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On Potential and Field Fluctuations in Two-Dimensional Classical Charged Systems

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We supplement a previous paper on three-dimensional systems by studying the electric potential and field fluctuations in two-dimensional Coulomb systems. The novelty in two dimensions is that the fluctuations of the potential at a point are infinite in the thermodynamic limit. However, the potential difference between two points has finite fluctuations, which resemble the ones which occur in the three-dimensional case. The field fluctuations are also rather similar in both cases. The correlations do not have a fast decay. Explicit results are obtained for a solvable model; the fluctuations of the potential are Gaussian with an infinite variance.

KEY WORDS: Coulomb systems; two dimensions; potential fluctuations; field fluctuations.

1. INTRODUCTION

The potential and field fluctuations in three-dimensional classical charged systems have been recently studied.[(1)] In the present paper, we briefly consider the same problem for two-dimensional systems, and show that they exhibit a rather similar behavior, with, however, some modifications. One of the motivations for studying two-dimensional systems is that, in a special case, the one-component plasma at some particular temperature, a variety of exact results can be obtained explicitly[(2–4)] and be used as checks and illustrations for the more general results.

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We use here the same notations as in Ref. 1. Equations from Ref. 1 are referred to as (I.1.1), etc. The two-body potential between two particles of charges e_{α_1} and e_{α_2} located at x_1 and x_2 now is some short-range potential plus

$$\phi(x_1\alpha_1, x_2\alpha_2) = -e_{\alpha_1}e_{\alpha_2}\ln(|x_1 - x_2|/L) \tag{1.1}$$

where L is some length scale which will be irrelevant in the following.

In Section 2, we study the general properties of the two-dimensional case. In Section 3, we illustrate these properties in the special solvable case of a one-component plasma at a temperature such that the dimensionless coupling constant be $\Gamma = 2$.

2. THE GENERAL TWO-DIMENSIONAL CASE

2.1. Average Potential

Before discussing the fluctuations, let us note that the average potential itself, in two dimensions as well as in three dimensions, can be different from zero even in an homogeneous state. If the infinite system is obtained as the thermodynamic limit of a finite system, this finite system in general will have a nonzero charge density near its boundaries (even if there is no net surface charge, a double electric layer will be formed near the boundaries). Inside the system, far from the boundaries, the macroscopic potential will be constant but in general different from zero, and this will remain true when the boundaries recede to infinity.

2.2. Potential Fluctuations

Let us now consider the fluctuations at neutral points. For three-dimensional systems, the whole analysis of Ref. 1 was based upon the assumption of the existence of a thermodynamic limit for the potential fluctuations. In a two-dimensional system, we cannot assume that the potential fluctuations have a thermodynamic limit; on the contrary, it can be seen that these fluctuations diverge logarithmically as the size of the system becomes infinite, by the following argument which is closely related to the one developed in the concluding remarks of Ref. 1.

Let us consider for instance a system in a disk Λ of radius R_0, and the potential fluctuations at its center

$$W_\Lambda(0,0) = \langle [V(0)]_\Lambda [V(0)]_\Lambda \rangle_\Lambda = \int_\Lambda dx \ln|x| \int_\Lambda dy \ln|y| S_\Lambda(x,y) \tag{2.1}$$

The "dangerous" contributions to (2.1) come from the boundary region. Suppose for instance that x is close to the boundary. Although the total excess charge carried by the particle at x vanishes,

$$\int_{\Lambda} dy\, S_{\Lambda}(x, y) = 0 \tag{2.2}$$

this excess charge will have in general a nonvanishing dipole moment

$$\int_{\Lambda} dy\, y S_{\Lambda}(x, y) = O(1) \tag{2.3}$$

localized near the boundary and therefore the potential at the origin created by this excess charge will be $O(1/R_0)$ (instead of $O(1/R_0^2)$ in three dimensions):

$$\int_{\Lambda} dy \ln|y| S_{\Lambda}(x, y) = O\left(\frac{1}{R_0}\right) \tag{2.4}$$

This estimate (2.4) (valid for x near the boundary) must be multiplied by $\ln|x|$ and integrated upon x in the boundary region which has an area of order R_0. Thus, the corresponding contribution to (2.1) is $O(\ln R_0)$ [instead of $O(1/R_0)$ in three dimensions], and $W_{\Lambda}(0, 0)$ diverges like $\ln R_0$ when R_0 becomes infinite.

However, a similar argument indicates that the fluctuations of the potential difference between two points do have a well-defined thermodynamic limit. We now consider the quantity

$$\begin{aligned} A_{\Lambda}(x, y) &= \langle [V(x) - V(y)]^2 \rangle_{\Lambda} \\ &= \langle V(x) - V(y) \rangle_{\Lambda}^2 + \int_{\Lambda} dz\, (\ln|x - z| - \ln|y - z|) \\ &\quad \times \int_{\Lambda} dt\, (\ln|x - t| - \ln|y - t|) S_{\Lambda}(z, t) \end{aligned} \tag{2.5}$$

We assume that the average potential difference $\langle V(x) - V(y) \rangle_{\Lambda}$ goes to zero in the thermodynamic limit (i.e., the plasma is a conductor inside which there cannot be a macroscopic electrical field). Let us investigate the behavior of the last term of (2.5). If z is near the boundary, we now have

$$\int_{\Lambda} dt\, (\ln|x - t| - \ln|y - t|) S_{\Lambda}(z, t) = O\left(\frac{1}{R_0^2}\right) \tag{2.6}$$

and the corresponding "dangerous" contribution to (2.5) is $O(1/R_0^2)$ and vanishes as $R_0 \to \infty$. This indicates that $A_{\Lambda}(x, y)$ has a thermodynamic limit.

Assuming the existence of this limit, it would be possible to adapt the analysis of Ref. 1 to the two-dimensional case. Here, we shall rather use a more heuristic argument, as follows. Since the contributions to (2.5) from z

near the boundary vanish as $R_0 \to \infty$, we can disregard these contributions and choose z far from the boundary. Assuming that $S_\Lambda(z,t)$ has good cluster properties, the relevant values of t will also be far from the boundary; thus $S_\Lambda(z,t)$ can be replaced by its bulk value, i.e., its thermodynamic limit $S(z-t)$, and the integration domain of t extended to infinity. Since

$$\int dt\,(\ln|x-t| - \ln|y-t|)S(z-t)$$

has a fast decay when $|z| \to \infty$ [faster than any inverse power law if $S(z-t)$ decays faster than any inverse power law], we can finally extend also the integration domain of z to infinity.

Therefore, the fluctuations of the potential difference have a well-defined limit, the analog of the second expression in (I.3.7):

$$\begin{aligned} A(x) &= \lim_{\Lambda \to \mathbb{R}^2} A_\Lambda(x+y, y) \\ &= \int dz\,(\ln|x-z| - \ln|z|)\int dt\,(\ln|x-t| - \ln|t|)S(z-t) \end{aligned} \tag{2.7}$$

Applying twice the convolution theorem of Fourier transforms, we obtain the analog of (I.3.10)

$$A(x) = 2\int dk\,\left[1 - \exp(ikx)\right]\frac{\tilde{S}(k)}{|k|^4} \tag{2.8}$$

where the Fourier transform $\tilde{S}(k)$ is defined as

$$\tilde{S}(k) = \int dx\,\exp(ikx)S(x) \tag{2.9}$$

Alternatively, we can write the analog of the first expression in (I.3.7), as shown in Appendix A:

$$A(x) = \pi\int dy\,(|y|^2\ln|y| - |x-y|^2\ln|x-y|)S(y) \tag{2.10}$$

2.3. Field Fluctuations

The field fluctuations are now obtained by deriving the fluctuations of the potential difference:

$$e^{rs}(x-y) = \langle E^r(x)E^s(y)\rangle = -\frac{1}{2}\frac{\partial^2}{\partial x^r\,\partial y^s}A(x-y) \tag{2.11}$$

Thus, we obtain from (2.8) the analog of (I.4.6)

$$e^{rs}(x) = \frac{1}{2}\frac{\partial^2}{\partial x^r\,\partial x^s}A(x) = \int dk\,\exp(ikx)\frac{\hat{k}^r\hat{k}^s}{|k|^2}\tilde{S}(k) \tag{2.12}$$

and from (2.10) the analog of (I.4.1)

$$e^{rs}(x) = -\pi \int dy \left[\delta_{rs} \ln|x-y| + \frac{(x-y)^r}{|x-y|} \frac{(x-y)^s}{|x-y|} \right] S(y) \quad (2.13)$$

where we have used the perfect screening property

$$\int dy\, S(y) = 0 \quad (2.14)$$

Note that the scalar product is somewhat simpler

$$\langle E(x)E(0) \rangle = \sum_{r=1}^{2} e^{rr}(x) = 2\int dk \exp(ikx) \frac{\tilde{S}(k)}{|k|^2} = -2\pi \int dy \ln|x-y| S(y) \quad (2.15)$$

2.4. Asymptotic Behaviors

The asymptotic expressions of the above quantities, as $|x| \to \infty$, are

$$\begin{aligned} A(x) &= -\pi \ln|x| \int dy\, |y|^2 S(y) + o(\ln|x|) \\ &= 2k_B T \ln|x| + o(\ln|x|) \end{aligned} \quad (2.16)$$

(where the second equality follows from the second Stillinger–Lovett sum rule) and

$$e^{rs}(x) = k_B T(\delta_{rs} - 2\hat{x}^r \hat{x}^s) \frac{1}{|x|^2} + o\left(\frac{1}{|x|^2}\right) \quad (2.17)$$

Thus both $A(x)$ and $e^{rs}(x)$ do not have a fast decay. However, from (2.15) we see that the trace of $e^{rs}(x)$ decays faster than any inverse power law if $S(y)$ has that property.

2.5. Fluctuations at a Charged Particle

Finally, let us consider the fluctuations at a charged particle. The potential fluctuations will diverge. However, the field fluctuations will be finite, and will obey the analogs of (I.5.10) and (I.5.11) in two dimensions, where 4π has to be replaced by 2π.

3. THE ONE-COMPONENT PLASMA AT $\Gamma = 2$

The general results of Section 2 will now be illustrated in a simple case, the two-dimensional one-component plasma (particles of charge e in a uniform background); when the dimensionless coupling constant $\Gamma = e^2/k_B T$ has the special value $\Gamma = 2$, this model is exactly solvable.

3.1. Average Potential

We consider a system of N particles plus a background, with zero total charge, in a circular box of radius R_0. We choose the unit of length in such a way that the background density be $\rho = 1/\pi$ (for a neutral system, $R_0 = \sqrt{N}$ in these units). We want to compute the average electrostatic potential $\langle V(0)\rangle_N$ at the center of the box

$$\langle V(0)\rangle_N = -e\int_{|x|\leqslant\sqrt{N}} dx \ln|x|[\rho_N(x) - \rho] \tag{3.1}$$

At $\Gamma = 2$, the one-body density is

$$\rho_N(x) = \rho \exp(-|x|^2) \sum_{n=0}^{N-1} \frac{|x|^{2n}}{\gamma(n+1, N)} \tag{3.2}$$

where

$$\gamma(n+1, N) = \int_0^N dt\, t^n \exp(-t) \tag{3.3}$$

is the incomplete gamma function. As shown in Appendix B, the thermodynamic limit of (3.1) exists and is

$$\langle V(0)\rangle = \frac{e}{4}(2\ln 2 - 1) \tag{3.4}$$

Thus, even for this system which has no net charge, the potential has a nonzero thermodynamic limit. This value can be understood as being the potential difference across the double electric layer which builds itself near the walls. Actually, in the limit of a semi-infinite system bounded by an uncharged wall, the density (3.2) becomes at a distance l from the wall[3]

$$\rho(l) = \frac{2}{\sqrt{\pi}}\rho\int_0^\infty dt \frac{\exp\left[-(t - l\sqrt{2})^2\right]}{1 + \Phi(t)} \tag{3.5}$$

where Φ is the error function, and the potential difference across the barrier indeed is

$$2\pi e\int_0^\infty dl\, l[\rho(l) - \rho] = \frac{e}{4}(2\ln 2 - 1) \tag{3.6}$$

in agreement with (3.4).

In the thermodynamic limit, the net charge density $e[\rho_N(x) - \rho]$ becomes zero for any finite value of $|x|$, and the potential becomes constant. Therefore, its limit is (3.4) also at any other point.

3.2. Potential Fluctuations

We now compute the potential fluctuations at the origin (for a finite system). From (2.1),

$$\langle [V(0)]_N [V(0)]_N \rangle_N = e^2 \int_{|x| \leqslant \sqrt{N}} dx \int_{|y| \leqslant \sqrt{N}} dy \ln|x| \ln|y| \left[\rho_N^T(x, y) + \rho_N(x) \delta(x - y) \right] \quad (3.7)$$

At $\Gamma = 2$, the one-body density $\rho_N(x)$ is given by (3.2) and the truncated two-body density $\rho_N^T(x, y)$ is

$$\rho_N^T(x, y) = -\rho^2 \exp(-|x|^2 - |y|^2) \times \sum_{n,p=0}^{N-1} \frac{|x|^{n+p} |y|^{n+p}}{\gamma(n+1, N) \gamma(p+1, N)} \exp[i(n-p)\varphi] \quad (3.8)$$

where φ is the angle between the vectors x and y. As shown in Appendix B, for large N,

$$\langle [V(0)]_N [V(0)]_N \rangle_N \sim \frac{e^2}{4} (\ln N + C + 1) \quad (3.9)$$

where C is Euler's constant, and thus it diverges logarithmically in the thermodynamic limit.

It may be noted that the same result is still obtained with a different way to the thermodynamic limit. If we take $R_0 \gg \sqrt{N}$, the particles form a blob at the center of the circular background, leaving unneutralized background around. We can take the limit $R_0 \to \infty$ first, for a fixed value of N. Then, instead of (3.2) and (3.8), we obtain similar expressions with $\gamma(n+1, N)$ replaced by $\Gamma(n+1)$, and $\gamma(p+1, N)$ replaced by $\Gamma(p+1)$. The calculation of the fluctuations, in Appendix B, is insensitive to these replacements, and therefore (3.9) is still valid in this case.

Although $\langle [V(0)]^2 \rangle$ diverges, if we now look at the reduced variable

$$v = \frac{[V(0)]_N}{(e/2)(\ln N)^{1/2}} \quad (3.10)$$

it is amusing to see that we can obtain its complete probability distribution in the thermodynamic limit, and that this distribution turns out to be exactly Gaussian. This can be shown as follows. The probability distribution of v is, for a finite system of N particles,

$$Q_N(v) = \left\langle \delta \left[v + 2 \frac{\sum_{n=1}^{N} (\ln|x_n| - \langle \ln|x_n| \rangle_N)}{(\ln N)^{1/2}} \right] \right\rangle_N \quad (3.11)$$

the Fourier transform of which is

$$T_N(k) = \int_{-\infty}^{\infty} dv \exp(ikv) Q(v) \tag{3.12}$$

As shown in Appendix B, in the thermodynamic limit, the characteristic function $T_N(k)$ becomes a Gaussian,

$$T(k) = \lim_{N\to\infty} T_N(k) = \exp\left(-\frac{k^2}{2}\right) \tag{3.13}$$

and therefore the probability distribution $Q_N(v)$ has the limit

$$Q(v) = \lim_{N\to\infty} Q_N(v) = \frac{1}{(2\pi)^{1/2}} \exp\left(-\frac{v^2}{2}\right) \tag{3.14}$$

which is also Gaussian, with a variance

$$\langle v^2 \rangle = 1 \tag{3.15}$$

in agreement with (3.9) and (3.10).

The Gaussian behavior (3.14) has some similarity with the Gaussian behavior found in three dimensions for the contributions from distant regions; here, the divergence comes from the remote particles, which therefore give the dominant contribution, and it is not a surprise that this contribution is Gaussian.

We now consider the fluctuations of the potential differences. Using (2.7) or (2.10) with[2]

$$S(y) = e^2\left[-\frac{1}{\pi^2}\exp(-|y|^2) + \frac{1}{\pi}\delta(y)\right] \tag{3.16}$$

or (2.8) with

$$\tilde{S}(k) = \frac{e^2}{\pi}\left[1 - \exp\left(-\frac{|k|^2}{4}\right)\right] \tag{3.17}$$

we find

$$A(x) = \frac{e^2}{2}\left[2\ln|x| - (|x|^2+1)\mathrm{Ei}(-|x|^2) - \exp(-|x|^2) + C + 1\right] \tag{3.18}$$

where

$$\mathrm{Ei}(-|x|^2) = -\int_{|x|^2}^{\infty} dt \frac{\exp(-t)}{t} \tag{3.19}$$

is the exponential-integral function. Since here $e^2 = 2k_BT$, (3.18) does have the asymptotic behavior (2.16).

3.3. Field Fluctuations

From the first expression (2.12) and (3.18), one obtains

$$e^{rs}(x) = \frac{e^2}{2}\left\{(\delta_{rs} - 2\hat{x}^r\hat{x}^s)\left[1 - \exp(-|x|^2)\right]\frac{1}{|x|^2} - \delta_{rs}\mathrm{Ei}(-|x|^2)\right\} \qquad (3.20)$$

The asymptotic behavior of (3.20) is in agreement with (2.17). Furthermore

$$\langle E(x)E(0)\rangle = \sum_{r=1}^{2} e^{rr}(x) = -e^2\mathrm{Ei}(-|x|^2) \qquad (3.21)$$

and this expression indeed decays faster than any inverse power law.

Alternatively, (3.20) can be obtained directly from (2.12) or (2.13), and (3.21) from (2.15).

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APPENDIX A

For deriving (2.10) from (2.7), we use the identity

$$\ln|x| = -1 + \tfrac{1}{4}\Delta(|x|^2\ln|x|) \qquad (A1)$$

Thus,

$$\begin{aligned}
&\int dt\,(\ln|x-t| - \ln|t|)S(z-t)\\
&= \frac{1}{4}\int dt\,\Delta_t(|x-t|^2\ln|x-t| - |t|^2\ln|t|)S(z-t)\\
&= \frac{1}{4}\int dt\,(|x-t|^2\ln|x-t| - |t|^2\ln|t|)\Delta_t S(z-t)\\
&= \frac{1}{4}\Delta_z\int dt\,(|x-t|^2\ln|x-t| - |t|^2\ln|t|)S(z-t) \qquad (A2)
\end{aligned}$$

where we have performed an integration by parts. Using (A2) in (2.7), we obtain

$$A(x) = \frac{1}{4}\int dz\,(\ln|x-z| - \ln|z|)\Delta_z \times\left[\int dt\,(|x-t|^2\ln|x-t| - |t|^2\ln|t|)S(z-t)\right]$$
$$= \frac{\pi}{2}\int dz\left[\delta(x-z) - \delta(z)\right] \times\left[\int dt\,(|x-t|^2\ln|x-t| - |t|^2\ln|t|)S(z-t)\right] \qquad \text{(A3)}$$

where we have again performed an integration by parts and used

$$\Delta\ln(z) = 2\pi\delta(z) \qquad \text{(A4)}$$

From (A3), one readily obtains (2.10) after regrouping equal contributions.

APPENDIX B

For computing (3.1), we use

$$\int_{|x|\leqslant\sqrt{N}} dx\,\ln|x|\exp(-|x|^2)|x|^{2n} = \frac{\pi}{2}\frac{\partial}{\partial n}\gamma(n+1, N) \qquad \text{(B1)}$$

and obtain

$$\langle V(0)\rangle_N = -\frac{e}{2}\left[\sum_{n=0}^{N-1}\frac{\partial}{\partial n}\ln\gamma(n+1,N) - N\ln N + N\right] \qquad \text{(B2)}$$

For taking the thermodynamic limit of (B2), we rewrite the series as

$$\sum_{n=0}^{N-1}\frac{\partial}{\partial n}\ln\gamma(n+1,N) = \sum_{n=0}^{N-1}\psi(n+1) + \sum_{n=0}^{N-1}\frac{\partial}{\partial n}\ln\frac{\gamma(n+1,N)}{\Gamma(n+1)} \qquad \text{(B3)}$$

where $\psi(n+1)$ is the logarithmic derivative of the γ function:

$$\psi(n+1) = \frac{d}{dn}\ln\Gamma(n+1) = \frac{d}{dn}\ln\int_0^\infty dt\,t^n\exp(-t) \qquad \text{(B4)}$$

For computing the first term in the right-hand side of (B3), we use

$$\psi(n+1) = -C + \sum_{p=1}^{n}\frac{1}{p} \qquad \text{(B4')}$$

where C is Euler's constant; rearranging the double summation upon n and p, we obtain

$$\sum_{n=0}^{N-1}\psi(n+1) = N\sum_{n=1}^{N-1}\frac{1}{n} - NC - N + 1 \qquad \text{(B5)}$$

In the large N limit,

$$\sum_{n=1}^{N-1} \frac{1}{n} = \ln N + C - \frac{1}{2N} + O\left(\frac{1}{N^2}\right) \tag{B6}$$

For computing the second term in the right-hand side of (B3), we note that the relevant values of n are such that $N - n = O(\sqrt{N})$, and we use the asymptotic formula valid in that case,

$$\frac{\gamma(n+1,N)}{\Gamma(n+1)} \sim \frac{1}{2}\left[1 + \Phi\left(\frac{N-n}{(2N)^{1/2}}\right)\right] \tag{B7}$$

where

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt \exp(-t^2) \tag{B8}$$

is the error function. In the large N limit, the sum upon n becomes an integral, and thus

$$\begin{aligned} \lim_{N\to\infty} \sum_{n=0}^{N-1} \frac{\partial}{\partial n} \ln \frac{\gamma(n+1,N)}{\Gamma(n+1)} &= \lim_{N\to\infty} \left[\ln \frac{\left[1 + \Phi\left((N-n)/(2N)^{1/2}\right)\right]}{2} \right]_{n=0}^{n=N} \\ &= -\ln 2 \end{aligned} \tag{B9}$$

The above results lead to (3.4)

For computing (3.7), we use again (B1) and

$$\int_{|x| \leqslant \sqrt{N}} dx \,(\ln|x|)^2 \exp(-|x|^2)|x|^{2n} = \frac{\pi}{4} \frac{\partial^2}{\partial n^2} \gamma(n+1,N) \tag{B10}$$

and obtain

$$\begin{aligned} \langle [V(0)]_N [V(0)]_N \rangle_N &= \frac{e^2}{4} \sum_{n=0}^{N-1} \frac{\partial^2}{\partial n^2} \ln \gamma(n+1,N) \\ &= \frac{e^2}{4} \left[\sum_{n=0}^{N-1} \frac{d^2}{dn^2} \ln \Gamma(n+1) + \frac{\partial^2}{\partial n^2} \ln \frac{\gamma(n+1,N)}{\Gamma(n+1)} \right] \end{aligned} \tag{B11}$$

From (B7), one readily sees that the second term in the last expression (B11) has a vanishing thermodynamic limit. For studying the first term, we use

$$\frac{d^2}{dn^2} \ln \Gamma(n+1) = \frac{\pi^2}{6} - \sum_{p=1}^{n} \frac{1}{p^2} \tag{B12}$$

and rearrange the double summation upon n and p into

$$\sum_{n=0}^{N-1} \frac{d^2}{dn^2} \ln \Gamma(n+1) = \sum_{p=1}^{N-1} \frac{1}{p} + N \sum_{p=N}^{\infty} \frac{1}{p^2} \sim \ln N + C + 1 \tag{B13}$$

from which (3.11) results.

For studying (3.12), we write it as

$$T_N(k) = \left\langle \exp\left[\frac{-ik}{(\ln N)^{1/2}} \sum_{n=1}^{N} \left(\ln|x_n|^2 - \langle \ln|x_n|^2 \rangle_N\right) \right] \right\rangle_N \tag{B14}$$

Since the N-body distribution function, after it has been integrated on all angles,[2] is proportional to

$$\exp\left(- \sum_{n=1}^{N} |x_n|^2 \right) \sum_P \prod_{n=0}^{N-1} |x_{\alpha_n}|^{2n}$$

where the sum runs on all permutation $P = (\alpha_0, \alpha_1, \alpha_2, \alpha_{N-1})$ of $(1, 2, 3, \ldots, N)$, one readily obtains from (B1) and (B14)

$$\ln T_N(k) = \sum_{n=0}^{N-1} \left[\ln \Gamma\left(n + 1 - \frac{ik}{(\ln N)^{1/2}} \right) - \ln \Gamma(n+1) + \frac{ik}{(\ln N)^{1/2}} \psi(n+1) \right] \tag{B15}$$

(We consider here the case in which the radius of the background has already been made infinite, and therefore only complete γ functions are involved.) Expanding (B15) in powers of k, we find

$$\ln T_N(k) = \frac{1}{2!} \left(\frac{-ik}{(\ln N)^{1/2}} \right)^2 \sum_{n=0}^{N-1} \psi'(n+1) + \frac{1}{3!} \left(\frac{-ik}{(\ln N)^{1/2}} \right)^3 \times \sum_{n=0}^{N-1} \psi''(n+1) + \cdots \tag{B16}$$

It is easy to see that $\sum \psi''(n+1)$ and similar sums involving higher derivatives of ψ have finite limits as $N \to \infty$. However, from (B13) it results that $\sum \psi'(n+1)$ behaves like $\ln N$. Therefore only the first term in the right-hand side of (B16) contributes to the limit $N \to \infty$, which is given by (3.13).

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