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Classical Coulomb Fluids in a Confined Geometry

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It has already been argued that a classical (three-dimensional) Coulomb fluid confined between two parallel walls exhibits ideal gas features when the distance between the walls becomes small; this is confirmed in the present paper. Two-dimensional models of Coulomb fluids (with a logarithmic interaction), confined in a strip, are also studied. These models do not become ideal gases in the narrow strip limit. The correlation functions are also studied. There is a special temperature at which exact results are obtained. At that temperature, the two-dimensional, two-component plasma (two-dimensional Coulomb gas), which is a conductor when unconfined, becomes a dielectric as soon as it is confined in a strip of noninfinite width. This can be understood as a displacement of the Kosterlitz–Thouless transition by the confinement.

KEY WORDS: Coulomb fluids; walls; two-dimensional models; Kosterlitz–Thouless phase transition.

1. INTRODUCTION

When a fluid with short-range interactions is confined between two parallel plates, it obeys an exact limiting law which says that the number density approaches the fugacity as the distance between the plates tends to zero.[1,2] At the same time, the pair correlation function becomes a dilute-gas one.[3] The limiting law about the density has been recently shown to hold also for ionic fluids.[4]

The present paper is another contribution to studying the properties of (classical) Coulomb fluids confined between two parallel hard walls. We consider specific simple models, about which we ask two questions:

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(a) Do the density and the fugacity approach each other when the distance between the walls goes to zero?

(b) How are the correlations affected by the presence of the walls?

Since we live in a three-dimensional world, we first revisit the three-dimensional fluids; this is done in Section 2.

In Section 3, we consider two-dimensional models (with a logarithmic interaction). They have the advantage that there is a special temperature at which they are exactly solvable. For many purposes, these two-dimensional models are believed to mimic rather well (at least qualitatively) the three-dimensional Coulomb fluids. However, the models with a logarithmic interaction have the "pathological" feature that they do *not* become ideal gases in the low-density limit, and that makes them special in the present context. Another specific feature of the two-dimensional case is the existence of the Kosterlitz–Thouless phase transition; how this transition is affected by the presence of walls is also discussed in Section 3.

2. THREE-DIMENSIONAL COULOMB FLUIDS

2.1. General Considerations

The equilibrium state of a fluid confined between two parallel hard walls can be characterized by its temperature T and its average volume density ρ: if the distance between the walls[2] is $2a$, ρ is defined by the requirement that the number of particles between two wall unit areas facing each other be $2a\rho$.

The limiting laws as a becomes small can be understood as follows. In the small-a limit the properties of the three-dimensional fluid are expected to approach those of a two-dimensional fluid having a *surface* density $\rho_s = 2a\rho$. If $a \to 0$ for a fixed value of ρ, $\rho_s \to 0$ and the fluid approaches a two-dimensional *dilute* gas. If the interactions are such that "dilute" implies "weakly coupled," in the small-a limit the excess chemical potential will vanish, i.e., the fugacity z of the three-dimensional fluid will approach its density ρ; at the same time, the pair correlation function will approach the simple form appropriate to a weakly coupled gas.

The above reasoning certainly applies to short-range interactions. It also applies to a Coulomb fluid. Indeed, for a two-dimensional fluid made of particles of charge $\pm e$, with an interaction potential e^2/r, at a surface number density ρ_s and a temperature T, a dimensionless coupling constant

[2] If the particles are hard spheres of diameter σ, it is convenient to call the distance between the walls $2a + \sigma$.

can be defined as $\varepsilon = 2\pi\rho_s e^4/(k_B T)^2$ (k_B is Boltzmann's constant); therefore the coupling does vanish with ρ_s and thus with a.

It is even possible to say how the small-a limit is approached for a Coulomb fluid, because the small-ε behavior of the two-dimensional fluid is known: it is given by using the Debye–Hückel theory with a suitable modification at short distances. Let us consider two specific models.

2.2. One-Component Plasma

The model is a system of identical particles of charge e embedded in a uniform background of opposite charge. The interaction potential between two particles at a distance r from one another is e^2/r. For the two-dimensional fluid, the Debye–Hückel approximation gives[5,6] a pair correlation function

$$h(r) = -\frac{e^2}{k_B T r}\int_0^\infty dx\,\frac{x}{x + r/\lambda}\,J_0(x) \tag{2.1}$$

where $\lambda = k_B T/2\pi e^2 \rho_s$; J_0 is a Bessel function. At large distances, i.e., for $r \gg \lambda$, $h(r)$ has only a power law decay, behaving like $-k_B T/4\pi^2 e^2 \rho_s^2 r^3$ (this decay is actually valid for arbitrary coupling strength). At small distances, (2.1) behaves like $-e^2/k_B T r$; this unacceptable result [the true pair distribution function $1 + h(r)$ should remain positive] is a well-known deficiency of the linearized Debye–Hückel theory. A more correct behavior of $h(r)$ at small distances would be $\exp(-e^2/k_B T r) - 1$.

Using (2.1) for computing the Coulomb energy per unit area of the two-dimensional fluid,

$$E_{\text{exc}} = \frac{1}{2}\rho_s^2 \int \frac{e^2}{r}\,h(r)\,d^2\mathbf{r} \tag{2.2}$$

would give a divergent result because of the incorrect small-r behavior of (2.1). It is necessary to take into account higher-order corrections to (2.1); to lowest order in ε (ε is defined in Section 2.1), these corrections amount to suppressing in (2.2) the contributions from $r \lesssim e^2/k_B T$, with the result[5,6]

$$\frac{E_{\text{exc}}}{k_B T} = \frac{1}{2}\rho_s \varepsilon \ln \varepsilon + O(\varepsilon)$$

Correspondingly, the excess chemical potential μ_{exc} is such that

$$\frac{\mu_{\text{exc}}}{k_B T} = \varepsilon \ln \varepsilon + O(\varepsilon)$$

Therefore, as $a \to 0$, the fugacity z of the confined three-dimensional fluid tends to its density as

$$z = \rho \exp \frac{\mu_{\text{exc}}}{k_{\text{B}} T} \sim \rho(1 + \varepsilon \ln \varepsilon)$$

with $\varepsilon = 4\pi\rho a e^4/(k_{\text{B}} T)^2$.

2.3. Two-Component Plasma

The model is a system of positive and negative particles of opposite charges $\pm e$. In addition to the Coulomb interaction $\pm e^2/r$, it is necessary to assume that there is some short-range repulsion (otherwise the system would be unstable against the collapse of pairs of oppositely charged particles). Here we shall assume that the particles are small, charged hard spheres of diameter σ.

Now (2.1) still holds for the pair correlation function between particles of the same sign, with, in the definition of λ, ρ_s meaning the *total* surface number density; the pair correlation function between particles of opposite signs is $-h(r)$. Again, these pair correlation functions hold for the confined three-dimensional fluid as $2a \to 0$.

The calculation of the Coulomb energy and of the excess chemical potential is similar to what is done in Section 2.2, except that the small-r cutoff is now provided by the hard core, at $r = \sigma$, if we assume that $\sigma \gg e^2/k_{\text{B}} T$. For the confined three-dimensional fluid, we now find the limiting law

$$z = \frac{1}{2} \rho \exp \frac{\mu_{\text{exc}}}{k_{\text{B}} T} \sim \frac{1}{2} \rho(1 + \varepsilon \ln Ca)$$

where ρ is the *total* number density, $\varepsilon = 4\pi\rho a e^4/(k_{\text{B}} T)^2$, and C is some constant independent of a. The case $\sigma \ll e^2/k_{\text{B}} T$ is more involved, and is not discussed here.

3. TWO-DIMENSIONAL COULOMB FLUIDS

3.1. General Considerations

We now deal with two-dimensional Coulomb fluids. The interaction potential between two particles of charges e and e' at a distance r from one another is $-ee' \ln(r/L)$, where L is some arbitrary length scale which defines the zero of the potential; let us recall that the choice of a logarithmic potential is dictated by the requirement that the usual laws of electrostatics (Poisson's equation, etc.) do hold in the two-dimensional world. The fluid will now be confined within a strip of width $2a$, bounded by hard walls, and we are especially interested in studying the small-width limit.

A peculiarity of the logarithmic interaction is that e^2 now has the dimensions of an energy, and therefore the dimensionless coupling constant is $\Gamma = e^2/k_B T$, *independent* of the density. As a consequence, the dilute gas is *not* in general a weakly coupled one, and the reasoning of Section 2.1 cannot be applied to the present case. However, it remains true that, as the strip width $2a$ tends to zero, the properties of the two-dimensional fluid should approach those of a one-dimensional fluid (with a logarithmic interaction), about which much is known. Let us see how this works, for the two models of the one-component plasma and the two-component plasma.

3.2. One-Component Plasma

The two-dimensional one-component plasma [again defined as a system of identical particles of charge e in a uniform background of opposite charge, but now with a logarithmic interaction $-e^2 \ln(r/L)$] has been widely studied. For the unconfined system, the equation of state has a simple form,[7] dictated by scaling considerations. In the high-temperature limit (near $\Gamma = 0$), the Debye–Hückel approximation[7,8] should be valid. More interestingly, there is a special temperature such that $\Gamma = e^2/k_B T = 2$, at which the whole thermodynamics and all the correlation functions can be computed exactly.[9,10]

Exact results can also be obtained for the two-dimensional one-component plasma confined in a strip, both in the Debye–Hückel regime ($\Gamma \ll 1$) and at $\Gamma = 2$, as follows.

3.2.1. Debye–Hückel regime ($\Gamma \ll 1$). We consider a strip of two-dimensional plasma confined between two hard walls parallel to the y direction and located at $x = \pm a$. The background charge density is $-e\rho$. What has been done in the case when there is only one hard wall[11,12] can be easily generalized to the present problem.

Let us first obtain the pair correlation function. To lowest order in Γ, the particle number density ρ can be considered as having the constant value ρ within the strip. Let $\psi(x_1, x_2, y)$ be the mean electrical potential at the point $\mathbf{r}_1 = (x_1, y)$ when there is a particle at the point $\mathbf{r}_2 = (x_2, 0)$.

Combining in the usual way the Poisson equation and the linearized Boltzmann equation, one obtains

$$\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial y^2}-\kappa^2\right)\psi(x_1, x_2, y) = -e\delta(\mathbf{r}_1-\mathbf{r}_2) \qquad (|x_1|, |x_2| < a)$$

$$\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial y^2}\right)\psi(x_1, x_2, y) = 0 \qquad (|x_1| > a, |x_2| < a)$$

where $\kappa^2 = 2\pi e^2\rho/k_B T = 2\pi\Gamma\rho$. The boundary conditions are that $\psi \to 0$ when $|\mathbf{r}_1| \to \infty$, and ψ and $\partial\psi/\partial x_1$ are continuous at $x_1 = \pm a$. In terms of the Fourier transform on y defined by

$$\psi(x_1, x_2, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\psi}(x_1, x_2, k)\, e^{iky}\, dk \tag{3.1}$$

the problem reduces to a one-variable differential equation

$$\left[\frac{d^2}{dx_1^2}-(\kappa^2+k^2)\right]\hat{\psi}(x_1, x_2, k) = -e\delta(x_1-x_2) \qquad (|x_1|, |x_2| < a)$$

$$\left(\frac{d^2}{dx_1^2}-k^2\right)\hat{\psi}(x_1, x_2, k) = 0 \qquad (|x_1| > a, |x_2| < a)$$

Taking into account the boundary and continuity conditions, one finds

$$\begin{aligned}\hat{\psi}(x_1, x_2, k) = e\frac{\pi}{K}\{&e^{-K|x_1-x_2|} + 2[(K+|k|)^2 e^{2Ka} - (K-|k|)^2 e^{-2Ka}]^{-1}\\ &\times[(K^2-k^2)\cosh K(x_1+x_2)\\ &+(K-|k|)^2 e^{-2Ka}\cosh K(x_1-x_2)]\}\end{aligned} \tag{3.2}$$

where $K = (\kappa^2+k^2)^{1/2}$. Using (3.2) in (3.1) gives a parametric representation of ψ. The pair correlation function h is related to ψ by the linearized Boltzmann equation:

$$h(x_1, x_2, y) = -(e/k_B T)\,\psi(x_1, x_2, y) \tag{3.3}$$

From the potential ψ, one can compute the thermodynamic properties. A particle located at $\mathbf{r}_2$ feels a potential (the self-potential being subtracted)

$$\Phi(x_2) = [\psi(\mathbf{r}_1, \mathbf{r}_2) + e\ln(|\mathbf{r}_1-\mathbf{r}_2|/L)]_{\mathbf{r}_1=\mathbf{r}_2}$$

and therefore the potential energy per unit length of the strip is

$$\begin{aligned} E_{\text{exc}} &= \frac{1}{2} e\rho \int_{-a}^{a} \Phi(x)\, dx \\ &= e^2 \rho a \left[\ln \frac{2}{\kappa L} - \gamma + \int_0^\infty \frac{dk}{K} \right. \\ &\quad \times \left. \frac{(K^2 - k^2)(Ka)^{-1} \sinh 2Ka + 2(K-k)^2 e^{-2Ka}}{(K+k)^2 e^{2Ka} - (K-k)^2 e^{-2Ka}} \right] \end{aligned} \tag{3.4}$$

where γ is Euler's constant. The other thermodynamic quantities can be obtained from $E_{\text{exc}}(\rho, T)$.

More explicit results can be given in the case of interest for us, i.e., when the strip width $2a$ becomes small. The excess energy (3.4) then has the behavior

$$E_{\text{exc}} = e^2 \rho a \left(\ln \frac{1}{2\pi \Gamma \rho a L} - \gamma \right) + o(a) \tag{3.5a}$$

Correspondingly, the excess free energy per unit length is

$$F_{\text{exc}} = e^2 \rho a \left(\ln \frac{1}{2\pi \Gamma \rho a L} - \gamma + 1 \right) + o(a) \tag{3.5b}$$

and the excess chemical potential is

$$\mu_{\text{exc}} = \frac{1}{2} e^2 \left(\ln \frac{1}{2\pi \Gamma \rho a L} - \gamma \right) + o(1) \tag{3.5c}$$

Thus, as $2a \to 0$, the excess chemical potential does *not* vanish (it actually diverges) and the fugacity z of the confined two-dimensional plasma does *not* approach its density ρ. Actually, the confined two-dimensional plasma approaches the one-dimensional one-component plasma (with a logarithmic interaction), the thermodynamics of which is exactly known,[(13)] and the expressions (3.5) in which one sets $2\rho a = \lambda$ are indeed the small-Γ results[3] for the one-dimensional plasma with a line density (number of particles per unit length) λ.

The small-a behavior of the pair correlation function can also be obtained. From (3.1)–(3.3), one finds

$$h(x_1 = 0, x_2 = 0, y) \sim -\frac{e^2}{k_B T} \int_0^\infty dk \frac{1 + 2ak}{\kappa^2 a + k(1 + 2ak)} \cos ky \tag{3.6}$$

[3] In the comparison, one must allow for different choices of the zero of the energy in ref. 13 and in the present paper.

The neglect of ak in (3.6) is not expected to change h in the region $y \gg a$. In this region

$$h(x_1=0, x_2=0, y) \sim -\frac{e^2}{k_B T}\int_0^\infty dk \frac{1}{\kappa^2 a + k} \cos ky$$

This is the result one would have obtained by using the Debye–Hückel approximation directly in the one-dimensional system with a line density $\lambda = 2\rho a$.

3.2.2. Exact Results at $\Gamma = 2$. At the special temperature such that $\Gamma = 2$, the problem of the two-dimensional one-component plasma confined in a strip was solved some time ago[14] in the canonical formalism.

When the strip width $2a$ becomes small, for a fixed value of the background charge density $-e\rho$, using the formulas in ref. 14, we find that the excess energy per unit length has the behavior

$$F_{\text{exc}} = e^2 \rho a \left(\ln \frac{1}{4\pi\rho a L} + 1 \right) + o(a) \tag{3.7a}$$

and correspondingly the excess chemical potential is

$$\mu_{\text{exc}} = \frac{e^2}{2} \ln \frac{1}{4\pi\rho a L} + o(1) \tag{3.7b}$$

Here also the excess chemical potential does not vanish as $a \to 0$, and the two-dimensional plasma does not become an ideal gas. The two-dimensional plasma approaches the one-dimensional plasma: if we set $2\rho a = \lambda$ in (3.7), we do recover the corresponding quantities of the one-dimensional plasma[13] with a line density λ.

3.2.3. Screening. The charge–charge correlation function of a *conducting* fluid confined between two parallel walls obeys a sum rule which is a consequence of screening: the sum rule is derived[15,16] under the assumption that the fluid perfectly screens an infinitesimal external charge. Conversely, the failure to obey this sum rule would mean that the fluid is not a conductor, at least in that screening sense.

The sum rule is expected to hold for a three-dimensional Coulomb fluid confined in a slab.

For a two-dimensional fluid in a strip bounded by the lines $x = \pm a$, the charge–charge correlation between points $\mathbf{r}_1$ and $\mathbf{r}_2$ is a function $S(x_1, x_2, y)$, where y is the y component of $\mathbf{r}_1 - \mathbf{r}_2$. The Fourier transform with respect to y is defined by

$$S(x_1, x_2, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{S}(x_1, x_2, k)\, e^{iky}\, dk$$

With these notations, the two-dimensional version of the sum rule is

$$\hat{s}(k) = \int_{-a}^{a} dx_1 \int_{-a}^{a} dx_2\, \hat{S}(x_1, x_2, k) \sim \frac{k_B T}{\pi} |k| \qquad (k \to 0) \qquad (3.8)$$

An equivalent statement about

$$s(y) = \int_{-a}^{a} dx_1 \int_{-a}^{a} dx_2\, S(x_1, x_2, y)$$

is: the "monotonic" part of the asymptotic form of $s(y)$ as $|y| \to \infty$ is $-k_B T/(\pi^2 y^2)$, the Fourier transform of (3.8). However, there may also be in the asymptotic form of $s(y)$ oscillating terms[15] generated by possible singularities of $\hat{s}(k)$ at nonzero values of k.

In the one-component plasma, S is related to the pair correlation function by

$$S(x_1, x_2, y) = e^2[\rho\delta(\mathbf{r}_1 - \mathbf{r}_2) + \rho^2 h(x_1, x_2, y)]$$

In the Debye–Hückel regime ($\Gamma \ll 1$) using (3.1)–(3.3), one can check that the sum rule (3.8) is indeed satisfied. The sum rule is also satisfied[15] at $\Gamma = 2$. Thus, at least for these two cases, the two-dimensional one-component plasma remains a conductor when confined in a strip.

3.3. Two-Component Plasma

The two-dimensional two-component plasma is a system of positive and negative charges $\pm e$. The interaction between two particles of the same species (of different species) is $-e^2 \ln r/L (+e^2 \ln r/L)$. Again, the dimensionless coupling constant is $\Gamma = e^2/k_B T$, independent of the density. The equation of state has the same simple form[7] as for the one-component plasma. Again, in the high-temperature limit (near $\Gamma = 0$), the Debye–Hückel approximation should be valid. Again, at the special temperature such that $\Gamma = 2$, the thermodynamics and all the correlation functions can be computed exactly.[17,18] However, for $\Gamma \geqslant 2$, the point-particle system would be unstable against the collapse of pairs of oppositely charged particles, and the particles are assumed to be small hard disks of diameter σ; the results for the thermodynamics at $\Gamma = 2$ are valid only in the small-σ limit.

The two-dimensional bulk two-component plasma has the very interesting feature of having a phase transition, the celebrated Kosterlitz–Thouless transition[19,20] between a conducting high-temperature phase and a dielectric low-temperature phase; in the low-density limit (or, equivalently, the small hard-disk limit), the transition takes place at $\Gamma = 4$.

Let us now consider the two-dimensional two-component plasma confined in a strip.

3.3.1. Debye–Hückel Regime ($\Gamma \ll 1$). The calculations for the two-component plasma are very similar to those done in Section 3.2.1 for the one-component plasma. In the definition of $\kappa^2 = 2\pi e^2 \rho / k_B T = 2\pi \Gamma \rho$, ρ must be taken as the total number density (particles of both signs being counted). Then the results of Section 3.2.1 are still valid. However, (3.3) is the pair correlation function between particles of the same sign; the pair correlation function between particles of opposite signs is $-h$.

3.3.2. Exact Results at $\Gamma = 2$. At the special temperature such that $\Gamma = 2$, the problem of the two-dimensional two-component plasma confined in a strip can be solved exactly, by a generalization of a previous work[21] about the plasma along one wall.

The grand canonical formalism is used. It is convenient to introduce an inverse length $m = 2\pi L z$, where z is the fugacity and L the length scale of the logarithmic potential. For the unconfined system $(2m)^{-1}$ turns out to be the correlation length. As shown in ref. 21, the thermodynamics, the density profile, and the correlations can be expressed in terms of Green functions $G_{s_1 s_2}(\mathbf{r}_1, \mathbf{r}_2)$ $(s_1, s_2 = \pm 1)$. The one-body density of the particles of sign s is

$$\rho_s(\mathbf{r}) = m G_{ss}(\mathbf{r}, \mathbf{r}) \tag{3.9}$$

(for obtaining nondivergent densities, one must introduce a small nonzero diameter σ for the particles). The two-body truncated densities are

$$\rho^{(2)T}_{s_1 s_2}(\mathbf{r}_1, \mathbf{r}_2) = -m^2 G_{s_1 s_2}(\mathbf{r}_1, \mathbf{r}_2)\, G_{s_2 s_1}(\mathbf{r}_2, \mathbf{r}_1) \tag{3.10}$$

(the two-body and higher *truncated* densities are finite even for point particles). It is useful to note the symmetry relations[4]

$$G_{ss}(\mathbf{r}_1, \mathbf{r}_2) = \overline{G_{ss}(\mathbf{r}_2, \mathbf{r}_1)}, \qquad G_{s-s}(\mathbf{r}_1, \mathbf{r}_2) = -\overline{G_{-ss}(\mathbf{r}_2, \mathbf{r}_1)}$$

In the present geometry (a strip along the y axis, bounded by the lines $x = \pm a$), the relevant variables are x_1, x_2, and y, and it is convenient to introduce the Fourier transform $\hat{G}$ defined by

$$G_{s_1 s_2}(x_1, x_2, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{G}_{s_1 s_2}(x_1, x_2, k)\, e^{iky}\, dk$$

[4] The first of these relations was misprinted in ref. 21.

Inside the strip, $\hat{G}_{++}$ is the solution of the differential equation

$$\left(\frac{d^2}{dx_1^2} - m^2 - k^2\right) \hat{G}_{++}(x_1, x_2, k) = -m\delta(x_1 - x_2)$$

while

$$\hat{G}_{-+}(x_1, x_2, k) = \frac{1}{m}\left(k - \frac{d}{dx_1}\right) \hat{G}_{++}(x_1, x_2, k) \tag{3.11}$$

The boundary conditions are

$$\hat{G}_{++}(x_1 = a, x_2, k) = 0, \qquad \hat{G}_{-+}(x_1 = -a, x_2, k) = 0 \qquad (k > 0)$$

$$\hat{G}_{++}(x_1 = -a, x_2, k) = 0, \qquad \hat{G}_{-+}(x_1 = a, x_2, k) = 0 \qquad (k < 0)$$

Similar equations hold for $\hat{G}_{--}$ and $\hat{G}_{+-}$.

One finds the solution

$$\begin{aligned} &\hat{G}_{++}(x_1, x_2, k) \\ &\quad = \frac{m}{2K}\{e^{-K|x_1 - x_2|} + [(K+k)\, e^{2Ka} + (K-k)\, e^{-2Ka}]^{-1} \\ &\quad\quad \times [(K-k)\, e^{-K(x_1 + x_2)} - (K+k)\, e^{K(x_1 + x_2)} \\ &\quad\quad -2(K-k)\, e^{-2Ka} \cosh K(x_1 - x_2)]\} \quad (k > 0) \end{aligned} \tag{3.12a}$$

$$\begin{aligned} &\hat{G}_{++}(x_1, x_2, k) \\ &\quad = \frac{m}{2K}\{e^{-K|x_1 - x_2|} + [(K+k)\, e^{-2Ka} + (K-k)\, e^{2Ka}]^{-1} \\ &\quad\quad \times [(K+k)\, e^{K(x_1 + x_2)} - (K-k)\, e^{-K(x_1 + x_2)} \\ &\quad\quad -2(K+k)\, e^{-2Ka} \cosh K(x_1 - x_2)]\} \quad (k < 0) \end{aligned} \tag{3.12b}$$

where $K = (m^2 + k^2)^{1/2}$. $\hat{G}_{-+}$ is given by (3.11).

When the width $2a$ becomes zero, the $G_{s_1 s_2}(x_1 = 0, x_2 = 0, k)$ have well-defined limits, provided one uses in (3.11) the "symmetrical" definition

$$\frac{d}{dx} e^{-K|x|}\big|_{x=0} = 0$$

One finds

$$\hat{G}_{++}(x_1 = 0, x_2 = 0, k) = 0, \qquad \hat{G}_{-+}(x_1 = 0, x_2 = 0, k) = \frac{1}{2}\frac{k}{|k|}$$

the Fourier transforms of which are

$$\hat{G}_{++}(x_1=0, x_2=0, y)=0, \qquad G_{-+}(x_1=0, x_2=0, y)=\frac{i}{2\pi y}$$

Using these functions in (3.10) together with the symmetry properties of G, one finds the two-body truncated densities

$$\rho_{ss}^{(2)T}(x_1=0, x_2=0, y)=0, \quad \rho_{s-s}^{(2)T}(x_1=0, x_2=0, y)=\frac{m^2}{4\pi^2 y^2}=\frac{(zL)^2}{y^2} \quad (3.13)$$

Thus, in the zero-width limit $2a=0$, the correlation functions have several remarkable features:

1. For a given fugacity z, the truncated two-body densities remain finite without the need of any short-distance cutoff in the Coulomb interaction.

2. There is no correlation between two particles of the same sign.

3. The truncated two-body density for two particles of different signs has the simple form $m^2/4\pi^2 y^2$.

Using $G_{ss}=0$ in (3.9) would yield $\rho_s=0$. However, this result $\rho_s=0$ is obtained by setting first $a=0$ in $\hat{G}_{ss}(x_1=0, x_2=0, k)$ and computing the Fourier transform $G_{ss}(x_1=0,\ x_2=0,\ y=0)$ afterward, while performing first the Fourier transformation would yield $\rho_s=\infty$. In fact, ρ_s is not a well-defined quantity for a point-particle system, and our approach to the small-width problem is not well adapted to the hard-disk system. Let us only note that here again we do *not* have $\rho_s=z$ in the small-width limit.

Finally, it is perhaps interesting to compare the two-dimensional system in the infinitely narrow strip limit and the one-dimensional two-component plasma (with a logarithmic interaction) for which exact results are available, in a lattice version of the model[17,22,23] (the presence of the lattice makes the system stable against collapse). Let us use for this one-dimensional model a notation adapted to the present paper: ζ is the fugacity normalized in such a way that ζ becomes the usual fugacity in the continuum limit (i.e., ζ is an inverse length), L is the length scale in the logarithmic interaction $\pm e^2 \ln(r/L)$, and τ is the lattice spacing (defined as the minimum distance between two particles of the *same* sign). Thus, the results for the one-dimensional lattice model at $\Gamma=2$ are as follows:

1. The line densities (numbers of particles of a given sign per unit length) are

$$\lambda_+=\lambda_-=\frac{1}{\tau}\frac{(\pi\zeta L)^2}{1+(\pi\zeta L)^2}$$

2. There is no correlation between particles of the same sign on different lattice sites.

3. The truncated two-body line density for two particles of different signs on different lattice sites at a distance y from one another is

$$\lambda^{(2)T}_{-+}(y) = \frac{(\zeta L)^2}{[1+(\pi\zeta L)^2]^2\, y^2} \tag{3.14}$$

A kind of continuum limit can be approached by choosing $\pi\zeta L \ll 1$, a condition which ensures that the average occupation number $\lambda_{\pm}\tau$ of a lattice site be small. Then $\lambda^{(2)T}_{-+}(y) \sim (\zeta L)^2/y^2$, and a comparison with (3.13) shows that the correlations of the two-dimensional system in the zero-width limit and of the one-dimensional system are consistent with each other.

3.3.3. Screening. The sum rule (3.8) provides a test for the conducting or dielectric nature of the system; the sum rule is obeyed if and only if the system is a conductor.

In the Debye–Hückel regime ($\Gamma \ll 1$), for the two-component plasma in a strip, by essentially the same calculation as for the one-component plasma, one can check that the sum rule (3.8) is indeed satisfied.

However, at $\Gamma = 2$, the story is a different one: the sum rule (3.8) is *not* satisfied. Indeed, from (3.10) and the symmetry properties of G,

$$S(x_1, x_2, y) = -2e^2m^2[|G_{++}(x_1, x_2, y)|^2 + |G_{-+}(x_1, x_2, y)|^2] \qquad (y \neq 0) \tag{3.15}$$

The Fourier transforms $\hat{G}_{s_1s_2}(x_1, x_2, k)$ are discontinuous at $k=0$ (without any other singularity on the k real axis). From (3.12) and (3.11) one finds

$$\hat{G}_{++}(x_1, x_2, 0^+) - \hat{G}_{++}(x_1, x_2, 0^-) = -\frac{\sinh m(x_1+x_2)}{\cosh 2ma}$$

$$\hat{G}_{-+}(x_1, x_2, 0^+) - \hat{G}_{-+}(x_1, x_2, 0^-) = \frac{\cosh m(x_1+x_2)}{\cosh 2ma}$$

and therefore one finds the asymptotic decays

$$G_{++}(x_1, x_2, y) \underset{|y|\to\infty}{\simeq} \frac{\sinh m(x_1+x_2)}{\cosh 2\,ma} \frac{1}{2\pi i y}$$

and

$$G_{-+}(x_1, x_2, y) \underset{|y|\to\infty}{\simeq} -\frac{\cosh m(x_1+x_2)}{\cosh 2ma} \frac{1}{2\pi i y}$$

From (3.15) and $e^2 = 2k_B T$,

$$S(x_1, x_2, y) \underset{|y| \to \infty}{\simeq} -k_B T m^2 \frac{\cosh 2m(x_1 + x_2)}{\cosh^2 2ma} \frac{1}{\pi^2 y^2}$$

and finally

$$s(y) = \int_{-a}^{a} dx_1 \int_{-a}^{a} dx_2\, S(x_1, x_2, y) \underset{|y| \to \infty}{\simeq} -\frac{k_B T \tanh^2 2ma}{\pi^2 y^2}$$

Thus, although $s(y)$ still decays like y^{-2}, the equivalent in coordinate space of the sum rule (3.8) fails by a factor $\tanh^2(2ma)$; this factor depends on the ratio between the strip width $2a$ and the correlation length $(2m)^{-1}$ of the unconfined system. For an unconfined system ($2a \to \infty$), $\tanh^2(2ma) \to 1$, the sum rule is satisfied, and the system is in its conducting phase, in agreement with the fact that, at $\Gamma = 2$, one is above the usual Kosterlitz–Thouless transition temperature by a factor 2. However, for a system confined in a strip of finite width, the sum rule fails, and therefore the system is no longer in its conducting phase. For small widths, one finds $s(y) \sim -k_B T(2ma)^2/\pi^2 y^2$, recovering the one-dimensional behavior one would obtain from (3.13) or from the continuum limit ($\pi\zeta L \ll 1$) of (3.14).

It was already known that the one-dimensional two-component plasma (with a logarithmic interaction) is not a conductor[23,24] at $\Gamma = 2$ (except for the special case of a half-filled lattice). We have just shown that the two-dimensional system at $\Gamma = 2$ acquires that specific feature of being nonconducting as soon as it is confined in a strip. In that sense, the strip behaves like a strictly one-dimensional system.

4. SUMMARY AND CONCLUSION

In our three-dimensional world, when a Coulomb fluid is confined between two parallel walls, the distance of which becomes small, ideal gas features do appear: on one hand, the density and the fugacity approach each other, and on the other hand the pair correlation function between two particles interacting through the Coulomb law $\pm e^2/k_B T r$ approaches the form $\exp[\mp e^2/k_B T r] - 1$, except at very large distances, where residual many-body effects screen the correlation function into an r^{-3} behavior.

The two-dimensional models, with a logarithmic interaction, which mimic rather well the three-dimensional fluids in other contexts, have rather specific properties when confined in a strip between parallel walls. When the distance between the walls becomes small, the two-dimensional models do *not* become ideal gases, and their density and fugacity do *not* approach each other.

The charge–charge correlation function of a *conducting* fluid confined between parallel walls obeys a sum rule (which expresses the perfect screening of an infinitesimal external charge) in two as well as in three dimensions. The two-dimensional one-component plasma does satisfy this sum rule when the coupling constant Γ is such that $\Gamma \ll 1$, and also when $\Gamma = 2$. The two-dimensional *two*-component plasma satisfies the sum rule for $\Gamma \ll 1$, but *not* at $\Gamma = 2$. Therefore, it seems that for the two-dimensional two-component plasma confined in a strip, there is a phase transition between a high-temperature conducting phase and a low-temperature nonconducting phase at some value of Γ within the range $(0, 2]$. Analogy with the one-dimensional lattice system[23,24] suggests that the confined two-dimensional system might be conducting at $\Gamma < 2$ and insulating at $\Gamma \geqslant 2$. The transition looks like a modification of the Kosterlitz–Thouless transition, which for an unconfined (low-density) two-dimensional two-component plasma takes place at $\Gamma = 4$. The phase diagram of the confined plasma certainly requires further investigation.

REFERENCES

1. M. S. Wertheim, L. Blum, and D. Bratko, in *Proceedings of the 1985 Symposium on Chemical Physics of Colloidal Phenomena*, S. H. Chen and R. Rajagopalan, eds. (Springer, New York, 1990).
2. A. Luzar, D. Bratko, and L. Blum, *J. Chem. Phys.* **86**:2955 (1987).
3. J. R. Henderson, *Mol. Phys.* **59**:89 (1986).
4. D. Bratko, D. J. Henderson, and L. Blum, *Phys. Rev. A* **44**:8235 (1991).
5. J. Chalupa, *Phys. Rev. B* **12**:4 (1975); erratum **13**:2243 (1976).
6. H. Totsuji, *J. Phys. Soc. Japan* **39**:253 (1975).
7. E. H. Hauge and P. C. Hemmer, *Phys. Norveg.* **5**:109 (1971).
8. A. Alastuey, *Ann. Phys. Fr.* **11**:653 (1986).
9. A. Alastuey and B. Jancovici, *J. Phys.* (Paris) **42**:1 (1981).
10. B. Jancovici, *Phys. Rev. Lett.* **46**:386 (1981).
11. R. L. Guernsey, *Phys. Fluids* **13**:2089 (1970).
12. A. S. Usenko and I. P. Yakimenko, *Sov. Tech. Phys. Lett.* **5**:549 (1979).
13. M. L. Mehta, *Random Matrices* (Academic Press, New York, 1967).
14. P. J. Forrester and E. R. Smith, *J. Phys. A: Math. Gen.* **15**:3861 (1982).
15. P. J. Forrester, B. Jancovici, and E. R. Smith, *J. Stat. Phys.* **31**:129 (1983).
16. Ph. A. Martin, *Rev. Mod. Phys.* **60**:1075 (1988).
17. M. Gaudin, *J. Phys.* (France) **46**:1027 (1985).
18. F. Cornu and B. Jancovici, *J. Stat. Phys.* **49**:33 (1987).
19. J. M. Kosterlitz and D. J. Thouless, *J. Phys. C* **6**:1181 (1973).
20. P. Minnhagen, *Rev. Mod. Phys.* **59**:1001 (1987).
21. F. Cornu and B. Jancovici, *J. Chem. Phys.* **90**:2444 (1989).
22. P. J. Forrester, *J. Stat. Phys.* **51**:457 (1988).
23. P. J. Forrester, *J. Stat. Phys.* **54**:57 (1989).
24. P. J. Forrester, *J. Stat. Phys.* **60**:203 (1990).