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The Two-Dimensional One-Component Plasma at $\Gamma = 2$: Behavior of Correlation Functions in Strip Geometry

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This paper considers a strip of two-dimensional one-component plasma of particles of charge q at a temperature T such that the coupling constant be $\Gamma = q^2/k_B T = 2$. The strip is of finite width and infinite length and bears charge densities on either edge. Inside the strip and on one side, the dielectric constant is 1; on the other side of the strip, it may be either 1 or 0 (in the latter case, image forces play an important role). The free energy as well as the one-particle and two-particle distribution functions can be exactly computed. They obey a variety of sum rules reflecting the Coulombic behavior of the system. At large separations the truncated two-particle distribution function behaves with algebraically decaying oscillations. The strip of finite width in fact is correlated along the strip much as a one-dimensional system is correlated.

KEY WORDS: Coulomb systems; plasmas; surface properties; strip geometry; correlations; sum rules.

1. INTRODUCTION

The recent development of exact results for the statistical mechanics of a two-dimensional one-component plasma[1-8] has provided the opportunity to test a variety of general theorems on systems with Coulomb interactions. This has led in turn to a variety of further developments in the theory of Coulombic systems,[8-12] particularly their properties near surfaces. It is of some interest then, to see how the properties of the two-dimensional one-component plasma change, when the system is confined to a strip of finite width, and also when image forces are present.

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A two-dimensional one-component plasma is described by a dimensionless coupling constant $\Gamma = q^2/k_B T$, where q is the charge of a particle, k_B Boltzmann's constant, and T the temperature.

One interesting property of Coulomb systems is the asymptotic behavior of the charge–charge correlation function. For the one-component plasma this is just the truncated two-particle distribution function

$$\rho^T_{(2)}(\mathbf{r}_1, \mathbf{r}_2) = \rho_{(2)}(\mathbf{r}_1, \mathbf{r}_2) - \rho_{(1)}(\mathbf{r}_1)\rho_{(1)}(\mathbf{r}_2) \tag{1.1}$$

In the bulk fluid phase, the correlations are expected to exhibit an exponential or faster decay. Such a decay has been rigorously proved for a two-dimensional one-component plasma at the special value of the coupling constant $\Gamma = 2$,[1] and at high temperature and low densities in all dimensions.[13]

Near a wall, longer-range correlations, which decay only algebraically, may occur. Consider a system confined to a half-space $x > 0$; the region $x < 0$ is filled with a continuum of dielectric constant ϵ_w, the dielectric constant of the background in $x > 0$ is chosen as $\epsilon_b = 1$. Let (x, y) be the coordinates of a particle [x measuring the distance from the wall and y being the coordinate(s) normal to x]. When $\epsilon_w = \epsilon_b = 1$, for a two-dimensional one-component plasma at $\Gamma = 2$[4,7] as well as in the weak-coupling limit ($\Gamma \to 0$) in ν dimensions ($\nu = 2, 3$),[14,7] $\rho^T_{(2)}(x_1, x_2; y_1 - y_2) \sim A(x_1, x_2)|y_1 - y_2|^{-\nu}$, for large y and x_1, x_2 finite. A heuristic argument has been given in Ref. 8 to show that this is true in general and that

$$\int_0^\infty dx_1 \int_0^\infty dx_2 A(x_1, x_2) = -\frac{k_B T \epsilon_w}{2(\nu - 1)^2 \pi^2 q^2} \qquad (\nu = 2, 3) \tag{1.2}$$

In special case $\epsilon_w = 0$,[4] $\rho_T^{(2)}$ decays as an oscillating exponential[8] for the two-dimensional one-component plasma at $\Gamma = 2$.

The present paper deals with a two-dimensional one-component plasma at $\Gamma = 2$ confined in a strip of finite width. Let ϵ_b again be the dielectric constant of the background in the strip, and let ϵ_1 and ϵ_2 be the dielectric constants of the continua on either side of the strip. The simplest soluble case in which all dielectric constants are equal, $\epsilon_1 = \epsilon_2 = \epsilon_b$, has already been investigated in Ref. 5. In the present paper, we also consider a case with image forces: a soluble problem is obtained if we make the (somewhat academic) choice $\epsilon_1 = 0$, $\epsilon_2 = \epsilon_b = 1$. Then there are images of the same sign and magnitude as the particles. In both cases it will be shown here that the correlations along the strip have an asymptotic behavior with algebraically decaying oscillations; this behavior resembles, as it should, that of a one-dimensional system.

Exact results for the system in a strip, in the case with image forces ($\epsilon_1 = 0$, $\epsilon_2 = \epsilon_b = 1$) are developed in Section 2. The asymptotic expansions of the correlations in a strip, for both cases, without ($\epsilon_1 = \epsilon_2 = \epsilon_b = 1$) and with image forces ($\epsilon_1 = 0$, $\epsilon_2 = \epsilon_b = 1$) are studied in Section 3. The results are discussed in Section 4.

2. EXACT RESULTS WITH IMAGE FORCES

Consider a two-dimensional system composed of an annulus of inner radius $R - L$ and outer radius $R + L$. The region $0 \leqslant r \leqslant R + L$ has dielectric constant 1 and the region $r > R + L$ has dielectric constant 0. The outer (inner) edge of the annulus bears linear charge density $-\sigma_+ q(-\sigma_- q)$ and the annular region bears a background charge density $-\eta q$ and contains N particles of charge q with exact charge neutrality holding so that

$$N = 2\pi(R - L)\sigma_- + 4\pi RL\eta + 2\pi(R + L)\sigma_+ \tag{2.1}$$

The pair potential between two charges q_1, q_2 at $\mathbf{r}_1, \mathbf{r}_2$ (in the annulus) is the solution to the two-dimensional Poisson equation with Neumann boundary condition at $r = R + L$, namely,

$$-(1/2)q_1 q_2 \log\left\{(\mathbf{r}_1 - \mathbf{r}_2)^2\left[1 - 2\mathbf{r}_1 \cdot \mathbf{r}_2/(R + L)^2 + \mathbf{r}_1^2\mathbf{r}_2^2/(R + L)^4\right]\right\}$$

Here $\mathbf{r}$ is measured in some arbitrary length unit taken as 1 for convenience. The net potential energy W of a configuration $\mathbf{r}_1, \ldots, \mathbf{r}_N$ of the particles is found to be given by

$$\begin{aligned}
\frac{-W(\mathbf{r}_1, \ldots, \mathbf{r}_N)}{k_B T}
&= \frac{\Gamma}{2}\Bigg\{-\frac{N}{2}\log(R + L)^2 + M^2 \log\frac{R - L}{R + L} \\
&\quad + NN^* - \frac{3}{4}\Sigma_B^2 + \frac{1}{2}\Sigma_B N^* - \Sigma_B \Sigma_- \\
&\quad + \sum_{k=1}^{N}\left\{M \log\frac{\mathbf{r}_k^2}{(R + L)^2} + \log\left[1 - \frac{\mathbf{r}_k^2}{(R + L)^2}\right] - \frac{N^* \mathbf{r}_k^2}{(R + L)^2}\right\} \\
&\quad + \sum_{k=1}^{N-1}\sum_{j=k+1}^{N}\log\left\{\frac{\mathbf{r}_{jk}^2}{(R + L)^2}\left[1 - \frac{2(\mathbf{r}_j \cdot \mathbf{r}_k)}{(R + L)^2} + \frac{\mathbf{r}_j^2 \mathbf{r}_k^2}{(R + L)^4}\right]\right\}\Bigg\}
\end{aligned} \tag{2.2}$$

Here $N^* = \pi\eta(R + L)^2$, $\Sigma_\pm = 2\pi(R \pm L)\sigma_\pm$, $\Sigma_B = 4\pi\eta RL$, $M = N^* - \Sigma_B - \Sigma_-$, and $\Gamma = q^2/k_B T$. Terms corresponding to the interaction of particles with their own images and with the images of other particles may be seen clearly. Notice that, with $\epsilon_1 = 0$, the problem $\nabla^2\phi = -2\pi\rho(r)$ for ϕ

is a Neumann problem and so only has a solution (unique up to an additive constant) if (2.1) holds. Exact results for this system are now developed with $\Gamma = 2$, using the methods of Ref. 4.

First, let $\mathbf{r}_j = (R + L)(z_j \cos\theta_j, z_j \sin\theta_j) = r_j(\cos\theta_j, \sin\theta_j)$ with $R - L \leqslant r_j \leqslant R + L$ and $0 \leqslant \theta_j \leqslant 2\pi$. Then define

$$a_j = z_j \exp(i\theta_j), \qquad a_{2N+1-j} = \frac{1}{z_j}\exp(i\theta_j), \qquad 1 \leqslant j \leqslant N \tag{2.3}$$

and the $2N \times 2N$ matrix

$$D_{2N}(k, j) = a_j^{k-1}, \qquad 1 \leqslant k, j \leqslant 2N \tag{2.4}$$

The determinant of the matrix is a $2N \times 2N$ van der Monde determinant and may be rearranged as

$$\begin{aligned}\operatorname{Det} D_{2N} &= \prod_{k=1}^{N}\left[z_k^{-(2N-1)}\exp\left[i\theta_k(2N-1)\right](1 - z_k^2)\right] \\ &\quad \times \prod_{k=1}^{N-1}\prod_{j=k+1}^{N}(\mathbf{z}_j - \mathbf{z}_k)^2\left(1 - 2\mathbf{z}_k \cdot \mathbf{z}_j + \mathbf{z}_k^2\mathbf{z}_j^2\right)\end{aligned} \tag{2.5}$$

with $\mathbf{z}_j = \mathbf{r}_j/(R + L)$. This gives a representation of the integrand in the canonical partition function and distribution functions which is particularly useful: the integrals on θ_j, when the determinant is given its permutation expansion, are all of the form $\int_0^{2\pi}\exp(iJ\theta_l)\,d\theta_l$, J being an integer. The remaining integrals may be performed, using incomplete gamma functions, and the thermodynamic limit taken, using the uniform asymptotic expansion of these functions.[6,4] It is useful to introduce the parameters $\alpha_\pm = \sigma_\pm(2\pi/\eta)^{1/2}$, $\kappa^2 = 2\pi\eta$ (this definition of κ^2 differs by a factor 2 of the one used in some other papers) and $Y = (1/2)(\alpha_+ + \alpha_- + 2\kappa L)$. In the limit $R \to \infty$, an infinite straight strip of width $2L$ is obtained.

The simplest exact (finite N) result is for the one-particle distribution function which gives

$$\rho_{(1)}(\mathbf{r}_1) = \frac{1}{\pi(R + L)^2} z_1^{2M}\exp(-N^* z_1^2)\sum_{l=1}^{N}\frac{z_1^{2l-2} - z_1^{2(2N-l)}}{F(l, M, N)} N^{*(M+l)} \tag{2.6a}$$

where

$$\begin{aligned}F(l, M, N) &= \gamma(M + l, N^*) - \gamma\left[M + l, N^*\left(\frac{R - L}{R + L}\right)^2\right] \\ &\quad - N^{*-(2N+1-2l)}\left[\gamma(M + 2N + 1 - l, N^*)\right. \\ &\qquad \left. - \gamma\left(M + 2N + 1 - l, N^*\left(\frac{R - L}{R + L}\right)^2\right]\right.\end{aligned} \tag{2.6b}$$

The passage to the thermodynamic limit allows sums like that in (2.6a) to be written as sum approximations to Riemann integrals with the approximation becoming exact in the thermodynamic limit.

The free energy per unit length of strip is

$$\begin{aligned}
&f(\sigma_-, \sigma_+, \eta; L) \\
&\quad = \lim_{R\to\infty} \frac{-k_B T}{2\pi R} \log Z_N(\Gamma = 2) \\
&\quad = \frac{-k_B T\kappa}{2\pi} \bigg(\int_0^{2Y} \log\big\{ \operatorname{erf}(\alpha_+ - t + 2\kappa L) - \operatorname{erf}(\alpha_+ - t) - \exp(4t\alpha_+) \\
&\qquad\qquad \times \big[\operatorname{erf}(\alpha_+ + t + 2\kappa L) - \operatorname{erf}(\alpha_+ + t) \big] \big\}\, dt \\
&\qquad\qquad - Y \log \frac{2\eta}{\pi^2} + \frac{8}{3} Y^3 - (\alpha_+ + \alpha_-) \\
&\qquad\qquad \times (\alpha_+ \alpha_- + 4\kappa L Y) - \frac{8}{3} (\kappa L)^3 \bigg)
\end{aligned} \tag{2.7}$$

where $\operatorname{erf}(t)$ is the error function

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t \exp(-u^2)\, du \tag{2.8}$$

The density at a distance x from the $\epsilon_1 = 0$ wall is found to be

$$\begin{aligned}
\rho_{(1)}(x) = {} & \frac{2\eta}{\sqrt{\pi}} \exp\big[-(\kappa x + \alpha_+)^2 \big] \int_{-2Y}^{2Y} dt \exp(-t^2) \sinh 2\kappa x t \\
& \times \big\{ \exp(-2\alpha_+ t) \big[\operatorname{erf}(\alpha_+ - t + 2\kappa L) - \operatorname{erf}(\alpha_+ - t) \big] \\
& \quad - \exp(2\alpha_+ t) \big[\operatorname{erf}(\alpha_+ + t + 2\kappa L) - \operatorname{erf}(\alpha_+ + t) \big] \big\}^{-1}
\end{aligned} \tag{2.9}$$

This density has no unexpected feature. In the limit $L \to \infty$, with x fixed, $\rho_{(1)}(x)$ becomes the density against a wall with $\epsilon_w = 0$;[4] in the limit $L \to \infty$, with $2L - x$ fixed, $\rho_{(1)}(x)$ becomes the density against a wall with $\epsilon_w = 1$.[4,6,7] For L finite these two limiting density profiles combine, with some distortion.

The two-particle distribution function is a little more difficult to evaluate, as with the $\epsilon_w = 0$ case in disk geometry.[4] If Eq. (2.9) be taken as the definition of a function of complex x, then the two-particle distribution function may be expressed in terms of it, just as for the disk case.[4] Consider the probability density for two particles, one at $(x_1, 0)$ and the other at (x_2, y) (x_i measuring the distance from the $\epsilon_1 = 0$ wall and the second coordinate being normal to x_i). In the thermodynamic limit, the truncated distribution function reduces to

$$\rho^T_{(2)}(x_1, x_2; y) = -\exp\{-\pi\eta[(x_1-x_2)^2+y^2]\} \times \{|\rho_{(1)}[\tfrac{1}{2}(x_1+x_2+iy)]|^2 - \exp(-2\kappa^2 x_1 x_2 - 4\alpha_+ \kappa x_2) \times |\rho_{(1)}[\tfrac{1}{2}(x_1-x_2+iy)]|^2\} \tag{2.10}$$

3. ASYMPTOTIC NATURE OF THE CORRELATIONS

3.1. Without Image Forces

We consider the same system as in Section 2, except that the dielectric constant is 1 everywhere. The one-body and two-body distribution functions of that system have been computed in Ref. 5. There too, the two-body truncated distribution function $\rho^T_{(2)}(x_1, x_2; y)$ was expressed in terms of the one-body distribution function $\rho_{(1)}(x)$, where the variable x had to be given the complex value $(x_1+x_2+iy)/2$. The function $\rho_{(1)}[(x_1+x_2+iy)/2]$, as a function of y, is the Fourier transform of some function of t, the support of which is the finite interval $(0, Y)$, and the asymptotic behavior as $|y| \to \infty$ is easily obtained by the usual procedure of integration by parts. The resulting asymptotic behavior of $\rho^T_{(2)}$, as $|y| \to \infty$, for fixed values of x_1 and x_2, is

$$\begin{aligned}\rho^T_{(2)}(x_1, x_2; y) \sim -\frac{4\eta^2}{\pi\kappa^2 y^2}\Bigg\{ & \frac{\exp[-(2Y-\alpha_+-\kappa x_1)^2-(2Y-\alpha_+-\kappa x_2)^2]}{[\operatorname{erf}(2Y-\alpha_+)-\operatorname{erf}(\alpha_-)]^2} \\ & + \frac{\exp[-(\alpha_++\kappa x_1)^2-(\alpha_++\kappa x_2)^2]}{[\operatorname{erf}(2Y-\alpha_-)-\operatorname{erf}(\alpha_+)]^2} \\ & - 2\frac{\exp[-2Y^2-(\alpha_+-Y+\kappa x_1)^2-(\alpha_+-Y+\kappa x_2)^2]}{[\operatorname{erf}(2Y-\alpha_+)-\operatorname{erf}(\alpha_-)][\operatorname{erf}(2Y-\alpha_-)-\operatorname{erf}(\alpha_+)]} \\ & \times \cos 2Y\kappa y\Bigg\} + 0\left(\frac{1}{y^3}\right)\end{aligned} \tag{3.1}$$

Note the appearance of an oscillating term $\cos 2Y\kappa y$.

It is of interest to consider the quantity

$$s(y)=\int_0^{2L}dx_1\int_0^{2L}dx_2\,\rho^T_{(2)}(x_1,x_2;y) \tag{3.2}$$

which is a linear density–linear density correlation function along the strip. From (3.1), one finds its asymptotic behavior

$$s(y)\sim\frac{-1}{2\pi^2y^2}\left\{1-\frac{\exp(-2Y^2)\left[\operatorname{erf}(Y-\alpha_+)+\operatorname{erf}(Y-\alpha_-)\right]^2}{\left[\operatorname{erf}(2Y-\alpha_+)-\operatorname{erf}(\alpha_-)\right]\left[\operatorname{erf}(2Y-\alpha_-)-\operatorname{erf}(\alpha_+)\right]}\right.$$
$$\left.\times\cos 2Y\kappa y\right\}+0\left(\frac{1}{y^3}\right) \tag{3.3}$$

3.2. With Image Forces

We come back to the case $\epsilon_1=0$, $\epsilon_2=\epsilon_b=1$. Again, $\rho_{(1)}[(x_1\pm x_2+iy)/2]$, as a function of y, is the Fourier transform of a function of t which has a finite support, $(-2Y,2Y)$ now. The resulting asymptotic behavior of $\rho^T_{(2)}$, as $|y|\to\infty$, for fixed values of x_1,x_2, is found to be

$$\rho^T_{(2)}(x_1,x_2;y)$$
$$\sim\frac{4\eta^2}{\pi}\exp\left[-(\kappa x_1+\alpha_+)^2-(\kappa x_2+\alpha_+)^2-4Y^2\right]$$
$$\times\left[\left[\frac{4S_0}{F(2Y)}\sinh 2\kappa Yx_1\sinh 2\kappa Yx_2\right]\frac{\cos 2\kappa Yy}{\kappa y}+\frac{1}{\kappa^2y^2}\right.$$
$$\times\left(\frac{-4\exp(-4Y^2)}{F^2(2Y)}\sinh 4\kappa Yx_1\sinh 4\kappa Yx_2+4S_0\sin 2\kappa Yy\right.$$
$$\times\left\{\left[\frac{4Y}{F(2Y)}+\frac{F'(2Y)}{F^2(2Y)}\right]\sinh 2\kappa Yx_1\sinh 2\kappa Yx_2\right.$$
$$-\frac{\kappa}{F(2Y)}(x_1\cosh 2\kappa Yx_1\sinh 2\kappa Yx_2$$
$$\left.\left.\left.+x_2\sinh 2\kappa Yx_1\cosh 2\kappa Yx_2)\right\}\right)\right]+O\left(\frac{1}{y^3}\right) \tag{3.4}$$

where

$$F(t) = \exp(-2\alpha_+ t)\left[\operatorname{erf}(\alpha_+ - t + 2\kappa L) - \operatorname{erf}(\alpha_+ - t)\right] - \exp(2\alpha_+ t)\left[\operatorname{erf}(\alpha_+ + t + 2\kappa L) - \operatorname{erf}(\alpha_+ t)\right] \tag{3.5}$$

and

$$S_0 = 2\pi\left(\frac{4}{\sqrt{\pi}}\left\{\exp(-\alpha_+^2) - \exp\left[-(\alpha_+ + 2\kappa L)^2\right]\right\} + 4\alpha_+\left[\operatorname{erf}(\alpha_+) - \operatorname{erf}(\alpha_+ + 2\kappa L)\right]\right)^{-1} \tag{3.6}$$

The resulting asymptotic behavior of $s(y)$ is

$$s(y) \sim \frac{\eta^2}{\kappa^2}\left(\exp(-2Y^2)\frac{F^2(Y)}{F(2Y)} S_0 \frac{\cos 2\kappa Y y}{\kappa y} + \frac{1}{\kappa^2 y^2} \times\left\{-1 + \exp(-2Y^2)\frac{S_0 F(Y)}{F(2Y)} \times\left[4Y + \frac{F'(2Y)}{F(2Y)} - 2G(Y)\right]\sin 2\kappa Y y\right\} + O(y^{-3})\right) \tag{3.7}$$

where

$$G(Y) = \exp(-2\alpha_+ Y) \times\left(\left[-\alpha_+ + Y\right]\left[\operatorname{erf}(2\kappa L + \alpha_+ - Y) - \operatorname{erf}(\alpha_+ - Y)\right] + \frac{1}{\sqrt{\pi}}\left\{\exp\left[-(\alpha_+ - Y)^2\right] - \exp\left[-(2\kappa L + \alpha_+ - Y)^2\right]\right\}\right) + \exp(2\alpha_+ Y)\left((-\alpha_+ - Y)\left[\operatorname{erf}(2\kappa L + \alpha_+ + Y) - \operatorname{erf}(\alpha_+ + Y)\right] + \frac{1}{\sqrt{\pi}}\left[\exp\left[-(\alpha_+ + Y)^2\right] - \exp\left[-(2\kappa L + \alpha_+ + Y)^2\right]\right]\right) \tag{3.8}$$

4. DISCUSSION

Consider a strip of width $2L$. Let there be charge densities $\pm\sigma q$ on the edges of the strip ($+\sigma q$ on the ϵ_1 edge), so that there is an electric field

$E = 2\pi\sigma q$ in the strip. Then it is to be expected[5] that

$$\frac{\partial f}{\partial E} = q\left[\int_0^{2L} x\rho_{(1)}(x)\,dx - 2\eta L^2 + 2L\sigma\right] \tag{4.1}$$

where $f(E, \beta, \eta; L)$ is the free energy of the system per unit length of strip. In Ref. 5, this equation was shown to hold for the case $\epsilon_1 = \epsilon_2 = \epsilon_b$. Using (2.7) and (2.9), we can also show it holds for the case $\epsilon_1 = 0$, $\epsilon_2 = \epsilon_b = 1$.

A fundamental property of Coulomb systems is the perfect-screening sum rule

$$\int_0^{2L} dx_2 \int_{-\infty}^{\infty} dy\, \rho_{(2)}^T(x_1, x_2; y) = -\rho_{(1)}(x_1) \tag{4.2}$$

This sum rule can be checked here for both cases $\epsilon_1 = \epsilon_2 = \epsilon_b$ and $\epsilon_1 = 0$, $\epsilon_2 = \epsilon_b = 1$. In the second case, integration of $\rho_{(2)}^T$ is not at all easy unless the representation [easily shown to be equivalent to Eq. (2.10)]

$$\begin{aligned} \rho_{(2)}^T(x_1, x_2; y) = {} & -\frac{4\eta^2}{\pi}\exp\left[-(\kappa x_1 + \alpha_+)^2 - (\kappa x_2 + \alpha_+)^2\right] \\ & \times \int_{-2Y}^{2Y} dt_1 \int_{-2Y}^{2Y} dt_2 \frac{\exp(-t_1^2 - t_2^2)}{F(t_1)F(t_2)} \\ & \times \sinh\left[\kappa(t_1 + t_2)x_1\right]\sinh\left[\kappa(t_1 + t_2)x_2\right] \\ & \times \left\{\exp\left[i\kappa y(t_1 - t_2)\right] + \exp\left[i\kappa y(t_1 + t_2)\right]\right\} \end{aligned} \tag{4.3}$$

is used. The y integration in (4.2) can be done first giving delta functions which allow the t_2 integration to be performed, then the x_2 integration. It should be noted that the perfect-screening sum rule (4.2) has been rigorously proved by Gruber, Lebowitz, and Martin,[9] using among other properties the assumption that $\rho_{(2)}^T(x_1, x_2, y)$ decays faster than, or at least as fast as[10] $|y|^{-\nu}$, where ν is the space dimensionality. This condition is not satisfied by (3.4), and thus it is not a necessary one.

A further sum rule for this system is

$$\frac{\partial \rho_{(1)}(x_1)}{\partial \sigma} = -4\pi\int_0^{2L} dx_2 \int_{-\infty}^{\infty} dy\,(x_2 - x_1)\rho_{(2)}^T(x_1, x_2; y) \tag{4.4}$$

which was derived in general, for the half-space problem, by Blum *et al.*[11] This sum rule can be shown to hold here for the strip problem as well, by using the same techniques as for proving (4.2), notwithstanding the weak decay (3.4).

It is now of interest to return to consider the functions $s(y)$ discussed in Section 3. It is possible to retrieve a one-dimensional system by taking the limit $L \to 0$, $\eta \to \infty$, for fixed values of σ_+ and σ_-, in such a way that

$2L\eta$ remains finite. Then

$$\mu = 2L\eta + \sigma_+ + \sigma_- = \frac{\kappa Y}{\pi} \tag{4.5}$$

which is the number of particles per unit length, has a finite limit. In this limit, one finds from (3.3)

$$s_1(y) \sim -\frac{1}{2\pi^2 y^2}(1 - \cos 2\pi\mu y) \tag{4.6}$$

and from (3.7)

$$s_0(y) \sim \mu^2\left[\frac{\cos 2\pi\mu y}{4\mu y} - \frac{1 + (\pi/2)\sin 2\pi\mu y}{4\pi^2\mu^2 y^2} + O\left(\frac{1}{y^3}\right)\right] \tag{4.7}$$

(the subscript 1 or 0 refers here to the value of ϵ_1). The right-hand side of (4.6) is exactly the correlation function obtained for the one-dimensional one-component log-potential system at $\Gamma = 2$ by Mehta and Dyson.[15] That is, the appearance of the cosine term in Eq. (3.3) seems due to the quasi-one-dimensional nature of the strip system. The result (4.7) is that obtained for the one-dimensional system,[15] for $\Gamma = 4$. The reason this temperature should occur is that the images double the effective coupling between charges when the charged particles are confined to a line against the $\epsilon_1 = 0$ wall. Of course this also means that the strip against an $\epsilon_1 = 0$ wall is also showing quasi-one-dimensional behavior. The images mean that the relevant one-dimensional behavior is that of a system at lower temperature ($\Gamma = 4$) than was relevant to the strip without images. These ideas suggest that for the two-dimensional one-component plasma in a strip with coupling constant Γ with dielectric constant ϵ_b in the strip and one side and with dielectric constant ϵ_1 on the other side, the integrated correlation function $s(y)$ will decay at large y in a fashion similar to the decay of the one-dimensional log-potential correlation function. However, the one-dimensional system must be considered at coupling constant $\Gamma[1 + (\epsilon_b - \epsilon_1)/(\epsilon_b + \epsilon_1)]$. Incidentally, the case $\epsilon_1 \to \infty$ is not covered by this argument, because the Coulomb interaction is then replaced by a monopole-dipole interaction, which may lead to completely different behaviors. The conjecture is interesting at $\Gamma = 1$, where it predicts that, as the strip system becomes infinitely narrow, the functions $s(y)$ should become[15]

$$s_1(y) \sim \mu^2\left(-\frac{1}{\pi^2\mu^2 y^2} - \frac{1 + \cos^2 \mu\pi y}{\pi^4\mu^4 y^4} + \cdots\right) \tag{4.8}$$

and

$$s_0(y) = -\frac{1}{2\pi^2 y^2}(1 - \cos 2\pi\mu y) \tag{4.9}$$

Thus we find oscillations with a wavelength μ^{-1}, which are reminiscent of a one-dimensional crystal. However, here there is no true long-range order, since the oscillations ultimately decay as $|y|$ becomes large. The strip systems which are considered here should not be confused with another system which has been studied by Choquard.(16) Choquard uses different boundary conditions: he considers a long thin rectangular strip with *periodic* boundary conditions across the strip (instead of hard walls), and he does find strict long-range crystalline order along the strip. His result can be easily understood, because each particle together with its periodic replicas approximately forms a charged rod, and the effective interaction between two such rods is the one-dimensional Coulomb (linear) potential. Thus, his system behaves like a one-dimensional one-component Coulomb potential system, which is indeed known to be crystalline at any temperature.(17) This kind of argument suggests that a three-dimensional Coulomb system restricted to a long thin prism with periodic boundary conditions across the prism might exhibit strict long-range crystalline order, but we have no hint for the behavior of this system for hard-wall boundary conditions.

The general argument which was given in Ref. 8 for deriving the asymptotic behavior of the correlations parallel to the wall in the half-space problem can be easily adapted to the strip problem; actually things are even simpler for a strip, because there is no bulk contribution to be subtracted. This argument is based on the linear-response relation, and the assumption that an external charge density periodic along the strip with a macroscopic wave number is perfectly screened. In this way, it can be shown that the Fourier transform

$$\hat{s}(k) = \int_{-\infty}^{\infty} e^{iky} s(y)\, dy \tag{4.10}$$

behaves, for small k values, as

$$\hat{s}(k) \sim -\mu + \frac{(\epsilon_1 + \epsilon_2) k_B T}{2\pi q^2} |k| + \cdots \tag{4.11}$$

This $|k|$ singularity in $\hat{s}(k)$ gives in $s(y)$ a contribution $-(\epsilon_1 + \epsilon_2) k_B T / 2\pi^2 q^2 y^2$, which is indeed present in the asymptotic forms (3.3), (3.7), (4.6), (4.7), (4.8), (4.9). However, the explicit calculations made here show that $\hat{s}(k)$ may have other singularities on the real-k axis. The oscillations found here with a wave number $2\pi\mu$ correspond to singularities of $\hat{s}(k)$ at $k = \pm 2\pi\mu$, which are the mathematical manifestation of a tendency to one-dimensional crystalline ordering. These singularities also contribute to the asymptotic form of $s(y)$. Unfortunately, it does not seem easy to predict the exact nature of these singularities at $k \neq 0$, and we cannot know *a priori* which kind of decay they will give for the cos $(2\pi\mu y)$ term in $s(y)$. It turns out that the dominant term in the asymptotic form of $s(y)$ is given by the $k=0$ singularity in (4.8), by the $k=\pm 2\pi\mu$ singularities in (3.7) or (4.7), and by the three of them in (3.3), (4.6), or (4.9).

For a three-dimensional fluid Coulomb system confined to a slab, there is no reason to expect singularities in $\hat{s}(\mathbf{k})$ for nonzero real values of $\mathbf{k}$. The only singularity presumably will be at $\mathbf{k}=0$, of the form

$$\hat{s}(\mathbf{k}) \sim -\mu + \frac{(\epsilon_1+\epsilon_2)k_B T}{4\pi q^2}|\mathbf{k}| \tag{4.12}$$

(μ is now the surface density on the slab), and the asymptotic behavior of $s(\mathbf{y})$ should again be simple, and of the form

$$s(\mathbf{y}) \sim -\frac{(\epsilon_1+\epsilon_2)k_B T}{8\pi^2 q^2 |\mathbf{y}|^3} \tag{4.13}$$

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