried by an infinitely thin surface layer.

### 5.1. Summary of Previous Results<sup>(8)</sup>

Again, for two points in the Poincaré disk at  $\mathbf{r} = (r, \varphi)$  and  $\mathbf{r}' = (r', 0)$ , one defines an auxiliary quantity  $G(\mathbf{r}, \mathbf{r}')$ , which now is just a scalar (instead of a  $2 \times 2$  matrix). In the case of a large disk of radius  $R = a\tau_0$ ,  $\tau_0 \rightarrow \infty$ ,

$$G(\mathbf{r}, \mathbf{r}') = \zeta \left( \frac{e^{\tau_0+1}}{4} \right)^\alpha \left( 1 - \frac{r^2}{4a^2} \right)^{(\alpha+1)/2} \left( 1 - \frac{r'^2}{4a^2} \right)^{(\alpha+1)/2} \times \sum_{\ell=0}^{\infty} \left( \frac{rr'}{4a^2} \right)^\ell \frac{e^{i\ell\varphi}}{1 + 4\pi a^2 \zeta e^\alpha \frac{\Gamma(\alpha, x)}{x^\alpha}} \quad (5.1)$$

where  $\alpha = 4\pi n_b a^2$ ,  $x = 4\ell e^{-\tau}$ , and  $\Gamma(\alpha, x)$  is the incomplete Gamma function

$$\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t} dt \quad (5.2)$$

The particle number density was found to be

$$n(r) = G(\mathbf{r}, \mathbf{r}) \quad (5.3)$$

By a similar calculation, one finds for the two-body density Ursell function

$$U(\mathbf{r}, \mathbf{r}') = -|G(\mathbf{r}, \mathbf{r}')|^2 \quad (5.4)$$

In (5.1),  $r/(2a) = \tanh(\tau/2)$ . Only the case  $\tau$  large ( $r$  close to the boundary of the disk) will be needed. Then  $1 - [r/(2a)]^2 \sim 4e^{-\tau}$  and  $[r/2a]^\ell \sim \exp(-2\ell e^{-\tau})$ . Let us assume that the relevant values of  $|\varphi|$  are small compared to 1. Then the sum on  $\ell$  can be replaced by an integral on  $x = 4\ell e^{-\tau}$ . This gives for the density as a function of the distance (in units of  $a$ ) from the boundary  $\lambda = \tau_0 - \tau$

$$n(\lambda) = G(\mathbf{r}, \mathbf{r}) = \zeta e^\alpha e^{(\alpha+1)\lambda} \int_0^\infty \frac{e^{-xe^\lambda} dx}{1 + 4\pi a^2 \zeta e^\alpha \frac{\Gamma(\alpha, x)}{x^\alpha}} \quad (5.5)$$

Integrating  $n(\lambda)$  gives the average number of particles

$$\langle N \rangle = \mathcal{A} \zeta e^\alpha \int_0^\infty \frac{\Gamma(\alpha, x) dx}{x^\alpha + 4\pi a^2 \zeta e^\alpha \Gamma(\alpha, x)} \quad (5.6)$$

## 5.2. Charge Fluctuations

For the one-component plasma the charge fluctuations are identical (for particles of charge +1) to the particle number fluctuations  $\langle Q^2 \rangle^T = \langle N^2 \rangle^T$ , since the background charge does not fluctuate. The charge fluctuations can be obtained either by integrating the correlation function (see Eq. (5.4)) or by using the thermodynamic relation

$$\langle N^2 \rangle^T = \zeta \frac{\partial \langle N \rangle}{\partial \zeta} \quad (5.7)$$

This gives for a large disk

$$\langle Q^2 \rangle^T = \frac{e^{\tau_0}}{4} \int_0^\infty \frac{g x^\alpha \Gamma(\alpha, x)}{(x^\alpha + g \Gamma(\alpha, x))^2} dx \quad (5.8)$$

where we have defined the dimensionless parameter  $g = 4\pi a^2 \zeta e^\alpha$ . For any finite value of  $\zeta$  and  $n_b$  the integral in the last equation is different from 1, thus the predictions of macroscopic electrostatics are not satisfied. This is indeed expected since in general we are out of the validity domain of macroscopic electrostatics. As explained above, we expect the results from macroscopic electrostatics to be valid only if the thickness of the layer of charge near the boundary is negligible compared to the macroscopic lengths: the radius of the disk  $R$  and the radius of curvature  $a$ . For the two-component plasma the thickness  $T$  of this layer is of order of the inverse of the fugacity  $m^{-1} = (4\pi a \zeta)^{-1}$ . For the one-component plasma we shall show that the situation is somehow different.

Thus, before proceeding to study the charge fluctuations in the large-fugacity limit, let us study first how the thickness  $T$  of the charged layer near the boundary depends on  $g$  in this limit, since the situation is not as simple as it is for the two-component plasma. We will show that indeed  $T$  vanishes when  $g \rightarrow \infty$ .

For simplicity let us consider the case when  $\alpha = 1$ . In units of  $a$  the thickness of the charged layer can be defined as the first moment of the density profile properly normalized

$$T = \frac{\int_0^\infty n(\lambda) \lambda e^{-\lambda} d\lambda}{\int_0^\infty n(\lambda) e^{-\lambda} d\lambda} = \frac{\int_0^\infty n(\lambda) \lambda e^{-\lambda} d\lambda}{n} \quad (5.9)$$

where  $n = \langle N \rangle / \mathcal{A}$  is the average particle density. The  $e^{-\lambda}$  factor comes from the area element  $dS = 2\pi a^2 \sinh \tau \, d\tau$  near the boundary. For  $\alpha = 1$  the density profile (5.5) becomes

$$n(\lambda) = n_b e^{2\lambda} \int_0^\infty \frac{g x e^{-x e^\lambda}}{x + g e^{-x}} dx \quad (5.10)$$

and the average density is given by

$$\frac{n}{n_b} = \int_0^\infty \frac{g e^{-x}}{x + g e^{-x}} dx \quad (5.11)$$

Let us define  $x_m$  as the principal solution of  $g = x_m e^{x_m}$ ; incidentally, the function  $x_m(g)$  is the Lambert function, which has many applications.<sup>(15)</sup> Now we write

$$\frac{n}{n_b} = \left( \int_0^{x_m} + \int_{x_m}^\infty \right) \frac{dx}{1 + \frac{x}{x_m} e^{x-x_m}} \quad (5.12)$$

In the first integral ( $x < x_m$ ) the second term in the denominator is negligible when  $g \rightarrow \infty$  and then the integrand is 1. After the change of variable  $x \rightarrow x_m + x$ , the second integral ( $x > x_m$ ) is easily shown to have the limit  $\ln 2$ . This gives in the limit  $g \rightarrow \infty$

$$\frac{n}{n_b} \sim x_m + \ln 2 \sim x_m \quad (5.13)$$

On the other hand, replacing expression (5.10) for the one-body density  $n(\lambda)$  into the first moment of the density and performing the integral over  $\lambda$  gives

$$\int_0^\infty n(\lambda) \lambda e^{-\lambda} d\lambda = n_b \int_0^\infty \frac{g \Gamma(0, x)}{x + g e^{-x}} dx \quad (5.14)$$

Again it is convenient to cut the integral in two intervals for  $x < x_m$  and  $x > x_m$ . As in the case for  $n$  when  $g \rightarrow \infty$  the second integral is negligible compared to the first. Then

$$\begin{aligned} \int_0^\infty n(\lambda) \lambda e^{-\lambda} d\lambda &\sim n_b \int_0^{x_m} \frac{e^x \Gamma(0, x)}{1 + \frac{x}{x_m} e^{x-x_m}} dx \\ &\sim n_b \int_0^{x_m} e^x \Gamma(0, x) dx = \gamma + e^{x_m} \Gamma(0, x_m) + \ln x_m \end{aligned} \quad (5.15)$$

where  $\gamma \simeq 0.577$  is the Euler constant. Since, when  $x_m \rightarrow \infty$ ,  $\Gamma(0, x_m) \sim e^{-x_m}/x_m$ , the dominant term for the first moment of the density is the third term in the preceding equation

$$\int_0^\infty n(\lambda) \lambda e^{-\lambda} d\lambda \sim n_b \ln x_m \quad (5.16)$$

Finally the thickness of the layer of charge near the boundary behaves as

$$T \sim \frac{1}{x_m} \ln x_m \rightarrow 0 \quad (5.17)$$

when  $x_m \rightarrow \infty$ . Remembering that  $x_m e^{x_m} = g = 4\pi a^2 \zeta e^\alpha$  one can notice that the dependence of the thickness on the fugacity is not trivial. It vanishes when  $\zeta \rightarrow \infty$  but very slowly contrarily to the case of the two-component plasma where  $T \sim \zeta^{-1}$ .

Now we will proceed to prove that, in the limit  $g \gg 1$ , the charge variance (5.8) is equal to the prediction of macroscopic electrostatics (3.5). To be as general as possible we consider again any value of  $\alpha$ . One can easily prove that the integrand in Eq. (5.8) is maximum when  $x = x_m$  where  $x_m$  is now given by  $g = x_m^\alpha / \Gamma(\alpha, x_m)$  for any value of  $\alpha$ . Doing the change of variable  $x \rightarrow x - x_m$  in the integral (5.8) and replacing  $g$  by its expression in term of  $x_m$  gives

$$\langle Q^2 \rangle^T = \frac{e^{\tau_0}}{4} \int_{-x_m}^\infty \frac{\left(1 + \frac{x}{x_m}\right)^\alpha \frac{\Gamma(\alpha, x+x_m)}{\Gamma(\alpha, x_m)}}{\left[\left(1 + \frac{x}{x_m}\right)^\alpha + \frac{\Gamma(\alpha, x+x_m)}{\Gamma(\alpha, x_m)}\right]^2} dx \quad (5.18)$$

When  $g \rightarrow \infty$  we have  $x_m \rightarrow \infty$ ,  $\Gamma(\alpha, x_m) \sim x_m^{\alpha-1} e^{-x_m}$  and

$$\frac{\Gamma(\alpha, x+x_m)}{\Gamma(\alpha, x_m)} \sim \left(1 + \frac{x}{x_m}\right)^{\alpha-1} e^{-x} \quad (5.19)$$

Then

$$\langle Q^2 \rangle^T \sim \frac{e^{\tau_0}}{4} \int_{-x_m}^\infty \frac{\left(1 + \frac{x}{x_m}\right)^\alpha \left(1 + \frac{x}{x_m}\right)^{\alpha-1} e^{-x}}{\left[\left(1 + \frac{x}{x_m}\right)^\alpha + \left(1 + \frac{x}{x_m}\right)^{\alpha-1} e^{-x}\right]^2} dx \quad (5.20)$$

We notice that for large values of  $|x|$  the integrand vanishes exponentially as  $e^{-|x|}$ . Then, since  $x_m \rightarrow \infty$ , we can replace the lower limit of the integral by  $-\infty$  and neglect  $x/x_m$  in front of 1. This gives

$$\langle Q^2 \rangle^T \sim \frac{e^{\tau_0}}{4} \int_{-\infty}^\infty \frac{e^{-x} dx}{(1 + e^{-x})^2} = \frac{e^{\tau_0}}{4} \quad (5.21)$$

Since  $\beta = 2$ , this is the expected result (3.5) obtained from macroscopic electrostatics considerations.

### 5.3. Surface Charge Correlations

Under the assumption that the relevant values of  $|\varphi|$  are small compared to 1, the same manipulations as the ones leading to (5.5) give

$$G(\mathbf{r}, \mathbf{r}') = \zeta e^\alpha e^{(\alpha+1)\frac{\lambda+\lambda'}{2}} \int_0^\infty dx \frac{e^{-x\frac{e^\lambda+e^{\lambda'}}{2}} e^{ix\frac{e^{\tau_0}}{4}\varphi}}{1+g\frac{\Gamma(\alpha,x)}{x^\alpha}} \quad (5.22)$$

where  $g = 4\pi a^2 \zeta e^\alpha$ ,  $\lambda = \tau_0 - \tau$ , and  $\lambda' = \tau_0 - \tau'$ .

Let us consider the case  $\varphi > 0$ . We are interested in the behavior of (5.22) as  $g \rightarrow \infty$ . This behavior will be shown to be determined by the pole of the integrand closest to the real axis in the upper half-plane. Let us assume that this pole has a large real part. Then, at this pole,  $\Gamma(\alpha, x)$  behaves as the first term  $x^{\alpha-1}e^{-x}$  of its asymptotic expansion<sup>(13)</sup> and the denominator in the integrand becomes  $1+ge^{-x}/x$ . Let us look for a zero of this denominator at  $x = i\pi + x_m$ . The equation for  $x_m$  is  $(i\pi + x_m)e^{x_m} = g$ , which becomes, in the large- $g$  limit,  $x_me^{x_m} = g$ . Thus,  $x_m$  is large and real and the assumption that the pole has a large real part is a posteriori verified. The same reasoning gives other poles at  $x = (2n+1)i\pi + x_m$ ,  $n \in \mathbb{Z}$ . The residue of the pole at  $x = i\pi + x_m$  is easily found to have a modulus behaving as  $\exp[-x_m(e^\lambda + e^{\lambda'})/2] \exp[-\pi e^{\tau_0}\varphi/4]$ .

Let  $I$  be the integral in (5.22).  $I$  is a part of an integral in the complex plane, along a contour  $C$ :  $C$  follows the positive real axis from 0 to  $+\infty$ , a large quarter of circle from  $+\infty$  to  $+i\infty$ , and comes back to the origin along the imaginary axis from  $+i\infty$  to 0. The contribution from the large quarter of circle at infinity is easily seen to vanish, and therefore the contour integral is  $I - I'$  with  $I'$  the integral with  $x = iy$  pure imaginary varying from 0 to  $i\infty$ :

$$I' = i \int_0^\infty dy \frac{e^{-iy\frac{e^\lambda+e^{\lambda'}}{2}} e^{-y\frac{e^{\tau_0}}{4}\varphi}}{1+g\frac{\Gamma(\alpha, iy)}{iy}} \quad (5.23)$$

For large  $g$ ,  $I'$  is easily seen to be of order  $1/g = e^{-x_m}/x_m$ .

The theorem of residues says that  $I = I' + 2\pi i \times \text{sum of the residues of the poles inside } C$ .  $I'$  is negligible (by a factor  $1/x_m$ ) compared to the residue of the pole at  $i\pi + x_m$ . The residues of the other poles have a factor

$\exp[-(2n+1)\pi e^{\tau_0}\varphi/4]$ ,  $n > 1$  which makes them also negligible. A similar reasoning holds in the case  $\varphi < 0$ , and finally

$$|G(\mathbf{r}, \mathbf{r}')| \sim 2\pi\zeta e^\alpha e^{(\alpha+1)\frac{\lambda+\lambda'}{2}} \exp\left[-x_m \frac{e^\lambda + e^{\lambda'}}{2}\right] \exp\left(-\pi \frac{e^{\tau_0}}{4} |\varphi|\right) \quad (5.24)$$

This form of  $|G|$  a posteriori justifies the assumption that the relevant values of  $\varphi$  are small compared to 1. Furthermore, in view of the fast decrease of  $|G|$  as a function of  $\lambda$  or  $\lambda'$  with a characteristic length  $1/x_m$  (compare with the thickness of  $n(r)$  which was found to be  $(1/x_m) \ln x_m$ ), a simpler form is

$$|G(\mathbf{r}, \mathbf{r}')| \sim 2\pi\zeta e^\alpha \exp\left[-x_m \left(1 + \frac{\lambda}{2} + \frac{\lambda'}{2}\right)\right] \exp\left(-\pi \frac{e^{\tau_0}}{4} |\varphi|\right) \quad (5.25)$$

The Ursell function is obtained by using (5.25) in (5.4):

$$U(\mathbf{r}, \mathbf{r}') \sim -(2\pi\zeta e^\alpha)^2 \exp[-x_m(\lambda + \lambda')] \exp(-2x_m) \exp\left(-\pi \frac{e^{\tau_0}}{2} |\varphi|\right) \quad (5.26)$$

The surface charge correlation is defined as

$$\langle \sigma(\varphi) \sigma(0) \rangle^T = a^2 \int_0^\infty d\lambda \int_0^\infty d\lambda' U(\mathbf{r}, \mathbf{r}') \quad (5.27)$$

Performing the integrations and using  $4\pi a^2 \zeta e^\alpha = x_m e^{x_m}$  reproduces the macroscopic result (3.13) at  $\beta = 2$ .

## 6. CONCLUSION

The charge fluctuations for a two-dimensional classical Coulomb fluid are drastically changed by the introduction of a negative curvature of space.

In the case of a flat disk communicating with a reservoir (grand-canonical ensemble), the total charge  $Q$  essentially does not fluctuate (bringing an additional charged particle from infinity would cost an infinite energy). In the macroscopic limit, one can define a surface charge density  $\sigma$  (charge per unit length on the boundary circle). The two-point correlation function of  $\sigma$  has an algebraic only decay (1.1), behaving as the inverse square distance between the two points (while the charge correlation function in the bulk has a faster than algebraic decay).

In the case of a disk on a pseudosphere (an infinite surface of constant negative curvature), in the macroscopic limit, the total charge  $Q$  does fluctuate with the variance (3.4). Furthermore the two-point correlation function of the surface charge density  $\sigma$  has a fast (exponential) decay (3.13) as a function of the angular distance  $|\varphi|$  between the two points.

This change of behavior of the surface charge correlation is related to the well-known fact that a negative curvature acts as a mass in the field equations. The curvature replaces the flat logarithmic Coulomb potential by the potential (2.11) which has an exponential decay at large distance  $s$ . For a flat disk, the algebraic decay of the two-point surface charge correlation is due to these field lines which connect the two points through the vacuum outside the disk. On a pseudosphere, these field lines outside the disk nevertheless carry an exponentially decaying interaction.

For retrieving the macroscopic limit from microscopic models, it is necessary that the thickness  $T$  of the surface charge density be negligible compared to the macroscopic length scales. On a pseudosphere with a radius of curvature  $a$ , in a disk of radius  $R$ , we have considered only the case  $R \gg a$ . The macroscopic behavior is expected to hold only when  $a \gg T$ . The two exactly solvable microscopic models which have been considered do exhibit the expected macroscopic features when this condition is satisfied.

## ACKNOWLEDGMENTS

J. M. Caillol brought to our attention the literature about the Lambert function. The authors acknowledge support from ECOS-Nord/COL-CIENCIAS-ICFES-ICETEX action C00P02 of French and Colombian cooperation.

## REFERENCES

1. E. H. Lieb and J. L. Lebowitz, *Adv. Math.* **9**:316 (1972).
2. E. H. Lieb and H. Narnhofer, *J. Stat. Phys.* **12**:291 (1975).
3. B. Jancovici, *J. Stat. Phys.* **110**:879 (2003).
4. B. Jancovici, *J. Stat. Phys.* **80**:445 (1995).
5. B. Jancovici, *J. Phys.: Condens. Matter* **14**:9121 (2002).
6. A. Erdélyi, *et al.*, *Higher Transcendental Functions*, Vol. II (McGraw-Hill, New York, 1953).
7. A. Erdélyi, *et al.*, *Tables of Integral Transforms*, Vol. I (McGraw-Hill, New York, 1954), p. 387.
8. R. Fantoni, B. Jancovici, and G. Téllez, *J. Stat. Phys.* **112**:27 (2003).
9. B. Jancovici and G. Téllez, *J. Stat. Phys.* **91**:953 (1998).
10. P. J. Forrester, *J. Chem. Phys.* **95**:4545 (1991).

11. B. Jancovici and G. Téllez, *J. Stat. Phys.* **82**:609 (1996).
12. F. Cornu and B. Jancovici, *J. Chem. Phys.* **90**:2444 (1989).
13. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1965).
14. J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon, Oxford, 1989).
15. R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, *Adv. Comput. Math.* **5**:329 (1996).