Derivation of an asymptotic expression in Beenakker's general fluctuation formula for random-matrix correlations near an edge

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We use a linear response argument and macroscopic electrostatics to derive the formula $\langle \sigma(x)\sigma(x')\rangle^T \sim -(x+x')/[\beta 2\pi^2 (xx')^{1/2}(x-x')^2]$, where $\sigma(x)$ denotes the charge density, for general logarithmic-potential Coulomb systems on a half-line $x>0$.

Beenakker[1] has recently obtained a general fluctuation formula for the variance of a linear statistic of the eigenvalues near the edge of a random matrix ensemble. Explicitly, with the scaled probability density function of the eigenvalues given by

$$\prod_{l=1}^{N} e^{-\beta V(\lambda_l)} \prod_{1\le j<k\le N} |\lambda_k-\lambda_j|^{\beta}, \quad \lambda_j \ge 0 \tag{1}$$

[the scale $\lambda_j \mapsto s(N)\lambda_j$ is to be chosen so that the number of eigenvalues near an edge remains finite and nonzero], Beenakker obtained the formula (see also Ref. 2)

$$\lim_{\alpha\to\infty} \lim_{N\to\infty} \mathrm{var}\left[\sum_{j=1}^{N} f(1/(1+\lambda_j/\alpha))\right] = \frac{1}{\beta\pi^2}\int_0^\infty dk |F(k)|^2 k \tanh(\pi k) , \tag{2}$$

where

$$F(k)=\int_{-\infty}^{\infty} dx\, e^{ikx} f\left[\frac{1}{1+e^x}\right] . \tag{3}$$

This formula was used[1] to calculate the variance of the conductance in mesoscopic wires.

To derive (2), Beenakker established a formula for the asymptotics of the truncated two-particle distribution between eigenvalues near the edge:

$$\rho^T(x,x') \sim -\frac{1}{2\pi^2\beta\sqrt{xx'}}\frac{x+x'}{(x-x')^2} . \tag{4}$$

Using (4), it is a relatively simple task to deduce (2). It is our purpose to provide a derivation of (4).

Like Beenakker,[1] our starting point is to observe that the probability density function (1) for eigenvalues is, up to a multiplicative constant, the Boltzmann factor for a one-component logarithmic potential Coulomb system confined to the vicinity of the origin by an external potential (which may be due to a neutralizing background) with Boltzmann factor $e^{-\beta V(x)}$. More generally, let us consider any logarithmic-potential Coulomb system confined to a half-line, and assume that the system is in its conductive phase [the two-component Coulomb system on a line undergoes a Kosterlitz-Thouless transition from a conductive to an insulator phase for $\beta \ge 2$ (Ref. 3)]. We will use a linear response argument and macroscopic electrostatics to deduce that the leading nonoscillatory term in the asymptotic expansion of the charge-charge correlation function near the edge of such systems has the universal form given by the right-hand side of (4). This technique is of general applicability to Coulomb systems in both two and three dimensions, and in a forthcoming publication[4] will be applied by one of us to deduce the asymptotic behavior of some surface-charge–surface-charge correlations.

By our assumptions, the system can be viewed as a conducting half-line, on the positive x axis say, imbedded in a plane and obeying the laws of $2d$ electrostatics. Let us now perturb the system by adding an external charge δq at a point $\mathbf{r}'$. The Hamiltonian of the system is then perturbed by an amount

$$\delta H = \delta q \Phi(\mathbf{r}') , \tag{5}$$

where $\Phi(\mathbf{r}')$ is the electrostatic potential at the point $\mathbf{r}'$ due to the half-line. We consider the canonical average of the potential $\Phi(\mathbf{r})$ at another point $\mathbf{r}$. By linear response theory we have

$$\langle \Phi(\mathbf{r})\rangle - \langle \Phi(\mathbf{r})\rangle_0 = -\beta\langle \Phi(\mathbf{r})\delta H\rangle^T = -\beta\delta q\langle \Phi(\mathbf{r})\Phi(\mathbf{r}')\rangle^T , \tag{6}$$

where $\langle\ \rangle_0$ refers to the value of the average before the perturbation and $\langle AB\rangle^T = \langle AB\rangle_0 - \langle A\rangle_0\langle B\rangle_0$. Now, if the microscopic detail is disregarded, the left-hand side of (6) can be obtained by macroscopic electrostatics and is just the potential at a point $\mathbf{r}$ due to the charge induced on a conducting half-line by the charge δq (Φ does not include the potential due to δq itself). With $\mathbf{r}$ specified by polar coordinates r,θ and $z=re^{i\theta}$ (and similarly $\mathbf{r}'$), it is thus easy to verify that

$$\begin{aligned}\langle \Phi(\mathbf{r})\rangle - \langle \Phi(\mathbf{r})\rangle_0 &= \delta q[\ln|z-z'| - \ln|z^{1/2}-z'^{1/2}| + \ln|z^{1/2}-\bar{z}'^{1/2}|] \\ &= \delta q\{\tfrac{1}{2}\ln[r^2+r'^2-2rr'\cos(\theta-\theta')] - \tfrac{1}{2}\ln[r+r'-2\sqrt{rr'}\cos(\theta-\theta')/2] \\ &\quad + \tfrac{1}{2}\ln[r+r'-2\sqrt{rr'}\cos(\theta+\theta')/2]\} .\end{aligned} \tag{7}$$

Substituting (7) in (6) gives an equation for the potential-potential correlation at points outside the half-line. To obtain from this the correlation between charge densities on the line, we again appeal to macroscopic electrostatics, which relates the surface charge density $\sigma(x)$ to the discontinuity of the perpendicular component of the electric field $E^+(x)-E^-(x)$ at the surface according to

$$E^+(x)-E^-(x)=2\pi\sigma(x)\ . \tag{8}$$

In polar coordinates, the perpendicular component of the electric field at the half-line is the angular component E_θ. Now in general

$$\langle E_\theta(\mathbf{r})E_{\theta'}(\mathbf{r}')\rangle^T=\frac{1}{rr'}\frac{\partial^2}{\partial\theta\partial\theta'}\langle\Phi(\mathbf{r})\Phi(\mathbf{r}')\rangle^T\ . \tag{9}$$

Thus from (7) and (6) we have

$$\beta\langle E^+(x)E_{\theta'}(\mathbf{r}')\rangle^T=\frac{1}{xr'}\frac{\partial}{\partial\theta'}\times\left[\frac{xr'\sin\theta'}{x^2+r'^2-2xr'\cos\theta'}-\frac{\sqrt{xr'}\sin\theta'/2}{x+r'-2\sqrt{xr'}\cos\theta'/2}\right], \tag{10}$$

where

$$E^+(x)=\lim_{\theta\to0^+}E_\theta(\mathbf{r})\ .$$

Taking the limits $\theta'\to0$ and $\theta'\to2\pi$ then gives

$$\beta\langle E^+(x)E^p(x')\rangle^T=\left[\frac{1}{x^2+x'^2-2xx'}-\operatorname{sgn}(p)\frac{1}{2\sqrt{xx'}}\times\frac{1}{x+x'-2\operatorname{sgn}(p)\sqrt{xx'}}\right], \tag{11}$$

where $p=+,-$ and

$$E^-(x')=\lim_{\theta\to2\pi^-}E_\theta(\mathbf{r}')\ . \tag{12}$$

And proceeding similarly we can also show

$$\beta\langle E^-(x)E^p(x')\rangle^T=\beta\langle E^+(x)E^{-p}(x')\rangle^T\ . \tag{13}$$

But from (9)

$$\beta\langle\sigma(x)\sigma(x')\rangle^T=\frac{\beta}{4\pi^2}\langle[E^+(x)-E^-(x)]\times[E^+(x')-E^-(x')]\rangle^T\ . \tag{14}$$

Substituting (13) and (11) in (14) then gives

$$\langle\sigma(x)\sigma(x')\rangle^T\sim-\frac{1}{2\pi^2\beta\sqrt{xx'}}\frac{x+x'}{(x-x')^2} \tag{15}$$

in precise agreement with the Beenakker formula (4). From our derivation it is clear that in (15) $\sigma(x)$ must be understood as a macroscopic charge density, i.e., averaged on a large number of adjacent particles.

Beenakker illustrated (4) using a known exact formula for the truncated two-particle distribution of the so-called Laguerre ensemble [$V(x)=x/2-(\alpha/2)\ln x$ in (1)] with $\beta=2$. To conclude this Brief Report, we remark that the exact results in Ref. 5 for the scaled edge distribution of the Gaussian ensemble [$V(x)=x^2/2$ in (1)] with $\beta=1$, 2, and 4 provide further illustrations. For example, with $\beta=2$ the scaled truncated two-body distribution is given by[5]

$$\rho^T(x,x')=-\left[\frac{\mathrm{Ai}(-x)\mathrm{Ai}'(-x')-\mathrm{Ai}(-x')\mathrm{Ai}'(-x)}{x-x'}\right]^2, \tag{16}$$

where $\mathrm{Ai}(x)$ denotes the Airy function and $\mathrm{Ai}'(x)$ its derivative. From the asymptotic expansion

$$\mathrm{Ai}(-x)\sim\frac{1}{\pi^{1/2}x^{1/4}}\cos(\tfrac{2}{3}x^{3/2}-\pi/4) \tag{17}$$

it is easy to verify that the large x,x' expansion of (16), averaged on oscillations, is given by the right-hand side of (4) with $\beta=2$, as expected.

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