Coulomb Energy and Nuclear Radius

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The apparent discrepancy between the values ($1.45\times10^{-13}A^{\frac{1}{3}}$ cm) for the nuclear radii derived from mirror nuclei and those ($1.2\times10^{-13}A^{\frac{1}{3}}$ cm) derived from μ-mesonic atoms was investigated. The conventional calculation of the Coulomb energy difference ΔE_c between mirror nuclei is improved in two respects: the usual uniform model is replaced by the more elaborate shell model with a finite square well potential, and the exchange terms are taken into account. An equivalent Coulomb radius R_c is defined by $R_c=(6/5)(Ze^2/\Delta E_c)$; an equivalent meson radius is defined by $R_M=(5/3)^{\frac{1}{2}}\langle r^2\rangle_{\text{Av}}^{\frac{1}{2}}$, where $\langle r^2\rangle_{\text{Av}}^{\frac{1}{2}}$ is the mean square radius of the electrical charge distribution. The two extreme cases of the pairs (F^{17},O^{17}) and (O^{15},N^{15}) are investigated. The computation gives $R_c/R_M=1.18$ in O^{17}, $R_c/R_M=1.07$ in N^{15}. These results are smaller by about 8 percent than the experimental ratios. However, the experimental discontinuity in R_c at the closure of the p shell is reproduced.

I. INTRODUCTION

THE experiments which determine the nuclear radii may be divided into two groups: fast nucleon scattering, α-decay lifetimes, yields of charged particle initiated nuclear reactions, probe the range of nuclear forces, whereas Coulomb energy differences for mirror nuclei, electron scattering, μ-mesonic atom x-rays probe the electrical charge distribution. It is difficult to compare values for the radius of the nucleus as given by methods which belong to different groups, as these methods do not measure the same quantity. On the other side, it is desirable to get consistency inside the same group. The present paper is concerned with the determination of the electrical charge distribution.

Whereas Coulomb energy differences between mirror nuclei were usually explained in terms of a radius of the order of $1.45\times10^{-13}A^{\frac{1}{3}}$ cm, electron scattering,[1] and more recently, μ-mesonic x-rays[2] were fitted by a smaller radius of the order of only $1.2\times10^{-13}A^{\frac{1}{3}}$ cm. Actually, mirror nuclei experiments are concerned with light nuclei, while the most accurate measurements with μ-mesonic atoms are carried out with heavier nuclei. However, the results in each range of A extrapolate very well, and it seems very likely at the present time that the radius from meson experiments would be $1.2\times10^{-13}A^{\frac{1}{3}}$ cm for lighter nuclei. To explain this discrepancy, several authors pointed out that the assumed nuclear model of a uniformly charged sphere was too crude an approximation.

Actually, the electron scattering and the μ-mesonic atom x-rays measure essentially the mean quadratic radius $\langle r^2\rangle_{\text{Av}}^{\frac{1}{2}}$ of the charge distribution,[1,3] which is related to the radius R of the uniform sphere model by

$$\langle r^2\rangle_{\text{Av}}^{\frac{1}{2}}=(3/5)^{\frac{1}{2}}R, \tag{1}$$

whereas the Coulomb energy difference ΔE_c between mirror nuclei of charges Z and $Z+1$ is related to the same radius R by

$$\Delta E_c=(6/5)(Ze^2/R). \tag{2}$$

If the assumption of a constant charge density is dropped, (1) and (2) may be considered as mere definitions of an equivalent meson radius

$$R_M=(5/3)^{\frac{1}{2}}\langle r^2\rangle_{\text{Av}}^{\frac{1}{2}}, \tag{3}$$

and of an equivalent Coulomb radius

$$R_c=(6/5)(Ze^2/\Delta E_c). \tag{4}$$

R_M and R_c are now just a convenient way of expressing $\langle r^2\rangle_{\text{Av}}^{\frac{1}{2}}$ and ΔE_c; one can expect that R_M and R_c will, in general, have different values.

It has been proved[3,4] that, if the same charge distribution is assumed for every proton, any departure from the uniform model would lead to $R_c<R_M$, thus making the discrepancy between the two experiments worse. However, qualitative considerations[3] showed that there was some hope to improve things by using a shell model, in which the charge distribution is not the same for all protons, and also by taking into account the exchange terms in the calculation of ΔE_c. If, in a first approximation, the electrostatic field is considered as a perturbation, ΔE_c is the electrostatic energy of the last proton in the Coulomb field of the other protons. The last proton has a smaller binding energy and is often in a higher angular momentum state than the other ones; its wave function extends toward the periphery of the nucleus, where the electrical potential is smaller, and ΔE_c is lowered. The exchange terms also lower ΔE_c. Hence, for a given R_M, a shell model is expected to give a smaller ΔE_c and a bigger R_c than the uniform model. However, a quantitative check of this argument will prove that these effects are not large enough to remove the discrepancy.

It will be shown in the present paper that one has to expect, on the basis of the above model and an $R_M=(5\langle r^2\rangle_{\text{Av}}/3)^{\frac{1}{2}}=1.2\times10^{-13}A^{\frac{1}{3}}$ cm, larger Coulomb energies (and hence smaller R_c) than the experimental data give.

Two extreme cases have been investigated: the pairs (F^{17},O^{17}) and (O^{15},N^{15}). In F^{17}, the last proton is in a d state outside the closed s and p shells, thus providing

[1] Lyman, Hanson, and Scott, Phys. Rev. **84**, 626 (1951).
[2] V. L. Fitch and J. Rainwater, Phys. Rev. **92**, 789 (1953).
[3] L. N. Cooper and E. M. Henley, Phys. Rev. **92**, 801 (1953).
[4] F. Bitter and H. Feshbach, Phys. Rev. **92**, 837 (1953).

a highly "favorable" case for computing a small ΔE_c and hence a large R_c. Conversely, in O^{15}, the last proton is part of a closed p shell, and provides an "unfavorable" case.

II. THE SQUARE WELL MODEL

A square well potential was assumed; its radius was arbitrarily taken as $R=1.4\times10^{-13}A^{\frac{1}{3}}$ cm. The initial choice of R is not very important, the ratios R_M/R and R_c/R are not expected to depend significantly on the absolute value of R. The same will be true then of R_M/R_c which is the quantity given by the experimental data.

Let $\psi_l(r)$ by r times the radial part of the wave function for a nucleon of orbital angular momentum l. It is normalized to unity so that

$$\int_0^\infty |\psi_l(r)|^2 dr=1. \tag{5}$$

It is convenient to express lengths in units of the well radius R, and to set $\rho=r/R$. Inside the well, ψ_l is of the form

$$\psi_l(\rho)=A_l R\rho j_l(\xi_l\rho) \quad \text{for} \quad \rho\leqslant 1. \tag{6}$$

A_l is a normalization coefficient; ξ_l is given in terms of the reduced mass M of the nucleon, its binding energy $|E|$, and the well depth V, by

$$\xi_l=R[2M(V-|E|)/\hbar^2]^{\frac{1}{2}}; \tag{7}$$

j_l is a spherical Bessel function as defined by Schiff.[5] The functions j_l are tabulated.[6]

If the binding energy plus centrifugal barrier of a proton is large as compared to the Coulomb barrier, the deformation of the wave function by the Coulomb barrier can be neglected and the wave function outside the well is correctly represented by

$$\psi_l(\rho)=B_l R\rho h_l(i\eta_l\rho) \quad \text{for} \quad \rho\geqslant 1. \tag{8}$$

B_l is a normalization coefficient;

$$\eta_l=R[2M|E|/\hbar^2]^{\frac{1}{2}}; \tag{9}$$

h_l is a spherical Hankel function.[5] The functions h_l are not tabulated but are easily expressed in terms of rational fractions and exponential functions.

The preceding argument does not apply in the case of the last nucleon of F^{17}, the binding energy of which is very small. In this case one can argue in the following way. The only difference between the Hamiltonians of F^{17} and O^{17} is the Coulomb energy C of the last nucleon. The Hamiltonians are, respectively, $H+C$ and H; the corresponding wave functions are Ψ_F and Ψ_O, which satisfy

$$(H+C)\Psi_F=E_F\Psi_F, \tag{10a}$$

$$H\Psi_O=E_O\Psi_O, \tag{10b}$$

where $|E_F|$ and $|E_O|$ are the binding energies of F^{17} and O^{17}. As far as ground states are concerned, the wave functions minimize energies so that

$$E_O\equiv\langle\Psi_O|H|\Psi_O\rangle<\langle\Psi_F|H|\Psi_F\rangle, \tag{11a}$$

$$E_F\equiv\langle\Psi_F|H+C|\Psi_F\rangle<\langle\Psi_O|H+C|\Psi_O\rangle, \tag{11b}$$

and

$$\langle\Psi_F|C|\Psi_F\rangle<E_F-E_O<\langle\Psi_O|C|\Psi_O\rangle. \tag{12}$$

It could be shown that $\langle\Psi_O|C|\Psi_O\rangle-\langle\Psi_F|C|\Psi_F\rangle$ is much less than 2 percent of E_F-E_O. Consequently, a perturbation theory calculation gives fair results with either Ψ_F or Ψ_O. Actually, $\langle\Psi_F|C|\Psi_F\rangle$ was calculated providing thus a lower limit for E_F-E_O.

The Coulomb wave function used for Ψ_F is

$$\psi_2(\rho)=B_2\mathfrak{F}(R\rho). \tag{13}$$

The well depth is adjusted to fit the experimental binding energy of the last nucleon: this binding energy determines by (9) and (8) or by (13) the wave function outside the well, except for the normalization coefficient B_l. The continuity requirement at $\rho=1$,

$$\left[\frac{1}{\rho j_l(\xi_l\rho)}\frac{d}{d\rho}\rho j_l(\xi_l\rho)\right]_{\rho=1}=\left[\frac{1}{\rho h_l(i\eta_l\rho)}\frac{d}{d\rho}\rho h_l(i\eta_l\rho)\right]_{\rho=1}, \tag{14}$$

or, in the special case of the last proton of F^{17},

$$\left[\frac{1}{\rho j_2(\xi_2\rho)}\frac{d}{d\rho}\rho j_2(\xi_2\rho)\right]_{\rho=1}=\left[\frac{1}{\mathfrak{F}(R\rho)}\frac{d}{d\rho}\mathfrak{F}(R\rho)\right]_{\rho=1}, \tag{15}$$

determines ξ_l, and by (7), the well depth V.

The other wave functions are then determined. Their ξ_l and η_l parameters are obtained from (14) and from the relation

$$\xi_l^2+\eta_l^2=R^2 2MV/\hbar^2. \tag{16}$$

The normalization coefficients A_l and B_l are required to fit (5) and the requirement that ψ_l be continuous at $\rho=1$. The integrals on Bessel or Hankel functions,

TABLE I. The Coulomb function $\mathfrak{F}(r)$. $\mathfrak{F}(r)$ is r times the radial part of the wave function for a d proton of reduced mass 16/17, in the field of a central charge $Z=8$, with a binding energy 0.586 Mev.

$10^{13}r$ cm	$\mathfrak{F}$	$10^{-12}(d\mathfrak{F}/\mathfrak{F}dr_{cm})$	$10^{13}(1/\mathfrak{F}^2)\int_r^\infty \mathfrak{F}^2 dr_{cm}$
3.077	1.195	−7.638	0.7535
3.692	0.7621	−6.708	
4.308	0.5141	−5.991	0.9547
4.923	0.3612	−5.438	
5.538	0.2618	−5.004	1.124
6.154	0.1944	−4.655	
6.769	0.1473	−4.370	1.267
7.385	0.1134	−4.131	
8.000	0.08845	−3.930	1.391

[5] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1948), p. 77.

[6] *Tables of Spherical Bessel Functions* (Columbia University Press, New York, 1947).

which are required for fitting (5) can be evaluated analytically.[5]

The mean square radius $\langle r^2\rangle_{Av}^{1/2}$ of the charge distribution is then computed for the residual nucleus of charge Z by

$$\langle r^2\rangle_{Av} = (1/Z)R^2 \sum \int_0^\infty |\psi|^2 \rho^2 d\rho, \tag{17}$$

where the summation extends over all Z protons. The integrals in (17) are easily evaluated analytically by expressing Bessel and Hankel functions in terms of trigonometrical and exponential functions.

The Coulomb energy of the last proton is also computed, exchange terms being included, by Slater's method,[7] leading through (4) to the Coulomb equivalent radius. All integrals involved in this calculation had to be carried out numerically.

III. APPLICATION TO TWO EXTREME CASES

A. $F^{17} \rightarrow O^{17}$

The last nucleon, decaying from a proton to a neutron, is in a d state, outside closed s and p shells.

Well depth

The well radius is initially chosen as $R = 3.6\times10^{-13}$ cm. The binding energy of the last proton was taken as $|E| = 0.586$ Mev. Outside the well the wave function will be a Coulomb function $\mathfrak{F}(r)$; inside the well, the wave function is taken as a Bessel wave function. The Coulomb wave function $\mathfrak{F}(r)$ was computed by Breit and his collaborators[8] and kindly made available to the author for the above energy, a reduced mass 16/17; an angular momentum $l = 2$. $\mathfrak{F}(r)$ is tabulated in Table I, together with its logarithmic derivative and its square integral. One obtains from (15), $\xi_2 = 4.59$, and from (7), $V = 41.2$ Mev.

Wave functions of F^{17}

The wave functions in this well are as follows:

s wave functions

$$\rho \leqslant 1: \quad \psi_0 = (10.29/R)^{\frac{1}{2}} \rho j_0(2.55\rho),$$
$$\rho \geqslant 1: \quad \psi_0 = (14200/R)^{\frac{1}{2}} \rho h_0(i3.80\rho);$$

p wave functions

$$\rho \leqslant 1: \quad \psi_1 = (20.50/R)^{\frac{1}{2}} \rho j_1(3.61\rho), \tag{18}$$
$$\rho \geqslant 1: \quad \psi_1 = (1343/R)^{\frac{1}{2}} \rho h_1(i2.88\rho);$$

d wave function,

$$\rho \leqslant 1: \quad \psi_2 = (30.64/R)^{\frac{1}{2}} \rho j_2(3.61\rho),$$
$$\rho \geqslant 1: \quad \psi_2 = (23.89/R^3)^{\frac{1}{2}} 10^{-13} \mathfrak{F}(R\rho).$$

[7] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, Cambridge, 1953), Chap. VI.

[8] Breit, Hull, Johnson, Huebner, Benedict, and Smolen (private communication).

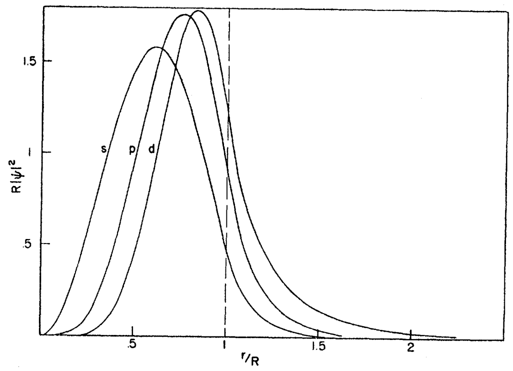

FIG. 1. The wave functions of F^{17}. ψ is r times the normalized radial wave function. The ordinate is $|\psi|^2$ in units of the reciprocal radius of the well.

The squares of these functions (probability of presence at distance r from the origin) are plotted in Fig. 1. 6.5 percent of the s charge distribution, 15 percent of the p one, and 29 percent of the d one are outside the well.

Effective meson radius of O^{17}

The mean square radius is found to be $0.6833R$ for an s wave function and $0.8090R$ for a p one. From (17) and (3) the effective meson radius is found to be $R_M = 1.01R$.

Effective Coulomb radius

In Condon and Shortley's notation, the direct Coulomb energy of the d proton is

$$2F^0(d,s) + 6F^0(d,p) = 8.62(e^2/R). \tag{19}$$

The exchange Coulomb energy is

$$-(1/5)G^2(d,s) - (2/5)G^1(d,p) - (9/35)G^3(d,p) = -0.52(e^2/R); \tag{20}$$

the total Coulomb energy is thus $\Delta E_c = 8.10(e^2/R)$, and by (4), the equivalent Coulomb radius is $R_c = 1.19R$.

We thus obtain $R_c/R_M = 1.18$.

B. $O^{15} \rightarrow N^{15}$

The β transition is from the configuration $s^2p^6; s^2p^5$ to $s^2p^5; s^2p^6$. It is convenient to look at it as a transition of a p hole in closed s and p shells.

Well depth

The well radius is chosen as $R = 3.45\times10^{-13}$ cm. The standard value $|E| = 8$ Mev was taken for the binding energy of the p nucleons. Corrections of reduced mass were neglected, and also any deformation of the wave functions by the Coulomb field. From (11), one obtains $\xi_1 = 3.48$, and, from (7), $V = 34.6$ Mev.

Wave functions

The wave functions in this well are:

s wave functions,

$$\rho\leqslant 1:\quad \psi_0=(9.47/R)^{\frac{1}{2}}\rho j_0(2.49\rho),$$
$$\rho\geqslant 1:\quad \psi_0=(3373/R)^{\frac{1}{2}}\rho h_0(i3.19\rho);\tag{21}$$

p wave functions.

$$\rho\leqslant 1:\quad \psi_1=(17.52/R)^{\frac{1}{2}}\rho j_1(3.48\rho),$$
$$\rho\geqslant 1:\quad \psi_1=(127.2/R)^{\frac{1}{2}}\rho h_1(i2.07\rho).$$

The squares of these functions are plotted on Fig. 2. 8.8 percent of the s charge distribution, 22 percent of the p one are outside the well.

Effective meson radius of N^{15}

The mean square radius is found to be $0.7030R$ for an s wave function and $0.8767R$ for a p one. From (17) and (3) the effective meson radius is found to be $R_M=1.07R$.

Effective Coulomb radius

The direct Coulomb energy of a p proton is

$$2F^0(d,s)+5F^0(p,p)=7.78(e^2/R).\tag{22}$$

The exchange Coulomb energy is

$$-(1/3)G^1(p,s)-(2/5)G^2(p,p)=-0.48(e^2/R).\tag{23}$$

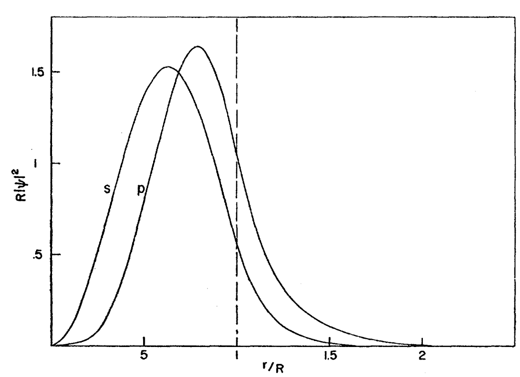

FIG. 2. The wave functions of O^{15}. ψ is r times the normalized radial wave function. The ordinate is $|\psi|^2$ in units of the reciprocal radius of the well.

TABLE II. Values of R_c/R_M.

	Square well	Oscillator well	Experimental
$F^{17}\rightarrow O^{17}$	1.18	1.20	1.27
$O^{15}\rightarrow N^{15}$	1.07	1.08	1.17

The total Coulomb energy is thus $\Delta E_c=7.30(e^2/R)$, and by (4), the Coulomb radius is $R_c=1.15R$. We thus get $R_c/R_M=1.07R$.

IV. THE OSCILLATOR WELL MODEL

It is interesting to see how much the final results R_c/R_M are dependent on the special potential well which is assumed. The same calculations can be carried out, exactly and easily, using Talmi's methods,[9] for the less accurate model of an infinite oscillator well. The results are then $R_c/R_M=1.20$ for F^{17}, and $R_c/R_M=1.08$ for O^{15}.

V. DISCUSSION OF RESULTS AND COMPARISON WITH EXPERIMENTAL DATA

The theoretical results and the experimental values are listed in Table II. The experimental values for R_c/R_M are obtained from the experimental R_c and an assumed $R_M=1.2A^{\frac{1}{3}}\times 10^{-13}$ cm. It is seen that the shell model reproduces the experimental discontinuity in R_c at the closure of the p shell. However, the value of R_c/R_M remains too small by 8 or 9 percent. The oscillator potential's results are very close to the square well's and show that it is not possible to suppress the discrepancy by changing the potential. The results are not very sensitive to the exact form of the potential. Thus, there is still an unexplained discrepancy between the values given for nuclear radii by mirror nuclei and by μ-mesonic atoms.

VI. ACKNOWLEDGMENTS

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[9] I. Talmi, Helv. Phys. Acta **25**, 185 (1952).