

Electric-field correlation in quantum charged fluids coupled to the radiation field

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In a recent paper [S. El Boustani, P. R. Buenzli, and P. A. Martin, Phys. Rev. E **73**, 036113 (2006)] about quantum charges in equilibrium with radiation, among other things the asymptotic form of the electric-field correlation has been obtained by a microscopic calculation. It has been found that this correlation has a long-range algebraic decay of the form $1/r^3$ (except in the classical limit). The macroscopic approach, in the Course of Theoretical Physics of Landau and Lifshitz, gives no such decay. In this Brief Report we revisit and complete the macroscopic approach of Landau and Lifshitz and suggest that, perhaps, the use of a classical electromagnetic field by El Boustani *et al.* was not justified.

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I. INTRODUCTION

We consider the equilibrium quantum statistical mechanics of an infinite and homogeneous system of nonrelativistic charged particles coupled to electromagnetic radiation. For this system, we investigate the correlation function of the electric field at equal times $\langle E_i(\mathbf{r}_1)E_j(\mathbf{r}_2) \rangle$ where $E_i(\mathbf{r}_1)$ is the quantum operator for the i th Cartesian component ($i = 1, 2, 3$) of the electric field at point \mathbf{r}_1 and $\langle \dots \rangle$ denotes a quantum statistical average at temperature T ; in the present homogeneous medium, this correlation function depends on \mathbf{r}_1 and \mathbf{r}_2 only through $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 = (x_1, x_2, x_3)$. We want to compute the form of this correlation function, when r becomes large compared to the microscopic scale.

A macroscopic approach to this problem had been made a long time ago by Landau and Lifshitz (LL) in their famous Course of Theoretical Physics [1,2]. Actually, their theory has been written for any medium, characterized by a complex frequency-dependent dielectric function $\epsilon(\omega)$. Recently, a microscopic theory has been elaborated by El Boustani, Buenzli, and Martin (BBM) [3]. They find an electric-field correlation function in disagreement with the one advocated by LL: while BBM find that the correlation function has a long-range power-law decay of the form $1/r^3$ (except in the classical limit), I could not extract this algebraic decay from the work of Landau and Lifshitz. The reason for this disagreement is an open problem.

As a first step for clarifying this problem, the present paper revisits and completes the macroscopic approach of LL. Section II summarizes this approach, making it more explicit about the decay of the electric-field correlation function. In Sec. III, the formalism is applied to the special case of a one-component plasma. Section IV is a Conclusion where the above-mentioned discrepancy is discussed.

II. MACROSCOPIC APPROACH

A. The results of Landau and Lifshitz

LL have solved the macroscopic Maxwell equations, in a medium characterized by a complex frequency-dependent di-

electric function $\epsilon(\omega)$, in presence of a random field. This macroscopic approach is expected to be valid only for distances large compared to the microscopic scale. The fluctuation-dissipation theorem is used for obtaining the electric-field correlation as a Fourier transform with respect to \mathbf{r} and the time difference (here this time difference is zero),

$$\langle E_i(\mathbf{r}_1)E_j(\mathbf{r}_2) \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^3k}{8\pi^3} \exp(i\mathbf{k} \cdot \mathbf{r}) \mathcal{E}_{ij}(\mathbf{k}, \omega), \quad (1)$$

where, in terms of the wave vector \mathbf{k} and the frequency ω (see §76 and 77 of [2]),

$$\begin{aligned} \mathcal{E}_{ij}(\mathbf{k}, \omega) = & -4\pi\hbar \coth \frac{\hbar\omega}{2T} \operatorname{Im} \frac{\omega^2/c^2}{(\omega^2/c^2)\epsilon(\omega) - k^2} \\ & \times \left[\delta_{ij} - \frac{c^2}{\omega^2\epsilon(\omega)} k_i k_j \right]; \end{aligned} \quad (2)$$

\hbar is Planck's constant divided by 2π , c is the velocity of light in vacuum, T is the temperature in units of energy.

In order to make easier the comparison with the paper by BBM, we shall now separate (2) in its longitudinal and transverse parts.

B. Longitudinal and transverse correlations

Introducing the projectors $k_i k_j / k^2$ and $\delta_{ij} - (k_i k_j / k^2)$ on the longitudinal and transverse parts of the electric-field correlation functions, we can rewrite (2) as

$$\mathcal{E}_{ij}(\mathbf{k}, \omega) = \mathcal{E}_{ij}^l(\mathbf{k}, \omega) + \mathcal{E}_{ij}^t(\mathbf{k}, \omega), \quad (3)$$

where the longitudinal part is

$$\mathcal{E}_{ij}^l(\mathbf{k}, \omega) = -4\pi\hbar \coth \frac{\hbar\omega}{2T} \operatorname{Im} \frac{1}{\epsilon(\omega)} \frac{k_i k_j}{k^2} \quad (4)$$

and the transverse part is

$$\mathcal{E}_{ij}^t(\mathbf{k}, \omega) = -4\pi\hbar \coth \frac{\hbar\omega}{2T} \operatorname{Im} \frac{\omega^2/c^2}{(\omega^2/c^2)\epsilon(\omega) - k^2} \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right). \quad (5)$$

It should be noted that the cross correlation between the longitudinal and transverse parts of the electric field vanishes.

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Indeed, in Fourier space, in terms of the Fourier transforms of these fields $\mathbf{E}^l(\mathbf{k}, \omega)$ and $\mathbf{E}'(\mathbf{k}, \omega)$, the cross correlation tensor is proportional to $\langle \mathbf{E}^l(\mathbf{k}, \omega) \mathbf{E}'(-\mathbf{k}, -\omega) \rangle$, where \mathbf{E}^l is along \mathbf{k} and \mathbf{E}' is normal to \mathbf{k} . Since the medium is isotropic, the correlation tensor is unchanged if \mathbf{E}' is replaced by its opposite; therefore, the correlation tensor is equal to its opposite, i.e., it vanishes.

The integral on ω of (4) converges; in particular, near $\omega = 0$, the dielectric function $\epsilon(\omega)$ behaves like $4\pi i\sigma/\omega$, where σ is the static conductivity [1] (see Sec. III for the special case of a one-component plasma) and (4) is finite at $\omega = 0$. The Fourier transform of $4\pi/k^2$ is $1/r$ and the longitudinal part of (1) is

$$\langle E_i(\mathbf{r}_1) E_j(\mathbf{r}_2) \rangle_l = \left(\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hbar \coth \frac{\hbar\omega}{2T} \text{Im} \frac{1}{\epsilon(\omega)} \right) \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{r}. \quad (6)$$

At small k , in (5), the term $(\omega^2/c^2)/[(\omega^2/c^2)\epsilon(\omega) - k^2]$ can be replaced by $1/\epsilon(\omega)$. Indeed, if $\omega \neq 0$, this term can be expanded in powers of k^2 and $1/\epsilon(\omega)$ is the leading term. If $|\omega| < \omega_0$, where ω_0 is a sufficiently small constant, $\epsilon(\omega) = 4\pi i\sigma/\omega$, $\hbar \coth \frac{\hbar\omega}{2T} = 2T/\omega$, and

$$\int_{-\omega_0}^{\omega_0} \frac{d\omega}{2\pi} \mathcal{E}_{ij}^t(\mathbf{k}, \omega) = \frac{2T}{\pi\sigma} \left[\omega_0 - \frac{(ck)^2}{4\pi\sigma} \arctan \frac{4\pi\sigma\omega_0}{(ck)^2} \right] \times \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right). \quad (7)$$

At small k , the arctan in (7) behaves like $\pi/2$. Finally, for all values of ω , at small k

$$\mathcal{E}_{ij}^t(\mathbf{k}, \omega) \sim -4\pi\hbar \coth \frac{\hbar\omega}{2T} \left[\left(\text{Im} \frac{1}{\epsilon(\omega)} \right) + O(k^2) \right] \times \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right). \quad (8)$$

The asymptotic behavior of the transverse part of (1) is given by the most singular part at small k of (8), which is just opposite to (4), i.e., the asymptotic form of $\langle E_i(\mathbf{r}_1) E_j(\mathbf{r}_2) \rangle_t$ is opposite to (6). These asymptotic behaviors of the form $1/r^3$ cancel each other in the total correlation function (1). This cancellation had been previously noted [6,7] in the classical limit $T \rightarrow \infty$, but the present macroscopic approach predicts that this cancellation persists in the quantum regime at any temperature, contrarily to the prediction of the microscopic theory of BBM.

The integral on ω in (6) is simply related to the second moment of the charge correlation function. Indeed the charge density ρ is given by the Poisson equation $\text{div} \mathbf{E} = 4\pi\rho$. Only the longitudinal part of \mathbf{E} has a nonvanishing divergence. Therefore, since the Fourier transform (4) of $\langle E_i(\mathbf{r}_1) E_j(\mathbf{r}_2) \rangle_l$ is of the form $A k_i k_j / k^2$, the Fourier transform of $\langle \rho(\mathbf{r}_1) \rho(\mathbf{r}_2) \rangle$ is $(4\pi)^{-2} A k^2$, which means that A is $-(3\pi)^{-1}$ times the second moment of this charge correlation function. Taking the inverse Fourier transform of $A k_i k_j / k^2$ gives for the longitudinal field correlation

$$\langle E_i(\mathbf{r}_1) E_j(\mathbf{r}_2) \rangle_l = - \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{r} \left[- \frac{2\pi}{3} \int d^3 r' r'^2 \langle \rho(\mathbf{0}) \rho(\mathbf{r}') \rangle \right], \quad (9)$$

for r large compared to the microscopic scale, in agreement with a microscopic derivation [4,5].

In the classical limit $T \rightarrow \infty$, $\hbar \coth \frac{\hbar\omega}{2T} \sim 2T/\omega$ and the integral on ω in (6) (which also occurs in the transverse part with the opposite sign) has the simple value $-T$ [6]. This can be shown [7,8] by invoking that $\epsilon(\omega)$ has no zeros when ω is in the complex upper half-plane. The calculation is made in the Appendix. Therefore, in the classical limit,

$$\langle E_i(\mathbf{r}_1) E_j(\mathbf{r}_2) \rangle_l \sim -T \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{r} \quad (10)$$

and

$$\langle E_i(\mathbf{r}_1) E_j(\mathbf{r}_2) \rangle_t \sim T \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{r}; \quad (11)$$

these classical results can also be obtained from (9) since, in the classical case, the second moment in (9) obeys the Stillinger-Lovett sum rule [9]

$$- \frac{2\pi}{3} \int d^3 r' r'^2 \langle \rho(\mathbf{0}) \rho(\mathbf{r}') \rangle = T. \quad (12)$$

The asymptotic classical transverse correlation function (11) is the one of a free field. This is in agreement with the Bohrvan Leeuwen theorem [10–12] which says that, in equilibrium classical statistical mechanics, matter and radiation are uncoupled.

III. ONE-COMPONENT PLASMA

A. Drude dissipationless dielectric function

The above formalism simplifies in the special case of a one-component plasma, a system of one species of particles of charge e , mass m , and number density n , in a neutralizing homogeneous background. We are interested in small wave numbers, when the dissipation goes to zero [13] (i.e., the static conductivity is infinite), and the dielectric function can be taken as the Drude one

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega(\omega + i\eta)}, \quad (13)$$

where $\omega_p = (4\pi n e^2 / m)^{1/2}$ is the plasma frequency and the dissipation constant η is taken as infinitesimal.

Then, using $\text{Im}[1/\epsilon(\omega)] = -\pi(\omega/|\omega|)\omega_p^2 \delta(\omega^2 - \omega_p^2)$ in (4), one finds for the longitudinal correlation function

$$\langle E_i(\mathbf{r}_1) E_j(\mathbf{r}_2) \rangle_l = - \frac{1}{2} \hbar \omega_p \coth \frac{\hbar\omega_p}{2T} \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{r}. \quad (14)$$

In a similar calculation, using (13) in (5), $\delta(\omega^2 - \omega_p^2 - c^2 k^2)$ appears, and one finds for the transverse correlation function

$$\begin{aligned} \langle E_i(\mathbf{r}_1)E_j(\mathbf{r}_2) \rangle_t &= \int \frac{d^3k}{8\pi^3} \exp(i\mathbf{k} \cdot \mathbf{r}) \frac{1}{2} \hbar(\omega_p^2 + c^2k^2)^{1/2} \\ &\times \coth \frac{\hbar(\omega_p^2 + c^2k^2)^{1/2}}{2T} 4\pi \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right). \end{aligned} \quad (15)$$

For small k , the k -singularity in (15) comes from the term $k_i k_j / k^2$ with the replacement of $(\omega_p^2 + c^2k^2)^{1/2}$ by ω_p . Then, one finds for the asymptotic form of the transverse correlation function just the opposite of the longitudinal one (14).

B. Longitudinal and transverse modes

The above results for the electric-field correlation functions can also be obtained, perhaps in a more transparent way, from the modes of vibration of the plasma. In this subsection, we shall not use the dielectric function $\epsilon(\omega)$, but rather take into account explicitly in the equations the charge and electric-current densities.

For every wave vector \mathbf{k} there is a longitudinal mode; its frequency [14] is ω_p (neglecting a term of order k^2 , which is consistent with the previous use of an ϵ independent of \mathbf{k}). In such a mode the electric field, at position \mathbf{r} and time t , is of the form

$$\mathbf{E}_{\mathbf{k}}(\mathbf{r}, t) = \text{Re}(\mathbf{k}/k)E_0 \exp(i\mathbf{k} \cdot \mathbf{r} - i\omega_p t). \quad (16)$$

Since this mode is an oscillator, for studying it in quantum mechanics we can first use classical mechanics and quantize at the end. A collective velocity, $\mathbf{v}_{\mathbf{k}}$, is given by Newton's law:

$$m \frac{d\mathbf{v}_{\mathbf{k}}}{dt} = e\mathbf{E}_{\mathbf{k}}. \quad (17)$$

From (17) one easily finds that the temporal average of the kinetic energy density associated to $\mathbf{v}_{\mathbf{k}}$ is equal to the temporal average of the energy density of the electric field. Taking into account that the temporal average of the squared real electric field is $|E_0|^2/2$ and equating the temporal average of the total energy to the statistical average value for an oscillator (including the zero-point energy) gives

$$\frac{|E_0|^2}{8\pi} = \frac{1}{V} \frac{\hbar\omega_p}{2} \coth \frac{\hbar\omega_p}{2T}, \quad (18)$$

where V is the volume of the system. The contribution of this mode (16) to the electric-field correlation is $\text{Re}(|E_0|^2/2)\exp(i\mathbf{k} \cdot \mathbf{r})(k_i k_j / k^2)$, where $|E_0|^2$ is given by (18). Summing on all \mathbf{k} wave vectors, i.e., computing the integral $V \int d^3k / (8\pi^3) \dots$ reproduces (14).

For every wave vector \mathbf{k} , there are also two (there are two possible polarizations) transverse modes of frequency [14] $\omega_k = (\omega_p^2 + c^2k^2)^{1/2}$. For such a mode, the electric field is of the form

$$\mathbf{E}_{\mathbf{k}}(\mathbf{r}, t) = \text{Re} \mathbf{e} E_0 \exp(i\mathbf{k} \cdot \mathbf{r} - i\omega_k t), \quad (19)$$

where \mathbf{e} is a unit vector normal to \mathbf{k} , and the magnetic-induction field is

$$\mathbf{B}_{\mathbf{k}}(\mathbf{r}, t) = \frac{c}{\omega_k} \mathbf{k} \times \mathbf{E}_{\mathbf{k}}. \quad (20)$$

Again, a collective velocity is given by Newton's law (17) (the magnetic force can be neglected for nonrelativistic velocities). The temporal average of the electric-field energy density is $|E_0|^2/(16\pi)$. The temporal average of the kinetic energy density is $(\omega_p^2/\omega_k^2)|E_0|^2/(16\pi)$, from (17). The temporal average of the magnetic-induction energy density is $(c^2k^2/\omega_k^2)|E_0|^2/(16\pi)$, from (20). Thus, the temporal average of the total energy density again is $|E_0|^2/(8\pi)$, and

$$\frac{|E_0|^2}{8\pi} = \frac{1}{V} \frac{\hbar\omega_k}{2} \coth \frac{\hbar\omega_k}{2T}. \quad (21)$$

The contribution of the transverse mode (19) to the electric-field correlation is $\text{Re} e_i e_j (|E_0|^2/2)\exp(i\mathbf{k} \cdot \mathbf{r})$ where $|E_0|^2$ is given by (21). Adding the contribution of the other polarization replaces $e_i e_j$ by $\delta_{ij} - (k_i k_j / k^2)$. Summing on all \mathbf{k} wave vectors reproduces (15).

IV. CONCLUSION

The macroscopic approach of LL is in agreement with the microscopic calculation of BBM, for the longitudinal part of the electric-field correlation (incidentally, this agreement is an indication that the macroscopic approach can be correct). The disagreement is about the transverse part only, which exactly cancels (6) in the macroscopic approach while it obeys (11) in the microscopic approach [3], even in the quantum regime. I am tempted to believe that the cancellation predicted by the macroscopic calculation is essentially correct. Here are my reasons.

BBM [3] point out that the macroscopic theory of LL uses a local dielectric function $\epsilon(\omega)$ rather than a \mathbf{k} -dependent one. Indeed, it is tempting to use a longitudinal $\epsilon_l(k, \omega)$ in (4) and a transverse $\epsilon_t(k, \omega)$ in (5). However, the leading (singular) term of (4) or (5) at small k would be still obtained by taking these k -dependent dielectric functions at $k=0$ where both of them become $\epsilon(\omega)$. Thus, a singular term [15] in the small- k expansion of $\epsilon_l(k, \omega)$ and/or $\epsilon_t(k, \omega)$ would not change the prefactors of the terms $k_i k_j / k^2$ (this cancellation only concerns the asymptotic terms of the form $1/r^3$, however an algebraic decay faster than $1/r^3$ could remain). BBM make another criticism of the macroscopic approach of LL: it neglects the magnetic permeability. It would be strange that taking the magnetic permeability into account in the macroscopic approach would change the transverse correlation function into the one of a free field (however, it might bring small corrections to the results of LL [16]).

The final remark by BBM might be the key point: in their microscopic approach, they use a *classical* electromagnetic field. That their asymptotic transverse correlation is the one of a free field, decoupled from matter, even in the case of quantum particles, might be due to this feature of their calculation. In Sec. III B, I have argued that the transverse modes certainly feel the presence of matter. Perhaps the argument, in the third paragraph of the Introduction of BBM, in favor of using a classical electromagnetic field, has a flaw:

The frequency of a transverse mode is always larger than the plasma frequency ω_p , and the condition $\beta\hbar ck \ll 1$ does *not* imply that $\beta\hbar\omega_p \ll 1$.

The microscopic calculation of BBM is very elaborate, cleverly using an elegant path-integral formalism. It would certainly be very interesting to redo this microscopic calculation with a quantized electromagnetic field, if feasible.

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APPENDIX

This Appendix is based on a private message from B. U. Felderhof. In the classical limit, the integral in (6) becomes TI where I is

$$I = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \operatorname{Im} \frac{1}{\epsilon(\omega)}. \quad (\text{A1})$$

We want to show that $I = -1$.

Since $\epsilon^*(\omega) = \epsilon(-\omega^*)$, I can be rewritten as

$$I = \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \frac{1}{\epsilon(\omega)}. \quad (\text{A2})$$

Since $\epsilon(\omega)$ has no zeros in the upper complex half-plane [1], the integral along the real axis in (A2) can be changed into an integral along the half-circle C at infinity in the upper half-plane. Since ϵ is 1 at infinity, this integral is $\int_C d\omega/\omega = -i\pi$. Therefore, $I = -1$.

The same reasoning applies to the transverse part. Comparing (4) and (5), one sees that, on the half-circle at infinity, $1/\epsilon$ and $(\omega^2/c^2)/[(\omega^2/c^2)\epsilon - k^2]$ have the same limit 1. Furthermore, the term $k_i k_j/k^2$ in (4) and the term $\delta_{ij} - (k_i k_j/k^2)$ in (5) will have opposite inverse Fourier transforms for $r \neq 0$. Thus, (11) is just opposite to (10).

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- [1] L. Landau and E. Lifshitz, *Electrodynamics of Continuous Media*, § 88 (Pergamon Press, Oxford, 1960).
 - [2] E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics, Part 2*, § 77 (Pergamon Press, Oxford, 1980).
 - [3] S. El Boustani, P. R. Buenzli, and P. A. Martin, Phys. Rev. E **73**, 036113 (2006).
 - [4] J. L. Lebowitz and P. A. Martin, J. Stat. Phys. **34**, 287 (1984).
 - [5] P. A. Martin, Rev. Mod. Phys. **60**, 1075 (1988).
 - [6] B. U. Felderhof, Physica (Amsterdam) **31**, 295 (1965).
 - [7] B. U. Felderhof, J. Chem. Phys. **67**, 493 (1977).
 - [8] B. U. Felderhof, private communication.
 - [9] F. H. Stillinger and R. Lovett, J. Chem. Phys. **49**, 1991 (1968).
 - [10] N. Bohr, *Dissertation* (Copenhagen, unpublished, 1911).
 - [11] J. H. van Leeuwen, J. Phys. Radium **2**, 361 (1921).
 - [12] A. Alastuey and W. Appel, Physica A **276**, 508 (2000).
 - [13] D. Pines and P. Nozières, *The Theory of Quantum Liquids, I* (Benjamin, New York, 1966).
 - [14] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975).
 - [15] F. Cornu and P. A. Martin, Phys. Rev. A **44**, 4893 (1991).
 - [16] B. U. Felderhof, J. Phys. C **11**, 551 (1978).