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MAGNETIC PROPERTIES OF A NEARLY CLASSICAL ONE-COMPONENT PLASMA IN THREE OR TWO DIMENSIONS

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The magnetic properties of the one-component plasma, in three and two dimensions, are studied in the nearly classical case. Since magnetism is an essentially quantum effect, Planck's constant $\hbar$ can be used as a small parameter. A generalized Wigner–Kirkwood expansion in powers of $\hbar^2$ is derived. This expansion is used for computing the induced magnetization. The displacement of the liquid–solid phase transition of the model, when a magnetic field is applied, is discussed. The model is applicable to electrons deposited at the surface of liquid helium.

1. Introduction and summary

The one-component plasma, also called "jellium", is a system of identical particles of charge e and mass M embedded in a uniform neutralizing background of opposite charge. In the present paper, we study this model under nearly classical conditions, i.e. when quantum effects are small. In such a case, the model is a reasonable description for at least two physical systems. In the skies, the one-component plasma represents certain states of stellar matter, where the nuclei are the particles and a sea of degenerate electrons forms the background. In the laboratory, the two-dimensional version of the model represents electrons deposited on the surface of liquid helium[1]).

This paper deals with the equilibrium statistical mechanics of the one-component three- or two-dimensional plasma in a magnetic field, in the nearly classical case. This is an especially simple situation, because magnetism does not exist in classical physics; therefore, in the *nearly* classical case, Planck's constant $\hbar$ provides a small parameter for the theory of magnetism, even in a system with strong interactions which would be intractable in the fully quantum case. What is meant by "nearly classical" can be defined as follows. The state of the plasma is characterized by its temperature T (or alternatively $\beta = 1/kT$, where k is Boltzmann's constant), and its number density ρ in the three-dimensional case (in the two-dimensional case, the number *surface*

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density ρ is to be used). It is customary to define two classical lengths, a two-body average classical distance of closest approach βe^2, and an average interparticle distance which is $a = (3/4\pi\rho)^{1/3}$ in the three-dimensional case (and $a = (\pi\rho)^{-1/2}$ in the two-dimensional case); a commonly used dimensionless coupling parameter is $\Gamma = \beta e^2/a$. If a magnetic field $\boldsymbol{B}$ is present, a third classical length appears, which can be chosen as an average gyration radius $R = (Mc^2/\beta e^2 B^2)^{1/2}$. Quantum effects are related to the thermal de Broglie wavelength $\lambda = (2\pi\hbar^2\beta/M)^{1/2}$. A nearly classical plasma is now defined as being such that the dimensionless parameters $\lambda/\beta e^2$, λ/a, λ/R are sufficiently small (note that the condition that $\lambda/\beta e^2$ be small is a *low*-temperature requirement). There is no restriction however on the relative values of βe^2, a, R; we shall be especially interested in the non-trivial strongly-coupled case $\Gamma \gg 1$.

The equilibrium properties of nearly classical systems can be expanded in powers of $\hbar^2$; this is the well-known Wigner–Kirkwood expansion. In section 2, the Wigner–Kirkwood expansion is generalized to the case when a magnetic field is present. The result is used in section 3 for computing the induced magnetization of a nearly classical one-component plasma, and in section 4 for discussing the possible influence of a magnetic field on the liquid–solid phase transition. The exchange effects, which are not taken into account by the Wigner–Kirkwood expansion, are studied in section 5; they are found to be negligible. Possible applications of the theory are discussed in section 6.

2. Wigner–Kirkwood expansion

The Wigner–Kirkwood expansion can be obtained in a variety of ways. Here we use the Laplace transform method[2]), which is more convenient than the original derivations[3,4]), and we work out a generalization to the case when a uniform magnetic field $\boldsymbol{B}$ is present.

We deal with the most general case of a system of N particles of charge e. The system may be two- or three-dimensional; we call the dimensionality d. In the two-dimensional case, the field $\boldsymbol{B}$ is assumed to be normal to the plane of the system. The particles are submitted to some scalar potential V which includes their mutual interactions. For brevity, we call the dN-dimensional position vector in configuration space $\boldsymbol{r} = (\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N)$, the dN-dimensional gradient $\boldsymbol{\nabla}$, and $\boldsymbol{A}$ is a dN-dimensional vector potential, the components of which are $(1/2)\boldsymbol{B} \wedge \boldsymbol{r}_1, (1/2)\boldsymbol{B} \wedge \boldsymbol{r}_2, \ldots, (1/2)\boldsymbol{B} \wedge \boldsymbol{r}_N$.

The Hamiltonian is

$$H = \frac{1}{2M}\left(-i\hbar\boldsymbol{\nabla} - \frac{e}{c}\boldsymbol{A}\right)^2 + V(\boldsymbol{r}). \tag{2.1}$$

The spin magnetic moments are not taken into account in (2.1); they will be considered later.

We want to compute the density in configuration space, which can be written as an integral of the Wigner distribution:

$$\langle \boldsymbol{r}|\,\mathrm{e}^{-\beta H}|\boldsymbol{r}\rangle = \int \left[\frac{\mathrm{d}\boldsymbol{p}}{(2\pi\hbar)^{dN}}\,\mathrm{e}^{-(\mathrm{i}/\hbar)\boldsymbol{p}\cdot\boldsymbol{r}}\,\mathrm{e}^{-\beta H}\,\mathrm{e}^{(\mathrm{i}/\hbar)\boldsymbol{p}\cdot\boldsymbol{r}}\right] \tag{2.2}$$

(note that the possible exchanges between the particles are neglected in this expression for the density). It is convenient to work with the Laplace transformed operator

$$(H+z)^{-1} = \int_0^\infty \mathrm{d}\beta\,\mathrm{e}^{-\beta z}\,\mathrm{e}^{-\beta H}. \tag{2.3}$$

In order to generate an expansion in powers of $\hbar$, we write

$$H + z = D + \mathcal{O}, \tag{2.4}$$

where D is a c-number

$$D = \frac{1}{2M}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2 + V(\boldsymbol{r}) + z \tag{2.5}$$

and

$$\mathcal{O} = \frac{1}{2M}\left(-\mathrm{i}\hbar\nabla - \frac{e}{c}\boldsymbol{A}\right)^2 - \frac{1}{2M}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2; \tag{2.6}$$

and expand $(H+z)^{-1}$ as

$$(H+z)^{-1} = D^{-1} - D^{-1}\mathcal{O}D^{-1} + D^{-1}\mathcal{O}D^{-1}\mathcal{O}D^{-1} - \cdots. \tag{2.7}$$

Since, for any function $f(\boldsymbol{r})$,

$$\mathcal{O}[f(\boldsymbol{r})\,\mathrm{e}^{(\mathrm{i}/\hbar)\boldsymbol{p}\cdot\boldsymbol{r}}] = -\,\mathrm{e}^{(\mathrm{i}/\hbar)\boldsymbol{p}\cdot\boldsymbol{r}}\left[\frac{\mathrm{i}\hbar}{M}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)\cdot\nabla + \frac{\hbar^2}{2M}\nabla^2\right]f(\boldsymbol{r}), \tag{2.8}$$

the term of order n in $\mathcal{O}$ in (2.7) provides a term of order $\hbar^n$ in the expansion

$$\mathrm{e}^{-(\mathrm{i}/\hbar)\boldsymbol{p}\cdot\boldsymbol{r}}(H+z)^{-1}\,\mathrm{e}^{(\mathrm{i}/\hbar)\boldsymbol{p}\cdot\boldsymbol{r}}$$

$$= D^{-1}\sum_{n=0}^{\infty}\left\{\left[\frac{\mathrm{i}\hbar}{M}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)\cdot\nabla + \frac{\hbar^2}{2M}\nabla^2\right]D^{-1}\right\}^n \tag{2.9}$$

which can be truncated at some given order n. We now carry out the differentiations in (2.9), take the inverse Laplace-transform, and integrate on the momentum variables $\boldsymbol{p}$ (actually, it is more convenient to use the shifted variables $\boldsymbol{\pi} = \boldsymbol{p} - (e/c)\boldsymbol{A}$). Since $\hbar$ comes as $\mathrm{i}\hbar$ in (2.9) and the density (2.2) is real, the odd powers of $\hbar$ necessarily cancel out; furthermore, the odd powers of π give no contribution after the momentum integration is performed. However, one must be careful to carry out the differentiations before dropping the odd powers of π, because the operator ∇ may act upon π. The calculation is straightforward, but very tedious. To order $\hbar^4$, the result can be rewritten as an exponential

$$\begin{aligned}\langle \boldsymbol{r}|\,\mathrm{e}^{-\beta H}|\boldsymbol{r}\rangle = \frac{1}{\lambda^{dN}}\exp\Big\{-\beta V(\boldsymbol{r}) \\ + \hbar^2\Big[-\frac{\beta^2}{12M}\nabla^2 V + \frac{\beta^3}{24M}(\nabla V)^2 - \frac{N\beta^2 e^2 B^2}{24M^2c^2}\Big] \\ + \frac{\hbar^4}{4M^2}\frac{1}{6!}\Big[-12\beta^3(\nabla^2)^2 V + 16\beta^4\nabla V\cdot\nabla(\nabla^2 V) + 4\beta^4\nabla^2(\nabla V)^2 \\ - 6\beta^5\nabla V\cdot\nabla(\nabla V)^2 + \frac{N\beta^4 e^4 B^4}{M^2c^4} + 4\frac{\beta^4 e^2 B^2}{Mc^2}\nabla_\perp^2 V \\ - 2\frac{\beta^5 e^2 B^2}{Mc^2}(\nabla_\perp V)^2\Big] + \cdots\Big\},\end{aligned} \tag{2.10}$$

where $\nabla_\perp$ means that only the components of ∇ normal to $\boldsymbol{B}$ must be kept (in two dimensions, $\nabla_\perp$ is identical with ∇). In the special case $\boldsymbol{B} = 0$, (2.10) is equivalent to a known[5]) expansion of the density.

It is interesting to note the structure of (2.10): the term of order $\hbar^{2n}$ is an homogeneous polynomial of order $2n$ in $\boldsymbol{B}$ and ∇.

For calculating the free energy, we define the classical average of a function $f(\boldsymbol{r})$ as

$$\langle f\rangle = \frac{\int \mathrm{e}^{-\beta V(\boldsymbol{r})} f(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}}{\int \mathrm{e}^{-\beta V(\boldsymbol{r})}\,\mathrm{d}\boldsymbol{r}} \tag{2.11}$$

and write the partition function as

$$Z = \frac{1}{N!}\int \langle \boldsymbol{r}|\mathrm{e}^{-\beta H}|\boldsymbol{r}\rangle\,\mathrm{d}\boldsymbol{r} = \frac{1}{N!\lambda^{dN}}\Big[\int \mathrm{e}^{-\beta V(\boldsymbol{r})}\,\mathrm{d}\boldsymbol{r}\Big]\langle \mathrm{e}^f\rangle, \tag{2.12}$$

where $f(\boldsymbol{r})$ is the argument of the exponential in (2.10) without its first term $-\beta V(\boldsymbol{r})$. Using the cumulant expansion

$$\langle \mathrm{e}^f\rangle = \mathrm{e}^{\langle f\rangle + \frac{1}{2}(\langle f^2\rangle - \langle f\rangle^2) + \cdots}, \tag{2.13}$$

we obtain for the free energy, to order $\hbar^4$,

$$F = -kT\ln Z = F_c - kT[\langle f\rangle + \tfrac{1}{2}(\langle f^2\rangle - \langle f\rangle^2) + \cdots], \tag{2.14}$$

where F_c is the classical free energy

$$F_c = -kT \ln\left[\frac{1}{N!\lambda^{dN}} \int e^{-\beta V(r)} \, d\boldsymbol{r}\right]. \tag{2.15}$$

The averages in (2.14) can be expressed in a variety of ways, by integrations by parts such as

$$\beta \int e^{-\beta V} (\nabla V)^2 \, d\boldsymbol{r} = \int e^{-\beta V} (\nabla^2 V) \, d\boldsymbol{r}. \tag{2.16}$$

A possible final form for the free energy is

$$\begin{aligned} F = F_c &+ \hbar^2 \left[\frac{\beta}{24M} \langle \nabla^2 V \rangle + \frac{N\beta e^2 B^2}{24 M^2 c^2}\right] \\ &+ \frac{\hbar^4}{4M^2} \frac{1}{2 \cdot 6!} \Big[5\beta^2 \langle (\nabla^2)^2 V \rangle - 2\beta^3 \langle \nabla\nabla V : \nabla\nabla V \rangle \\ &- 5\beta^3 \langle (\nabla^2 V)^2 \rangle + 5\beta^3 \langle \nabla^2 V \rangle^2 - 2 \frac{N\beta^3 e^4 B^4}{M^2 c^4} \\ &- 4 \frac{\beta^3 e^2 B^2}{M c^2} \langle \nabla_\perp^2 V \rangle \Big] + \cdots. \end{aligned} \tag{2.17}$$

3. Magnetization and magnetic susceptibility

3.1. Three-dimensional plasma

For a three-dimensional one-component plasma,

$$\nabla^2 V = 4\pi N e^2 \rho - 4\pi e^2 \sum_{i \neq j} \delta(\boldsymbol{r}_{ij}). \tag{3.1}$$

The first term in (3.1) comes from the particle-background interactions, the second term from the particle–particle interactions; this second term actually plays no role, because it is always weighted by a classical Boltzmann factor which vanishes at zero interparticle distance. The replacement of $\nabla^2 V$ by $4\pi N e^2 \rho$ markedly simplifies (2.17). With no magnetic field present, (2.17) has been used for a numerical evaluation of the free energy[6]). Here, we want to compute the magnetization. Since, in the case of a fluid or a solid of cubic symmetry,

$$\langle \nabla_\perp^2 V \rangle = (2/3) \langle \nabla^2 V \rangle = (8/3) \pi N e^2 \rho, \tag{3.2}$$

the orbital part of the magnetization density is in those cases

$$\mathcal{M}_o = -\frac{\rho}{N}\frac{\partial F}{\partial B}$$
$$= -\frac{\hbar^2\beta e^2\rho B}{12M^2c^2}\left[1 - \frac{\hbar^2\beta^2e^2B^2}{60M^2c^2} - \frac{2\pi\hbar^2\beta^2e^2\rho}{45M} + \cdots\right], \tag{3.3}$$

and the zero-field orbital magnetic susceptibility is

$$\chi_o = \left(\frac{\partial \mathcal{M}_o}{\partial B}\right)_{B=0}$$
$$= -\frac{\hbar^2\beta e^2\rho}{12M^2c^2}\left[1 - \frac{2\pi\hbar^2\beta^2e^2\rho}{45M} + \cdots\right]. \tag{3.4}$$

These results have two remarkable features:

a) The leading term, of order $\hbar^2$, does not depend on the interaction V. Therefore, *as the classical limit is approached, the magnetic susceptibility and the magnetization are the same as for a system of non-interacting particles.*

b) The Coulomb interaction provides a correction of order $\hbar^4$. However, within that order, this correction is independent of the structure; it makes no difference whether the plasma is a fluid or a Wigner solid. The structure would manifest itself only in the next term, of order $\hbar^6$, the calculation of which would require knowing correlation functions for evaluating the relevant averages of derivatives of the interaction.

3.2. Two-dimensional plasma

In two dimensions, the Laplacian of $1/r$ is $1/r^3$, and there is no contribution from the background in the limit of an infinite system. Therefore,

$$\langle\nabla^2 V\rangle = \langle\nabla_\perp^2 V\rangle = \left\langle\sum_{i\neq j}\frac{e^2}{r_{ij}^3}\right\rangle$$
$$= Ne^2\rho\int_0^\infty \frac{1}{r^3}g(r)2\pi r\,\mathrm{d}r, \tag{3.5}$$

where $g(r)$ is the classical pair distribution function. We now obtain for the orbital magnetization per unit surface

$$\mathcal{M}_o = -\frac{\hbar^2\beta e^2\rho B}{12M^2c^2}\left[1 - \frac{\hbar^2\beta^2e^2B^2}{60M^2c^2} - \frac{\alpha\hbar^2\beta^2e^2}{30Ma^3} + \cdots\right], \tag{3.6}$$

where the dimensionless number α is

$$\alpha = a\int_0^\infty \frac{g(r)}{r^2}\mathrm{d}r; \tag{3.7}$$

the zero field orbital magnetic susceptibility per unit surface is

$$\chi_o = -\frac{\hbar^2\beta e^2\rho}{12M^2c^2}\left[1 - \frac{\alpha\hbar^2\beta^2e^2}{30Ma^3} + \cdots\right]. \tag{3.8}$$

The number α depends on the coupling parameter $\Gamma = \beta e^2/a$. For very small values of Γ,

$$g(r) \sim \exp[-\beta e^2/r], \tag{3.9}$$

and we find from (3.7)

$$\alpha = a/\beta e^2 = \Gamma^{-1}. \tag{3.10}$$

For larger values of Γ, we use in (3.7) pair distribution functions $g(r)$ obtained by numerical resolution of the hypernetted chain equation[7,8,11]); these pair distribution functions are in fair agreement with those obtained by computer simulation[9,10,11]). We find $\alpha = 1.782$ for $\Gamma = 1$, $\alpha = 0.981$ for $\Gamma = 10$, $\alpha = 0.834$ for $\Gamma = 100$. Finally, for very large values of Γ, a lattice model should give good results; if we assume that the particles are located on a triangular lattice, we can compute α by an Ewald sum method, with the result $\alpha = 0.796$.

Again, in the present two-dimensional case, the interactions do not affect the magnetic properties at order $\hbar^2$, and provide a correction at order $\hbar^4$. Now this correction is, in principle, no longer independent of the structure. However, the value $\alpha = 0.834$ obtained for the fluid model at $\Gamma = 100$ actually differs very little from the value $\alpha = 0.796$ obtained for a perfect lattice model.

As discussed in section 4, a fluid–solid phase transition occurs in the neighborhood of $\Gamma = 100$. Our results indicate that the fluid and the solid should have very similar magnetic properties; this is essentially because the structure is already very solid-like in the fluid near the transition. Therefore a measurement of the magnetization is *not* a proper tool for distinguishing between the solid and the liquid*.

3.3. Spin magnetism

If the particles carry a spin magnetic moment, it will contribute to the magnetization. In the nearly classical limit, when exchange effects are neglected, the spin behave as independent magnetic moments. For particles of spin $\frac{1}{2}$ and intrinsic magnetic moment μ, the spin magnetization density is

$$\mathcal{M}_s = \rho\mu \tanh \beta\mu B, \tag{3.11}$$

*Other authors[12]) have reached the opposite conclusion. However, they have treated the liquid phase as a perfect gas, while we take into account the interactions on an equal footing in both solid and liquid phases.

in three as well as in two dimensions, and the zero-field spin magnetic susceptibility is

$$\chi_s = \rho\beta\mu^2. \tag{3.12}$$

These spin contributions must be added to the orbital contributions (3.3), (3.4), or (3.6), (3.8). For electrons, μ is of course the Bohr magneton $e\hbar/2Mc$.

4. Fluid–solid phase transition

As Γ is increased, the classical one-component plasma undergoes a phase transition from a fluid state to a crystalline state, the Wigner solid. In three dimensions, a computer simulation[13]) indicates that the transition occurs at $\Gamma = 155 \pm 10$. In two dimensions, a computer simulation[9]) has located the transition at $\Gamma = 95 \pm 2$, and a more recent one[11]) at $\Gamma = 125 \pm 15$. Very recently, the experimental discovery of this transition for electrons at the surface of liquid helium, at $\Gamma = 137 \pm 15$, has been announced[14]). Several authors have pointed out that a magnetic field enhances the tendency to form a crystal[15]); however, they mainly considered the zero-temperature quantum system. Here, we discuss the nearly classical case.

4.1. Free energies

The conventional way of locating a phase transition is to compare the free energies in both phases. It results from section 3 that, in the three-dimensional case, an applied magnetic field shifts the free energy by an amount which is independent of the structure, within order $\hbar^4$; therefore, within that order, the free energy shift is the same for both phases, and the transition is not displaced. The same result holds, at least approximately, in the two-dimensional case, because, although the free energy depends on the structure at order $\hbar^4$, the structures of both phases are very similar near the transition.

One would have to go beyond order $\hbar^4$ to see, through the free energies, a displacement of the transition by a magnetic field.

4.2. Lindemann criterion

Another possible approach to the fluid–solid transition problem is to use the Lindemann criterion, which says that the solid melts when the root mean square displacement of a particle is some definite fraction of the average interparticle distance. This criterion is not well founded from a theoretical point of view, since it uses no information at all about the liquid phase;

however the Lindemann criterion turns out to be often a useful semi-quantitative empirical rule.

The mean square displacement of a particle in a crystal can be computed by a phonon analysis[15]). In the nearly classical limit, one can as well use the general expansion (2.10). In a crystal, in the harmonic approximation, the potential V has the general form

$$V = \tfrac{1}{2}\boldsymbol{u} \cdot \mathscr{A} \cdot \boldsymbol{u}, \tag{4.1}$$

where $\boldsymbol{u}$ is a $3N$ or $2N$-dimensional vector, the components of which are the cartesian components of the displacements of the N particles from their equilibrium positions; $\mathscr{A}$ is the $3N \times 3N$ or $2N \times 2N$ dynamical matrix. Using (4.1), we find for the argument of the exponential in (2.10) a constant plus a term

$$-\beta V' = -\tfrac{1}{2}\beta \boldsymbol{u} \cdot \mathscr{A}' \cdot \boldsymbol{u}, \tag{4.2}$$

where $\mathscr{A}'$ is the matrix

$$\mathscr{A}' = \mathscr{A} - \frac{\hbar^2\beta^2}{12M}\mathscr{A}^2 + \frac{\hbar^4\beta^4}{5!M^2}\mathscr{A}^3 + \frac{\hbar^4\beta^4 e^2 B^2}{6!M^3c^2}\mathscr{A}P_\perp\mathscr{A} \tag{4.3}$$

(in 3 dimensions $P_\perp$ is the projector on the plane normal to the field $\boldsymbol{B}$, in 2 dimensions $P_\perp$ is simply the unit matrix). The crystal vibrates like a classical crystal with a dynamical matrix $\mathscr{A}'$, the inverse of which is, within order $\hbar^4$,

$$\mathscr{A}'^{-1} = \mathscr{A}^{-1} + \frac{\hbar^2\beta^2}{12M}1 - \frac{\hbar^4\beta^4}{720M^2}\mathscr{A} - \frac{\hbar^4\beta^4 e^2 B^2}{6!M^3c^2}P_\perp + \cdots. \tag{4.4}$$

The mean square displacement of a particle is

$$\langle u_i^2\rangle = \frac{1}{N\beta}\operatorname{Tr}\mathscr{A}'^{-1}. \tag{4.5}$$

In three dimensions,

$$\operatorname{Tr} 1 = 3\mathrm{N}, \quad \operatorname{Tr}\mathscr{A} = \nabla^2 V = 4\pi N e^2 \rho, \quad \operatorname{Tr} P_\perp = 2N, \tag{4.6}$$

and we find from (4.4) and (4.5)

$$\langle u_i^2\rangle = \langle u_i^2\rangle_c + \frac{\hbar^2\beta}{4M} - \frac{\pi\hbar^4\beta^3 e^2\rho}{180M^2} - \frac{\hbar^4\beta^3 e^2 B^2}{360M^3c^2} + \cdots, \tag{4.7}$$

where $\langle u_i^2\rangle_c$ is the mean square displacement in the classical limit.

In two dimensions,

$$\operatorname{Tr} 1 = \operatorname{Tr} P_\perp = 2N, \quad \operatorname{Tr}\mathscr{A} = \nabla^2 V = \sum_{i\neq j}\frac{e^2}{R_{ij}^3} = \frac{2\alpha N e^2}{a^3}, \tag{4.8}$$

where R_{ij} is the distance between the lattice sites i and j; the lattice sum has already been considered in section 3.2, with the result $\alpha = 0.796$. We now find from (4.4) and (4.5)

$$\langle u_i^2 \rangle = \langle u_i^2 \rangle_c + \frac{\hbar^2 \beta}{6M} - \frac{\alpha \hbar^4 \beta^3 e^2}{360 M^2 a^3} - \frac{\hbar^4 \beta^3 e^2 B^2}{360 M^3 c^2} + \cdots. \tag{4.9}$$

The classical mean square displacement is divergent in two dimensions, for an infinite system. However, this divergence goes only as $\ln N$, and causes no serious trouble for a real finite physical system.

It is seen from (4.7) or (4.9) that, in three dimensions as well as in two dimensions, the presence of a magnetic field $\boldsymbol{B}$ causes some reduction in the mean square displacement of a particle, and therefore might decrease the value of Γ at which the transition occurs. Let us remark that the free energy approach predicted no effect at order $\hbar^4$, while the Lindemann criterion already indicates some effect at this order. Anyhow, the effect is necessarily very weak near the classical limit.

5. Exchange effects

In the present section, the exchange contributions to the magnetic susceptibility, which have been neglected up to now, will be shown to be indeed exponentially small. The reason for this is that free particles can be exchanged only when they approach one another at a distance of the order of the de Broglie wavelength λ, which is small in the nearly classical limit; in the interacting system, the Coulomb repulsion strongly inhibits such close encounters.

We compute the exchange free energy, following the same method as when there is no magnetic field[16]). For definiteness, we consider a system of charged fermions of spin $\frac{1}{2}$ and intrinsic magnetic moment μ, electrons for instance. It is now necessary to take the spins explicitly into account. We call the spin state of the ith particle σ_i ($\sigma_i = \pm 1$). An unsymmetrized state vector of the system can be labeled as $|\boldsymbol{r}_1\sigma_1\, \boldsymbol{r}_2\sigma_2 \ldots \boldsymbol{r}_N\sigma_N\rangle$. When a uniform magnetic field $\boldsymbol{B}$ is applied, the total hamiltonian is

$$\mathcal{H} = H - \mu B \sum_i \sigma_i, \tag{5.1}$$

where H is the spatial hamiltonian (2.1). The lowest-order exchange effects come from two-body exchanges (which occur only between particles in the same spin state), and it is enough to write for the partition function

$$Z = \frac{1}{N!} \sum_{\sigma_1 \ldots \sigma_N} \int \Big[\langle \boldsymbol{r}_1\sigma_1\boldsymbol{r}_2\sigma_2\boldsymbol{r}_3\sigma_3 \ldots \boldsymbol{r}_N\sigma_N |\, \mathrm{e}^{-\beta\mathcal{H}} | \boldsymbol{r}_1\sigma_1\boldsymbol{r}_2\sigma_2\boldsymbol{r}_3\sigma_3 \ldots \boldsymbol{r}_N\sigma_N \rangle$$

$$-\frac{N(N-1)}{2}\delta_{\sigma_1\sigma_2}\langle \boldsymbol{r}_2\sigma_2\boldsymbol{r}_1\sigma_1\boldsymbol{r}_3\sigma_3\ldots\boldsymbol{r}_N\sigma_N|\,\mathrm{e}^{-\beta\mathscr{H}}|\boldsymbol{r}_1\sigma_1\boldsymbol{r}_2\sigma_2\boldsymbol{r}_3\sigma_3\ldots\boldsymbol{r}_N\sigma_N\rangle\bigg]$$

$$\times\,\mathrm{d}\boldsymbol{r}_1\ldots\mathrm{d}\boldsymbol{r}_N$$

$$=\frac{1}{N!}\int\bigg[\langle \boldsymbol{r}_1\boldsymbol{r}_2\boldsymbol{r}_3\ldots\boldsymbol{r}_N|\,\mathrm{e}^{-\beta H}|\boldsymbol{r}_1\boldsymbol{r}_2\boldsymbol{r}_3\ldots\boldsymbol{r}_N\rangle(2\cosh\beta\mu B)^N$$

$$-\frac{N(N-1)}{2}\langle \boldsymbol{r}_2\boldsymbol{r}_1\boldsymbol{r}_3\ldots\boldsymbol{r}_N|\,\mathrm{e}^{-\beta H}|\boldsymbol{r}_1\boldsymbol{r}_2\boldsymbol{r}_3\ldots\boldsymbol{r}_N\rangle(2\cosh 2\beta\mu B)$$

$$\times(2\cosh\beta\mu B)^{N-2}\bigg]\,\mathrm{d}\boldsymbol{r}_1\ldots\mathrm{d}\boldsymbol{r}_N. \tag{5.2}$$

Expanding $\ln Z$ with respect to the exchange term, we obtain for the free energy

$$F=F_{\mathrm{d}}+F_{\mathrm{ex}}, \tag{5.3}$$

where the direct part F_{d} is given by

$$\beta F_{\mathrm{d}}=-\ln\bigg[\frac{1}{N!}(2\cosh\beta\mu B)^N \times\int\langle \boldsymbol{r}_1\boldsymbol{r}_2\boldsymbol{r}_3\ldots\boldsymbol{r}_N|\,\mathrm{e}^{-\beta H}|\boldsymbol{r}_1\boldsymbol{r}_2\boldsymbol{r}_3\ldots\boldsymbol{r}_N\rangle\bigg]\,\mathrm{d}\boldsymbol{r}_1\ldots\mathrm{d}\boldsymbol{r}_N \tag{5.4}$$

and the exchange part F_{ex} by

$$\beta F_{\mathrm{ex}}=\tfrac{1}{2}N(N-1)\frac{\cosh 2\beta\mu B}{2\cosh^2\beta\mu B}\times\frac{\int\langle \boldsymbol{r}_2\boldsymbol{r}_1\boldsymbol{r}_3\ldots\boldsymbol{r}_N|\,\mathrm{e}^{-\beta H}|\boldsymbol{r}_1\boldsymbol{r}_2\boldsymbol{r}_3\ldots\boldsymbol{r}_N\rangle\,\mathrm{d}\boldsymbol{r}_1\ldots\mathrm{d}\boldsymbol{r}_N}{\int\langle \boldsymbol{r}_1\boldsymbol{r}_2\boldsymbol{r}_3\ldots\boldsymbol{r}_N|\,\mathrm{e}^{-\beta H}|\boldsymbol{r}_1\boldsymbol{r}_2\boldsymbol{r}_3\ldots\boldsymbol{r}_N\rangle\,\mathrm{d}\boldsymbol{r}_1\ldots\mathrm{d}\boldsymbol{r}_N}. \tag{5.5}$$

The direct term (5.4) has been studied in section 2. We now turn to the exchange term (5.5).

In the nearly classical limit, the integral in the denominator of (5.5) can be replaced by Q/λ^{dN}, where Q is the classical configuration integral. For evaluating the exchange matrix element

$$\mathcal{N}=\langle \boldsymbol{r}_2\boldsymbol{r}_1\boldsymbol{r}_3\ldots\boldsymbol{r}_N|\,\mathrm{e}^{-\beta H}|\boldsymbol{r}_1\boldsymbol{r}_2\boldsymbol{r}_3\ldots\boldsymbol{r}_N\rangle \tag{5.6}$$

in the numerator of (5.5), it is enough to take into account the quantum effects only for the relative motion of those particles 1 and 2 which are exchanged. This relative motion is described by a reduced hamiltonian

$$H_{12}=\frac{1}{M}\Big(-\mathrm{i}\hbar\nabla_{12}-\frac{e}{4c}\boldsymbol{B}\wedge\boldsymbol{r}_{12}\Big)^2+\frac{e^2}{r_{12}}, \tag{5.7}$$

where ∇_{12} is the gradient with respect to the relative coordinate r_{12} (note that the coupling of the relative motion with the magnetic field is described by a reduced charge $e/2$). Particles 3 to N and the center of mass of particles 1 and 2 can be treated classically; in this limit, their kinetic energies contribute a factor $2^{d/2}\lambda^{-d(N-1)}$ to $\mathcal{N}$ as if no magnetic field was present. The total potential energy may be split as

$$V = \frac{e^2}{r_{12}} + W(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \ldots, \boldsymbol{r}_N); \tag{5.8}$$

in the nearly classical limit, $\mathcal{N}$ goes very quickly to zero as r_{12} increases, and the function W can be approximated by its value at $\boldsymbol{r}_1 = \boldsymbol{r}_2$, since it is regular there. Thus, we obtain

$$\mathcal{N} = 2^{d/2}\lambda^{-d(N-1)} \exp[-\beta W(\boldsymbol{r}_1, \boldsymbol{r}_1, \boldsymbol{r}_3, \ldots, \boldsymbol{r}_N]\langle -\boldsymbol{r}_{12}|\, \mathrm{e}^{-\beta H_{12}}|\boldsymbol{r}_{12}\rangle \tag{5.9}$$

and

$$\frac{\beta F_{\mathrm{ex}}}{N} = \rho(\lambda\sqrt{2})^d \frac{\cosh 2\beta\mu B}{4\cosh^2\beta\mu B}\, \mathrm{e}^C \int \langle -\boldsymbol{r}_{12}|\, \mathrm{e}^{-\beta H_{12}}|\boldsymbol{r}_{12}\rangle\, \mathrm{d}\boldsymbol{r}_{12}, \tag{5.10}$$

where C is a classical quantity, defined by

$$\rho^2\, \mathrm{e}^C = \frac{N(N-1)}{Q} \int \exp[-\beta W(\boldsymbol{r}_1, \boldsymbol{r}_1, \boldsymbol{r}_3, \ldots, \boldsymbol{r}_N)]\, \mathrm{d}\boldsymbol{r}_3 \ldots \mathrm{d}\boldsymbol{r}_N. \tag{5.11}$$

The quantity C is related to the short-range behavior of the classical pair distribution function

$$g(r) \underset{r\to 0}{\sim} \exp\left[-\frac{\beta e^2}{r} + C + \cdots\right]. \tag{5.12}$$

In (5.10), the many-body effects are taken care of by C. In three dimensions, a numerical evaluation of C as a function of the coupling parameter Γ has been previously obtained by an analysis[17]) of computer results about mixtures[18]); in the range $1 \leqslant \Gamma \leqslant 155$,

$$C = 1.0531\Gamma + 2.2931\Gamma^{1/4} - 0.5551 \ln \Gamma - 2.35. \tag{5.13}$$

This formula (5.13) is in good agreement with the simple estimate obtained from the ion-sphere model[19]),

$$C = \tfrac{9}{10}(2^{5/3} - 2)\Gamma = 1.0573\Gamma. \tag{5.14}$$

In two dimensions, there are no available computer results for deriving the analog of (5.13), and we must rely on the two-dimensional analog of the ion-sphere model which gives

$$C = (\sqrt{2} - 1)\left(4 - \frac{16}{3\pi}\right)\Gamma = 0.9537\Gamma. \tag{5.15}$$

The evaluation of the exchange free energy is now reduced to a two-body problem, the evaluation of the integral in (5.10). The free energy can be expanded in powers of the magnetic field; for calculating the zero-field susceptibility, it is enough to go to order B^2 for the free energy. From now on, for brevity, we denote by $\boldsymbol{r}$ the relative coordinate $\boldsymbol{r}_{12}$. The integral in (5.10) can be expanded as

$$\int \langle -\boldsymbol{r}|\,\mathrm{e}^{-\beta H_{12}}|\boldsymbol{r}\rangle\,\mathrm{d}\boldsymbol{r}$$

$$= \int \langle -\boldsymbol{r}|\,\mathrm{e}^{-\beta H_0}\left[1 + \frac{\beta e B}{2Mc}L_z + \frac{\beta^2 e^2 B^2}{8M^2c^2}L_z^2 - \frac{\beta e^2 B^2}{16Mc^2}r_\perp^2\right]|\boldsymbol{r}\rangle\,\mathrm{d}\boldsymbol{r}, \qquad (5.16)$$

where H_0 is the one-body pure Coulomb hamiltonian

$$H_0 = -\frac{\hbar^2}{M}\nabla^2 + \frac{e^2}{r}, \qquad (5.17)$$

L_z is the z-component of the angular momentum operator (the z-axis is taken along $\boldsymbol{B}$), $\boldsymbol{r}_\perp$ the component of $\boldsymbol{r}$ which is normal to $\boldsymbol{B}$. The term of order BL_z gives a vanishing contribution for symmetry reasons. To order B^2, we obtain from (5.10) and (5.16)

$$\frac{\beta F_{\mathrm{ex}}}{N} = \tfrac{1}{4}\rho(\lambda\sqrt{2})^d\,\mathrm{e}^C\left[I(1+\beta^2\mu^2B^2) + J\frac{\beta^2e^2}{8M^2c^2}B^2 - K\frac{\beta e^2}{16Mc^2}B^2\right], \qquad (5.18)$$

where

$$I = \int \langle -\boldsymbol{r}|\,\mathrm{e}^{-\beta H_0}|\boldsymbol{r}\rangle\,\mathrm{d}\boldsymbol{r}, \qquad (5.19a)$$

$$J = \int \langle -\boldsymbol{r}|\,\mathrm{e}^{-\beta H_0}\,L_z^2|\boldsymbol{r}\rangle\,\mathrm{d}\boldsymbol{r}, \qquad (5.19b)$$

$$K = \int \langle -\boldsymbol{r}|\,\mathrm{e}^{-\beta H_0}\,r_\perp^2|\boldsymbol{r}\rangle\,\mathrm{d}\boldsymbol{r}. \qquad (5.19c)$$

The integrals I, J, K are calculated in the Appendix. In the nearly classical limit, K is of a higher order in $\hbar$ than J, and can be neglected. Furthermore, the spin contribution, which is proportional to $I\mu^2$ (where μ is the Bohr magneton $e\hbar/2Mc$) is found to be of a higher order in $\hbar$ than the orbital contribution from J. Thus, the leading term in the exchange zero-field magnetic susceptibility comes from the orbital term L_z^2 and is

$$\chi_{\mathrm{ex}} = -\frac{\rho}{N}\left(\frac{\partial^2 F_{\mathrm{ex}}}{\partial B^2}\right)_{B=0}$$

$$= (1/24\pi\sqrt{3})\,\mathrm{e}^C\rho^2\lambda^{8/3}(\beta e^2)^{7/3}(e^2/Mc^2)\exp[-(27\pi^2\beta e^4M/16\hbar^2)^{1/3}] \qquad (5.20)$$

in three dimensions, and

$$\chi_{ex} = (\sqrt{2}/16\pi\sqrt{3})\, e^{C}\rho^2\lambda^{7/3}(\beta e^2)^{5/3}(e^2/Mc^2)$$
$$\times \exp[-(27\pi^2\beta e^4 M/16\hbar^2)^{1/3}] \qquad (5.21)$$

in two dimensions.

In both cases, χ_{ex} contains the exponential factor $\exp[-(27\pi^2\beta e^4 M/16\hbar^2)^{1/3}]$ and is negligible, in the nearly classical limit, when compared to the direct terms which involve power series of $\hbar^2$, as shown in section 3.

6. Possible applications and conclusion

6.1. Three-dimensional plasma

If we use in our formulas numbers appropriate to dense stellar matter, the magnetic effects for the nuclei turn out to be totally negligible. The magnetic susceptibility of the nearly classical nuclei is much smaller than the susceptibility of the degenerate electrons. As far as the fluid–solid phase transition is concerned, the Lindemann criterion does not indicate any appreciable displacement of the transition, even in those fields of 10^{12} or 10^{13} gauss which might exist in the solid crust of a neutron star[20]).

Perhaps measurable magnetic effects will be obtained in the future in one-component pure electron plasmas[21]).

6.2. Two-dimensional plasma

Magnetic effects might be more easily observed in two dimensions, for electrons at the surface of liquid helium, a system which seems to be a good example of a nearly classical one-component plasma. Typical numerical values are as follows. For $T = 0.5$ K, $\rho = 3\times 10^8\ \mathrm{cm}^{-2}$, the system should be near its phase transition since then $\Gamma = 103$; the characteristic lengths are $\beta e^2 = 3.34\times 10^{-3}$ cm, $a = 3.26\times 10^{-5}$ cm, $\lambda = 1.05\times 10^{-5}$ cm. From (3.8), the orbital susceptibility is found to be -1.17×10^{-16} cm, a value very close to the one for a system of free electrons, since the second term in the bracket of (3.8) is 0.05; it is clearly seen in this example how little the susceptibility is changed by the interactions, although they are very strong as measured by Γ. The reduction of the Lindemann ratio by the presence of a magnetic field is completely negligible; from (4.9), one finds that $\langle u_i^2\rangle/a^2$ diminishes only by 3×10^{-4} when a field $B = 10^4$ G is applied. Therefore the phase transition will not be displaced in any appreciable way by the field.

6.3. *Conclusion*

By studying nearly classical systems, one finds magnetic effects which are necessarily weak; this is the very reason for which these effects could be studied here by series expansions. In situations in which the quantum effects would be more important, our series will not converge well; their first terms might however be qualitatively useful, and serve as a starting point for more elaborate theories.

Appendix

In this Appendix, the integrals (5.19) are computed, through the use of the Coulomb time-independent Green's function[22]).

A.1. *Three dimensions*

The integral I has already been computed in a previous paper (where it was called A) [16]). Here, we use a slightly different approach, which will be more convenient for the other integrals. We write

$$\langle -\boldsymbol{r}|\exp(-\beta H_0)|\boldsymbol{r}\rangle = -\frac{1}{\pi}\,\mathrm{Im}\int_0^\infty \mathrm{d}(k^2)\exp(-\beta\hbar^2k^2/M)\left\langle -\boldsymbol{r}\left|\frac{1}{k^2-(M/\hbar^2)H_0}\right|\boldsymbol{r}\right\rangle \tag{A.1}$$

(where k is assumed to have an infinitesimal positive imaginary part). For the Green's function in (A.1), we use a partial wave expansion equivalent to eq. (1.10) in ref. 22 (since here we are dealing with a repulsive rather than attractive potential, the convergence difficulties of ref. 22 are not present, and we can use a real rather than a contour integral):

$$\langle \boldsymbol{r}_2|\frac{1}{k^2-(M/\hbar^2)H_0}|\boldsymbol{r}_1\rangle = -\frac{1}{4\pi(r_1r_2)^{1/2}}\int_1^\infty \mathrm{d}\xi(\xi+1)^{-\mathrm{i}\nu-1/2}(\xi-1)^{\mathrm{i}\nu-1/2}$$
$$\times\, \mathrm{e}^{\mathrm{i}k(r_1+r_2)\xi}\sum_{l=0}^\infty(2l+1)P_l(\cos\theta)I_{2l+1}(-2\mathrm{i}k(r_1r_2)^{1/2}(\xi^2-1)^{1/2}), \tag{A.2}$$

where θ is the angle between $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, $\nu = Me^2/2\hbar^2k$, P_l is a Legendre polynomial, I is a modified Bessel function. In the case $\boldsymbol{r}_1 = -\boldsymbol{r}_2 = \boldsymbol{r}$, the sum in (A.2) reduces to

$$\sum_{l=0}^\infty(-1)^l(2l+1)I_{2l+1}(-2\mathrm{i}kr(\xi^2-1)^{1/2}) = -\mathrm{i}kr(\xi^2-1)^{1/2}. \tag{A.3}$$

Using (A.2) and (A.3), and making the change of variable $\xi = \operatorname{cotanh}(t/2)$, we can compute the integral upon $\boldsymbol{r}$

$$\operatorname{Im} \int \langle -\boldsymbol{r}| \frac{1}{k^2 - (M/\hbar^2)H_0} |\boldsymbol{r}\rangle \, \mathrm{d}\boldsymbol{r} = \operatorname{Im} \frac{1}{4k^2} \int_1^\infty \frac{\mathrm{d}\xi}{\xi^3} \left(\frac{\xi - 1}{\xi + 1}\right)^{\mathrm{i}\nu}$$

$$= -\frac{1}{8k^2} \int_0^\infty \mathrm{d}t \sinh\frac{t}{2} \left(\cosh\frac{t}{2}\right)^{-3} \sin \nu t$$

$$= -\frac{\nu}{8k^2} \int_0^\infty \mathrm{d}t \left(\cosh\frac{t}{2}\right)^{-2} \cos \nu t = -\frac{\pi\nu^2}{4k^2 \sinh \pi\nu}. \tag{A.4}$$

From (A.1) and (A.4),

$$I = \int \langle -\boldsymbol{r}| \exp(-\beta H_0)|\boldsymbol{r}\rangle \, \mathrm{d}\boldsymbol{r} = (Me^2/4\hbar^2)^2$$

$$\times \int_0^\infty \mathrm{d}(k^2) k^{-4} \exp(-\beta\hbar^2 k^2/M)[\sinh(\pi Me^2/2\hbar^2 k)]^{-1}. \tag{A.5}$$

Finally, near the classical limit $\hbar \to 0$,

$$[\sinh(\pi Me^2/2\hbar^2 k)]^{-1} \sim 2 \exp(-\pi Me^2/2\hbar^2 k), \tag{A.6}$$

and the integral in (A.5) can be performed by the saddle-point method, with the result

$$I \sim (\beta e^4 M/3\pi\hbar^2)^{1/2} \exp[-(27\pi^2\beta e^4 M/16\hbar^2)^{1/3}]. \tag{A.7}$$

For the integral J, we use, instead of (A.2),

$$\int \langle -\boldsymbol{r}| \frac{1}{k^2 - (M/\hbar^2)H_0} L_z^2 |\boldsymbol{r}\rangle \, \mathrm{d}\boldsymbol{r} = \frac{1}{3} \int \langle -\boldsymbol{r}| \frac{1}{k^2 - (M/\hbar^2)H_0} L^2 |\boldsymbol{r}\rangle \, \mathrm{d}\boldsymbol{r}$$

$$= -\int \mathrm{d}\boldsymbol{r} \frac{\hbar^2}{12\pi r} \int_1^\infty \mathrm{d}\xi (\xi + 1)^{-\mathrm{i}\nu - 1/2} (\xi - 1)^{\mathrm{i}\nu - 1/2} \, \mathrm{e}^{2\mathrm{i}kr\xi}$$

$$\times \sum_{l=0}^\infty (-1)^l l(l+1)(2l+1) I_{2l+1}(-2\mathrm{i}kr(\xi^2 - 1)^{1/2}) \tag{A.8}$$

and the sum

$$\sum_{l=0}^\infty (-1)^l l(l+1)(2l+1) I_{2l+1}(-2\mathrm{i}kr(\xi^2 - 1)^{1/2}) = -\mathrm{i}k^3 r^3 (\xi^2 - 1)^{3/2}. \tag{A.9}$$

The remaining part of the calculation is similar to the previous one. The result is

$$J = \int \langle -\boldsymbol{r}| \exp(-\beta H_0) L_z^2 |\boldsymbol{r}\rangle \, \mathrm{d}\boldsymbol{r}$$
$$\sim -(2/27)^{1/2}\hbar^2(\beta e^4 M/2\pi\hbar^2)^{7/6} \exp[-(27\pi^2\beta e^4 M/16\hbar^2)^{1/3}]. \tag{A.10}$$

The integral K is obtained by similar calculations:

$$K = \int \langle -\boldsymbol{r}| \exp(-\beta H_0) r_\perp^2 |\boldsymbol{r}\rangle \, \mathrm{d}\boldsymbol{r}$$
$$\sim (2/3)^{3/2}\pi^{-2}(2\pi\hbar^2\beta/M)^{1/6}(\beta e^2)^{5/3} \exp[-(27\pi^2\beta e^4 M/16\hbar^2)^{1/3}]. \tag{A.11}$$

A.2. Two dimensions

In two dimensions, the equivalent of (A.2) is obtained by suitable modifications in the argument of ref. 22. One obtains

$$\langle \boldsymbol{r}_2| \frac{1}{k^2 - (M/\hbar^2)H_0} |\boldsymbol{r}_1\rangle = -(2\pi)^{-1} \int_1^\infty \mathrm{d}\xi (\xi+1)^{-\mathrm{i}\nu-1/2}(\xi-1)^{\mathrm{i}\nu-1/2}$$
$$\times \mathrm{e}^{\mathrm{i}k(r_1+r_2)\xi} \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m\theta} I_{2m}(-2\mathrm{i}k(r_1 r_2)^{1/2}(\xi^2-1)^{1/2}). \tag{A.12}$$

The sums which appear are now

$$\sum_{m=-\infty}^{\infty} (-)^m I_{2m}(-2\mathrm{i}kr(\xi^2-1)^{1/2}) = 1 \tag{A.13}$$

and

$$\sum_{m=-\infty}^{\infty} (-)^m m^2 I_{2m}(-2\mathrm{i}kr(\xi^2-1)^{1/2}) = -k^2r^2(\xi^2-1). \tag{A.14}$$

One finds

$$I \sim (4\beta e^4 M/27\pi\hbar^2)^{1/6} \exp[-(27\pi^2\beta e^4 M/16\hbar^2)^{1/3}], \tag{A.15}$$

$$J \sim -(2/3)^{1/2}\hbar^2(\beta e^4 M/2\pi\hbar^2)^{5/6} \exp[-(27\pi^2\beta e^4 M/16\hbar^2)^{1/3}], \tag{A.16}$$

$$K \sim (4\hbar^2\beta^3 e^4/3\pi^3 M)^{1/2} \exp[-(27\pi^2\beta e^4 M/16\hbar^2)^{1/3}]. \tag{A.17}$$

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