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NUCLEAR REACTION RATE ENHANCEMENT IN DENSE STELLAR MATTER

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ABSTRACT

The enhancement factor for the rate of thermonuclear reactions in dense stellar matter is calculated in the one-component, strongly coupled plasma model. The nuclear reaction rate is related to the short-range behavior of the quantum-mechanical pair correlation function. Within the range of parameters which is investigated here, the pair correlation function is expressed in terms of classical quantities, which are known from existing computer results; the classical potential of mean force is found to play an important role, from the practical point of view. Numerical results are given.

Subject headings: dense matter — nuclear reactions — plasmas — quantum mechanics

I. INTRODUCTION

The rate of fusion nuclear reactions in stellar matter is important for the evolution of the star. In dense ionized matter, the rate of nuclear reactions is enhanced by screening effects; the Coulomb barrier between two ions is lowered by the cloud of surrounding particles. In the present paper, we study this enhancement effect. We consider the case in which the medium can be well described by the one-component plasma model, a system of identical particles of charge Ze and mass M (nuclei) embedded in a uniform neutralizing background of opposite charge (degenerate electrons); a typical example would be a ^{12}C plasma at a temperature 10^8 K and a density 10^{10} g cm^{-3}. The enhancement of nuclear reactions in the framework of the one-component plasma model has already been discussed by Salpeter (1954), Salpeter and Van Horn (1969), DeWitt, Graboske, and Cooper (1973), Mitler (1977), Itoh, Totsuji, and Ichimaru (1977), and one of us (Jancovici 1977, hereafter Paper I).

In the present paper, we again study the one-component plasma in its fluid phase, as in Paper I, with, however, modifications which enable us to deal with lower temperatures and higher densities. We try to make the present paper self-contained by giving simplified derivations of some results which were given in Paper I.

The state of the plasma is defined by its temperature T (or alternatively $\beta = 1/kT$, where k is Boltzmann's constant) and its ion number density ρ [or alternatively the mean ionic distance $a = (3/4\pi\rho)^{1/3}$]. We shall also use the conventional dimensionless parameters $\Gamma = \beta(Ze)^2/a$ and $\tau = [27\pi^2\beta M(Ze)^4/4\hbar^2]^{1/3}$. We may consider Γ as a measure of the strength of the coupling in the plasma, and τ^{-1} as a measure of the importance of quantum effects (although the ratio Γ/τ will turn out to be a better measure of quantum effects). The range of values which will be discussed here is $\tau \gg 1$ and $0 \leq 3\Gamma/\tau \leq 1.6$. Although Γ itself can have any value in the fluid range $0 \leq \Gamma \leq 155$, the interesting case is the one of strongly coupled plasmas (large Γ). A graph relating Γ and τ to T and ρ for ^{12}C can be found in the paper of Itoh, Totsuji, and Ichimaru (1977).

In § II, we argue that the nuclear reaction rate is related to the short-range behavior of the quantum-mechanical pair correlation function. In § III, the enhancement factor is expressed in terms of known classical quantities. The results of numerical computations are presented in § IV. These results are summarized in § V, where we also make some comparisons between the present paper and previous ones. In the Appendix, we give a justification for an approximation made in the main text.

II. NUCLEAR REACTION RATE AND PAIR CORRELATION FUNCTION

Our starting point is to assume that the number, ν, of reactions per nucleus and per unit time is proportional to the probability of finding another nucleus in the immediate neighborhood of the first one. If ρ is the number density (number of nuclei per unit volume) and $g(r)$ the pair correlation function (also called pair distribution function), ν is proportional to $\rho g(r)$, where r is of the order of the nuclear diameter. For such small values of r, $g(r)$ is governed by the tunnel effect through the electrostatic potential barrier between the nuclei, and therefore $g(r)$ must be taken as the quantum-mechanical pair correlation function. In an infinitely dilute plasma, the pair correlation function is $g_0(r)$; only two-body collisions with a Coulomb potential $(Ze)^2/r$ contribute to it. The enhancement factor for the nuclear reaction rate is $E = g(r)/g_0(r)$.

Some other authors (Salpeter and Van Horn 1969; Mitler 1977; Itoh, Totsuji, and Ichimaru 1977) have claimed that one should take into account some kind of nonequilibrium effect; they essentially say that, while two nuclei

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are tunneling toward one another, the surrounding particles do not necessarily have enough time for relaxing to the equilibrium distribution which corresponds to every separation of the reacting pair. We, on the contrary, argue that the pair correlation function should be taken as the static one. Our point is that the transmission coefficient of the potential barrier is exceedingly small, which makes nuclear reactions very rare events. In a loose classical analogy, we might say that, in most collisions, the colliding particles tunnel through only a certain distance and are then reflected back. Therefore, as soon as r is larger than a few nuclear diameters, equilibrium is achieved and the probability of finding two nuclei at a distance r from one another is given by the equilibrium pair correlation function. Furthermore, this pair correlation function can be computed by taking into account the electrostatic forces alone (it is not necessary to make an explicit use of the nuclear forces). The extraction of nuclei from the entrance channel by nuclear reactions disrupts equilibrium and affects the wave functions only for distances smaller than a few nuclear diameters.

For small values of r, although $g(r)$ and $g_0(r)$ are rapidly varying functions of r, their ratio depends very little on r. Therefore, we come to the conclusion that, for computing the enhancement factor, we can forget about the nuclear forces, and then compute the ratio $g(r)/g_0(r)$ where this is more convenient, i.e., at $r = 0$.

III. QUANTUM-MECHANICAL PAIR CORRELATION FUNCTION AND CLASSICAL QUANTITIES

a) General Approach

We consider a system of N particles in a volume Ω. We are interested in configurations in which particles 1 and 2 are colliding; we call $\mathbf{r}$ the relative coordinate of particles 1 and 2, $\mathbf{R}$ the coordinate of their center of mass, and α the coordinates of the $N - 2$ other particles. A quantum state of the system can be labeled as $|\mathbf{r}\mathbf{R}\alpha\rangle$. The Hamiltonian is H. The pair correlation function at zero separation is given by the standard expression

$$g(0) = \Omega^2 \frac{\int \langle 00\alpha|\exp(-\beta H)|00\alpha\rangle d\alpha}{\int \langle \mathbf{r}\mathbf{R}\alpha|\exp(-\beta H)|\mathbf{r}\mathbf{R}\alpha\rangle d\mathbf{r}d\mathbf{R}d\alpha}; \tag{1}$$

for convenience, in the numerator of equation (1), the center of mass $\mathbf{R}$ of the colliding particles has been located at the origin.

In the range of parameters that we study, the thermal de Broglie wavelength, $\lambda = (2\pi\hbar^2/MkT)^{1/2}$, is small enough for the *thermodynamics* of the system to be describable, at least in first approximation, by classical statistical mechanics (corrections to this approximation will be briefly discussed in § V); this means that those configurations which contribute a nonnegligible weight to the denominator of equation (1) are classical, and this denominator can be replaced by Q/λ^{3N}, where Q is the classical configuration integral. The numerator of equation (1), however, is dominated by configurations in the neighborhood of $\mathbf{r} = 0$, where the potential is very steep, and thus quantum effects are essential for the relative motion of particles 1 and 2; on the other hand, most of the motion of the center of mass of 1 and 2, and of the other particles, occurs in regions of configuration space where the potential is smooth, and these motions can be considered classical. Therefore, it is enough to keep the kinetic energy associated with $\mathbf{r}$ and the total potential energy explicitly in the Hamiltonian H, while the kinetic energies associated with $\mathbf{R}$ and α only contribute a multiplicative factor $2^{3/2}\lambda^{-3(N-1)}$. Furthermore, it is convenient to split the total potential energy as

$$(Ze)^2/r + W(\mathbf{r}, \alpha),$$

where $W(\mathbf{r}, \alpha)$ is the sum of all interactions except the one between particles 1 and 2, for the configuration $(\mathbf{r}, 0, \alpha)$. Thus equation (1) reduces to

$$g(0) = (\Omega^2/Q)2^{3/2}\lambda^3 \int \langle 0|\exp\{-\beta[-(\hbar^2/M)\Delta + (Ze)^2/r + W(\mathbf{r}, \alpha)]\}|0\rangle d\alpha, \tag{2}$$

where the Laplacian Δ refers to the relative coordinate $\mathbf{r}$, and where the diagonal matrix element is taken at $\mathbf{r} = 0$.

In the framework of our approximations, the quantum many-body problem has now been reduced to a quantum one-body problem, which, however, involves a complicated potential $W(\mathbf{r}, \alpha)$ depending on the coordinates α of $N - 2$ particles. In principle, we must first compute the matrix element in equation (2) for every value of the set of parameters α, and afterwards perform the integration upon α. This program will be fulfilled through a systematic method of successive approximations. First, note that, in the case of an infinitely dilute plasma, $g(0)$ becomes $g_0(0)$ and is obtained by using equation (2) with $Q = \Omega^N$ and $W = 0$; therefore, the enhancement factor is

$$E = \frac{g(0)}{g_0(0)} = \frac{\Omega^2}{Q} \frac{\int \langle 0|\exp\{-\beta[-(\hbar^2/M)\Delta + (Ze)^2/r + W(\mathbf{r}, \alpha)]\}|0\rangle d\alpha}{\langle 0|\exp\{-\beta[-(\hbar^2/M)\Delta + (Ze)^2/r]\}|0\rangle}. \tag{3}$$

Throughout this discussion, we have neglected all exchange effects. The exchange of particles 1 and 2 would provide a multiplicative factor of 2 in both the numerator and the denominator of equation (3), making no change in E. Exchanges involving the other particles α are negligible (Jancovici 1978).

For evaluating the matrix elements in equation (3), we use the path integral approach (Feynman and Hibbs 1965). Each of these matrix elements can be formally expressed as a functional integral on all paths that go from the origin back to the origin in a "time" $\beta\hbar$. In the small-λ limit which is considered here, the functional integral is dominated by a factor $\exp(-S/\hbar)$, where S is the classical action for a particle of mass $M/2$ following a classical trajectory from the origin back to the origin, in a time $\beta\hbar$, in the *reversed* potential (keeping only this classical trajectory is equivalent to using the WKB method at the energy which gives the dominant contribution to the nuclear reaction rate). For the denominator of equation (3) the action is

$$S_0 = \int_0^{\beta\hbar} [(M/4)\dot{\mathbf{r}}^2 + (Ze)^2/r]dt\,, \tag{4}$$

where the trajectory $\mathbf{r}(t)$ minimizes S_0. For the numerator of equation (3), the action is

$$S = \int_0^{\beta\hbar} [(M/4)\dot{\mathbf{r}}^2 + (Ze)^2/r + W(\mathbf{r}, \alpha)]dt\,, \tag{5}$$

where the trajectory $\mathbf{r}(t)$ now minimizes S. Using equations (4) and (5) in equation (3), we obtain

$$E = (\Omega^2/Q)\int \exp[-(S - S_0)/\hbar]d\alpha\,. \tag{6}$$

b) Perturbation Theory

Let us consider first the case of a rather low density, when the many-body potential $W(\mathbf{r}, \alpha)$ can be considered as a perturbation to the two-body potential $(Ze)^2/r$. We then obtain an approximate expression for S by using in equation (5) the same trajectory as the one appropriate to S_0. This trajectory is a Kepler orbit reduced to a radial line segment of arbitrary direction on which the particle travels from the origin to an aphelion value

$$2A = (4\hbar^2\beta^2Z^2e^2/\pi^2M)^{1/3} \tag{7}$$

and back to the origin, following a law $\mathbf{r}(t)$ which has the parametric representation

$$t = (\beta\hbar/2\pi)(\xi - \sin\xi)\,, \qquad r = A(1 - \cos\xi)\,, \tag{8}$$

where the parameter ξ goes from 0 to 2π. We then obtain from equation (6)

$$E = (\Omega^2/Q)\int \exp\left\{-\hbar^{-1}\int_0^{\beta\hbar} W[\mathbf{r}(t), \alpha]dt\right\}d\alpha\,, \tag{9}$$

where $r(t)$ is defined by equation (8).

At this point, it may be of interest to note that

$$2A/a = 3\Gamma/\tau\,, \tag{10}$$

and that this ratio can be considered as another measure of the importance of quantum effects. Furthermore, one may remark that $2A$ is the distance of the turning point to the origin in the conventional WKB approach, in the case of a pure Coulomb potential $(Ze)^2/r$.

From the complicated many-body potential $W(\mathbf{r}, \alpha)$, it is customary to build a two-body potential $w(r)$ defined by

$$\exp[-\beta w(r)] = (\Omega^2/Q)\int \exp[-\beta W(\mathbf{r}, \alpha)]d\alpha\,. \tag{11}$$

$(Ze)^2/r + w(r)$ is the classical potential of mean force, and is related to the classical pair correlation function $g_c(r)$ by

$$g_c(r) = \exp\{-\beta[(Ze)^2/r + w(r)]\}\,. \tag{12}$$

Much numerical information on $g_c(r)$ and $w(r)$ is available from the computer calculations of Hansen (1973) and Hansen, Torrie, and Vieillefosse (1977). However, $w(r)$ does not appear directly in equation (9), since the integration on t must be performed *before* the integration on α. Fortunately, it is possible to introduce $w(r)$ in an approximate way, because a particle which follows the trajectory (8) spends a large fraction of its time near its time-averaged position

$$\bar{r} = (\beta\hbar)^{-1}\int_0^{\beta\hbar} r(t)dt = 3A/2\,; \tag{13}$$

thus, as a first approximation, we can replace $\mathbf{r}(t)$ in equation (9) by $\bar{\mathbf{r}}$, with the result

$$E = \exp\,[-\beta w(\bar{r})]\,. \tag{14}$$

We can go beyond that first approximation by expanding $W(\mathbf{r}, \alpha)$ in a Taylor series around $\bar{r} = 3A/2$ and using equation (8) for computing the time integral. We obtain

$$\hbar^{-1}\int_0^{\beta\hbar} W[\mathbf{r}(t), \alpha]dt = \beta[W(\bar{\mathbf{r}}, \alpha) + (A^2/8)W''(\bar{\mathbf{r}}, \alpha) - (A^3/48)W'''(\bar{\mathbf{r}}, \alpha) + (A^4/128)W''''(\bar{\mathbf{r}}, \alpha) + \cdots]\,, \tag{15}$$

where W'', W''', etc., are second, third, etc. derivatives of W taken with respect to r in the direction $\mathbf{r}$. We expect this expansion to converge well if W does not vary too rapidly with $\mathbf{r}$. Defining the conditional configuration average $\langle f\rangle$ of any function $f(\mathbf{r}, \alpha)$, for a given value of $\mathbf{r}$, by

$$\langle f\rangle = \frac{\int \exp\,[-\beta W(\mathbf{r}, \alpha)]f(\mathbf{r}, \alpha)d\alpha}{\int \exp\,[-\beta W(\mathbf{r}, \alpha)]d\alpha}\,, \tag{16}$$

and using equations (15) and (11) in equation (9), we obtain

$$E = \{\exp\,[-\beta w(\bar{r})]\}\langle \exp\,[-\beta(A^2/8)W''(\bar{\mathbf{r}}, \alpha) + \beta(A^3/48)W'''(\bar{\mathbf{r}}, \alpha) - \beta(A^4/128)W''''(\bar{\mathbf{r}}, \alpha) + \cdots]\rangle\,. \tag{17}$$

Finally, we use the cumulant expansion

$$\langle \exp f\rangle = \exp\,[\langle f\rangle + (1/2)(\langle f^2\rangle - \langle f\rangle^2) + \cdots] \tag{18}$$

for rewriting equation (17) as

$$\begin{aligned} E = \exp\{&-\beta w(\bar{r}) - \beta(A^2/8)\langle W''(\bar{\mathbf{r}}, \alpha)\rangle + \beta(A^3/48)\langle W'''(\bar{\mathbf{r}}, \alpha)\rangle \\ &- \beta(A^4/128)\langle W''''(\bar{\mathbf{r}}, \alpha)\rangle + \beta^2(A^4/128)[\langle W''(\bar{\mathbf{r}}, \alpha)^2\rangle - \langle W''(\bar{\mathbf{r}}, \alpha)\rangle^2] + \cdots\}\,. \end{aligned} \tag{19}$$

c) *Improved Perturbation Theory*

At low densities, we could use first-order perturbation theory for W; at higher densities, this is no longer legitimate and equation (9) is not valid. However, since the mean potential w defined by equation (11) provides the leading term of equation (19), we may hope to improve the convergence by computing the action, exactly, this time, with w, and using first-order perturbation theory for the *difference* $W - w$. It will be confirmed in the Appendix that this modified first-order perturbation theory is indeed a sufficient approximation.

We call s the exact action associated with w:

$$s = \int_0^{\beta\hbar} [(M/4)\dot{r}^2 + (Ze)^2/r + w(r)]dt\,, \tag{20}$$

where $r(t)$ is the exact trajectory which minimizes s. We rewrite equation (6) as

$$E = \{\exp\,[-(s - S_0)/\hbar]\}(\Omega^2/Q)\int \exp\,[-(S - s)/\hbar]d\alpha\,, \tag{21}$$

and follow the same steps as previously for evaluating the integral in equation (21), still with the approximation of using the Kepler orbit (8) for computing $S - s$. We then find, instead of equation (19),

$$\begin{aligned} E = \exp\{&-(s - S_0)/\hbar - \beta(A^2/8)[\langle W''(\bar{\mathbf{r}}, \alpha)\rangle - w''(\bar{r})] \\ &+ \beta(A^3/48)[\langle W'''(\bar{\mathbf{r}}, \alpha)\rangle - w'''(\bar{r})] - \beta(A^4/128)[\langle W''''(\bar{\mathbf{r}}, \alpha) - w''''(\bar{r})] \\ &+ \beta^2(A^4/128)[\langle W''(\bar{\mathbf{r}}, \alpha)^2\rangle - \langle W''(\bar{\mathbf{r}}, \alpha)\rangle^2] \quad \cdots\}\,. \end{aligned} \tag{22}$$

Equation (22) gives the enhancement factor E in terms of classical quantities, the evaluation of which will be discussed in the next section.

IV. NUMERICAL COMPUTATIONS

a) *Actions*

The leading term of equation (22) involves the actions S_0 and s defined by equations (4) and (20). It is easy to compute S_0, using equations (4) and (8), with the result

$$S_0/\hbar = \tau = [27\pi^2\beta M(Ze)^4/4\hbar^2]^{1/3}; \tag{23}$$

$\exp(-S_0/\hbar)$ is the familiar barrier factor for a pure Coulomb potential.

The computation of s requires the knowledge of the mean potential $w(r)$. Numerical information on $w(r)$ can be obtained by analyzing computer results (Hansen 1973; Hansen, Torrie, and Vieillefosse 1977). In a large domain near and above $r = a$, $w(r)$ is approximately a linear function of r (DeWitt, Graboske, and Cooper 1973); the reaction rate enhancement calculation of Itoh, Totsuji, and Ichimaru (1977) is indeed based on a linear $w(r)$. Near the origin, however, $w(r)$ is a quadratic function of r, as shown in Paper I. Here, we use an overall approximate fit of $w(r)$, in the whole interval $0 \leq r \leq 1.6a$:

$$\beta w(r) = -C + \Gamma[(1/4)(r/a)^2 - 0.039(r/a)^4 + 0.0043(r/a)^6], \tag{24}$$

where C, as computed in Paper I, is, in the range $1 \leq \Gamma \leq 155$,

$$C = 1.0531\Gamma + 2.2931\Gamma^{1/4} - 0.5551 \ln \Gamma - 2.35. \tag{25}$$

Using the conservation of energy in the reversed potential, we can rewrite equation (20) as

$$s = 2\int_0^b \{M[(Ze)^2/r + w(r) - (Ze)^2/b - w(b)]\}^{1/2} dr + \beta\hbar[(Ze)^2/b + w(b)], \tag{26}$$

where b is the aphelion of the trajectory, which is to be determined by the minimization condition

$$\partial s/\partial b = 0 \tag{27}$$

(b will be of the order of magnitude of the unperturbed aphelion $2A$). When r/b is used in equation (26) as the integration variable, the integral can be expressed in terms of a dimensionless integral which depends on the parameter b/a; this integral has been numerically tabulated as a function of b/a. After b/a has been determined by equation (27), we obtain s as a function of Γ and τ. The result can be fitted, in the range $0 \leq 3\Gamma/\tau \leq 1.6$, by

$$-(s - S_0)/\hbar = C - (\tau/3)[(5/32)(3\Gamma/\tau)^3 - 0.014(3\Gamma/\tau)^4 - 0.0128(3\Gamma/\tau)^5]. \tag{28}$$

The coefficient 5/32 in equation (28) comes from an expansion with respect to $3\Gamma/\tau$; the other coefficients are empirical.

b) *Correction Terms*

The next term in equation (22) involves $\langle W'' \rangle - w''$. The value of $w''(3A/2)$ can easily be obtained from equation (24) [and is vanishingly small if $3A/2$ is in the linear region of $w(r)$]. The potential W is given by the explicit expression

$$\beta W(\mathbf{r}, \alpha) = \beta U(\alpha) + (\Gamma/4)(r/a)^2 + \Gamma a \sum_{j=3}^{N} [|\mathbf{r}_j - (\mathbf{r}/2)|^{-1} + |\mathbf{r}_j + (\mathbf{r}/2)|^{-1}], \tag{29}$$

where $U(\alpha)$ is the sum of the interactions which are independent of r and where $(\Gamma/4)(r/a)^2$ is the interaction of particles 1 and 2 with the background (it is convenient to take for the background a large sphere centered at the origin). The average value $\beta\langle W'' \rangle$, as defined by equation (16), is

$$\beta\langle W''(\bar{\mathbf{r}}, \alpha)\rangle = \Gamma/2a^2 + \frac{\Gamma a\rho}{2g_c(\bar{r})} \int g_3(-\mathbf{r}/2, \mathbf{r}/2, \mathbf{r}') \frac{3\{[(\bar{\mathbf{r}}/2) - \mathbf{r}']\cdot(\bar{\mathbf{r}}/\bar{r})\}^2 - [(\bar{\mathbf{r}}/2) - \mathbf{r}']^2}{|(\bar{\mathbf{r}}/2) - \mathbf{r}'|^5} d\mathbf{r}', \tag{30}$$

where g_3 is the three-body classical correlation function. The integral in equation (30) has been numerically computed for a few values of Γ, using the superposition approximation which expresses g_3 as a product of three pair correlation functions; these latter ones were taken from Hansen (1973). The results can be fitted by

$$-\beta(A^2/8)[\langle W''(\bar{\mathbf{r}}, \alpha)\rangle - w''(\bar{r})] = -\Gamma[0.0055(3\Gamma/\tau)^4 - 0.0098(3\Gamma/\tau)^5 + 0.0048(3\Gamma/\tau)^6]. \tag{31}$$

For $3\Gamma/\tau \leq 1.6$, equation (31) gives at most a correction of 2% to the leading term (eq. [28]). The following terms in equation (22) have been roughly estimated, and found to be smaller than a fraction of 1%.

TABLE 1

NUMERICAL EXAMPLES FOR ^{12}C AT $T = 10^8$ K ($\tau = 181.3$)*

Density (g cm^{-3})	Γ	$3\Gamma/\tau$	ln E	Enhancement Factor E
10^8.....	16.60	0.275	18.0–0.0	7×10^7
19^9.....	35.77	0.592	37.2–0.0	10^{16}
10^{10}.....	77.06	1.275	68.5–0.2	5×10^{29}
2×10^{10}.....	97.08	1.606	79.3–1.4	7×10^{33}

* In the column labeled ln E, the main and correction terms of equation (32) are separately displayed.

V. SUMMARY AND DISCUSSION

Our conclusion is that, in the range $0 \leq 3\Gamma/\tau \leq 1.6$, the enhancement factor is given by

$$E = \exp\{-(s - S_0)/\hbar - \beta(A^2/8)[\langle W''(\bar{\mathbf{r}}, \alpha)\rangle - w''(\bar{r})]\}, \tag{32}$$

within a precision of the order of 1% for the exponent. The first term of equation (32), which is the main one, is represented by equations (28) and (25); the second term, which is a correction, is represented by equation (31). A few numerical examples for ^{12}C are given in Table 1.

In Paper I, we made an expansion of $W(\mathbf{r}, \alpha)$ around $\mathbf{r} = 0$. Such an expansion is useful only for small values of $2A/a = 3\Gamma/\tau$. Here the expansion has been made around $r = 3A/2$; this is more appropriate to higher values of $2A/a$. In the limiting case of small values of $3\Gamma/\tau$, it is enough to keep in equation (32) the leading terms of equation (28), which give

$$E = \exp[C - (45/32)\Gamma^3/\tau^2], \tag{33}$$

and we recover the exponential of equation (35) of Paper I. The further truncation of equation (33) into $E = \exp C$ is essentially equivalent to the original approximation of Salpeter (1954).

In the present paper, we have neglected the quantum effects associated with the coordinates $\mathbf{R}$ and α, and we have used the classical limit for the partition function. Furthermore, we have approximated each path integral by the exponential of the action, although the complete semiclassical approximation would also bring a multiplicative factor involving second derivatives of the action. The leading contributions from these effects, neglected here, have been computed in Paper I. They contribute to the enhancement factor E the multiplicative factor of equation (35) of Paper I:

$$1 - [(35/18\pi) - (\pi/6)]\hbar^2\beta^2(Ze)^2\rho/M = 1 - 0.0562(3\Gamma/\tau)^3. \tag{34}$$

Even in the extreme case $3\Gamma/\tau = 1.6$, these effects only lower E by 23%, a negligible correction compared to the uncertainty in the *exponent* in equation (32).

Our numerical results are sometimes not drastically different from those of Itoh, Totsuji, and Ichimaru (1977) (although this approximate agreement is partly due to some cancellation between several causes of divergence). The main achievement of the present paper is the consideration of the problem of nuclear reaction rate enhancement from first principles, starting from its formulation as a quantum-mechanical many-body problem. There are many ways of defining an effective two-body potential, and there is no *a priori* reason for choosing the classical potential of mean force. We have devised a systematic method of successive approximations, in which a nuclear reaction rate based on the potential of mean force appears only as a first approximation; it is a fortunate simplifying feature that the corrections to that first approximation turn out to be rather small, for the ranges of parameters which are considered here. It must be noted, however, that our method of approximation is basically an expansion in the neighborhood of the classical limit; this expansion is not expected to converge for too-large values of the parameter $2A/a$, i.e., for too-high densities or too-low temperatures.

We are indebted to E. Schatzman for his continuous interest in our work, and to J. P. Hansen for having kindly made available to us unpublished tables of $g_c(r)$.

APPENDIX

HIGHER ORDER PERTURBATION THEORY

In the main text, we have computed the integral in equation (21) by using first-order perturbation theory for the actions S and s, i.e., by evaluating these actions along the unperturbed Kepler orbit (8). Here we check the validity of this approximation by estimating the second-order perturbation-theory term.

The exact action s is given by equation (26). For a simple estimate of the second-order perturbation-theory term, it is enough to replace $w(r)$ by its linear approximation

$$w(r) = w(\bar{r}) + w'(\bar{r})(r - \bar{r}) , \tag{35}$$

and then to make a straightforward expansion of equation (26) with respect to w, up to the second order. After b has been determined by equation (27), to the same order, one finds

$$s/\hbar = \tau + \beta w(\bar{r}) - (5/64)(2A/a)^3\Gamma^{-1}[\beta a w'(\bar{r})]^2 + \cdots . \tag{36}$$

In principle, a similar calculation cannot be done for the action S, because the trajectory which minimizes S is three-dimensional. The conservation of energy is not sufficient to allow S to be expressed as an integral on the potential similar to (26). However, since we are interested only in an order-of-magnitude estimate, we neglect the departure of the trajectory from a straight line, and obtain for S an expression similar to (36), with $w'(\bar{r})$ replaced by $W'(\bar{\mathbf{r}}, \alpha)$.

We then find for the correction term of equation (21)

$$(\Omega^2/Q)\int \exp[-(S - s)/\hbar]d\alpha = \exp\{(5/64)(2A/a)^3\Gamma^{-1}\beta^2a^2[\langle W'(\bar{\mathbf{r}}, \alpha)^2\rangle - w'(\bar{r})^2]\} , \tag{37}$$

where we have used the cumulant expansion (18) limited to its first term. By deriving equation (11) twice with respect to r, we obtain the identity

$$\beta[\langle W'^2\rangle - w'^2] = \langle W''\rangle - w'' , \tag{38}$$

and the second-order perturbation-theory correction (37) is equal to

$$\exp\{(5/8)A^3a^{-1}\Gamma^{-1}\beta[\langle W''(\bar{\mathbf{r}}, \alpha)\rangle - w''(\bar{r})]\} . \tag{39}$$

The exponent in equation (39) is smaller than equation (31) by a factor $5A/a\Gamma$; equation (31) was already a small correction, and therefore equation (39) can be safely neglected.

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