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Guest Charge and Potential Fluctuations in Two-Dimensional Classical Coulomb Systems

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Abstract A known generalization of the Stillinger-Lovett sum rule for a guest charge immersed in a two-dimensional one-component plasma (the second moment of the screening cloud around this guest charge) is more simply retrieved, just by using the BGY hierarchy for a mixture of several species; the zeroth moment of the excess density around a guest charge immersed in a two-component plasma is also obtained. The moments of the electric potential are related to the excess chemical potential of a guest charge; explicit results are obtained in several special cases.

Keywords Coulomb systems · Two dimensions · Potential fluctuations · Sum rules

1 Introduction

One of us (L.Š.) has derived a generalization of the Stillinger-Lovett sum rule for a guest charge immersed in a two-dimensional one-component plasma [1]: an exact simple expression for the second moment of the screening cloud around the guest charge was obtained, by using a mapping technique onto a discrete one-dimensional anticommuting-field theory. In the present paper, we first show that the same result can be obtained in a simpler way by just using the BGY hierarchy, which provides also more general results.

The excess chemical potential of a guest charge (which can be expressed in terms of the charge density of the screening cloud) has an expansion in powers of the guest-particle charge Ze, which allows to compute the average of powers (moments) of the electric potential.

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We consider a classical (i.e. non-quantum) system of charged particles located in an infinite two-dimensional (2D) plane of points $\mathbf{r} \in \mathrm{R}^2$. According to the laws of 2D electrostatics, the particles can be thought of as infinitely long charged lines in the 3D which are perpendicular to the 2D plane. The electrostatic potential v at a point $\mathbf{r}$, induced by a unit charge at the origin $\mathbf{0}$, is thus given by the 2D Poisson equation

$$\Delta v(\mathbf{r}) = -2\pi\delta(\mathbf{r}). \tag{1.1}$$

The solution of this equation, subject to the boundary condition $\nabla v(\mathbf{r}) \to 0$ as $|\mathbf{r}| \to \infty$, reads

$$v(r) = -\ln\left(\frac{r}{L}\right), \tag{1.2}$$

where $r = |\mathbf{r}|$ and the free length constant L, which determines the zero point of the potential, will be set for simplicity to unity. The Fourier component of this potential $\tilde{v}(\mathbf{k}) \propto 1/k^2$ exhibits the characteristic singularity at $k = 0$, which maintains many generic properties (like screening) of "real" 3D charged systems.

A general Coulomb system consists of M mobile species $\alpha = 1, 2, \ldots, M$ with the corresponding charges e_α (which may be integer multiples of the elementary charge e). Mobile particles may be embedded in a fixed uniform background of charge density ρ_b. The most studied models are the one-component plasma (OCP), which corresponds to $M = 1$ with $e_1 = e$ and ρ_b of opposite sign, and the symmetric two-component plasma (TCP), which corresponds to $M = 2$ with $e_1 = e$, $e_2 = -e$, $\rho_b = 0$. The interaction energy of a configuration $\{\mathbf{r}_i, e_{\alpha_i}\}$ of the charged particles plus the background is

$$E = \sum_{i<j} e_{\alpha_i} e_{\alpha_j} v(|\mathbf{r}_i - \mathbf{r}_j|) + \sum_i e_{\alpha_i} \phi_b(\mathbf{r}_i) + E_{b-b}, \tag{1.3}$$

where $\phi_b(\mathbf{r})$ is the one-body potential created by the background and the background-background energy term E_{b-b} does not depend on the particle coordinates. In the case of point particles, for many-component systems with at least two oppositely species, the singularity of the Coulomb potential (1.2) at the origin $\mathbf{r} = \mathbf{0}$ prevents, for small enough temperatures, the thermodynamic stability against the collapse of positive-negative pairs of charges. In those cases, one introduces to v a short-range repulsion which prevents the collapse.

The Coulomb system is considered in thermodynamic equilibrium, at inverse temperature $\beta = 1/(k_B T)$. The thermal average over an infinite neutral system will be denoted by $\langle \cdots \rangle$. In terms of the microscopic density of particles of species α, $\hat{n}_\alpha(\mathbf{r}) = \sum_i \delta_{\alpha,\alpha_i} \delta(\mathbf{r} - \mathbf{r}_i)$, the microscopic total number density and the microscopic total charge density are defined, respectively, by

$$\hat{n}(\mathbf{r}) = \sum_\alpha \hat{n}_\alpha(\mathbf{r}), \qquad \hat{\rho}(\mathbf{r}) = \sum_\alpha e_\alpha \hat{n}_\alpha(\mathbf{r}) + \rho_b. \tag{1.4}$$

The microscopic electrostatic potential created by the particle-background system at point $\mathbf{r}$ is given by

$$\hat{\phi}(\mathbf{r}) = \int \mathrm{d}\mathbf{r}' v(\mathbf{r} - \mathbf{r}') \hat{\rho}(\mathbf{r}'). \tag{1.5}$$

At the one-particle level, the homogeneous number density of species α and the total particle number density are given respectively by

$$n_\alpha = \langle \hat{n}_\alpha(\mathbf{r}) \rangle, \qquad n = \langle \hat{n}(\mathbf{r}) \rangle. \tag{1.6}$$

The charge density $\rho = \langle \hat{\rho}(\mathbf{r}) \rangle$ vanishes due to the charge neutrality of the system. At the two-particle level, one introduces the translationally invariant two-body densities

$$n^{(2)}_{\alpha\alpha'}(|\mathbf{r}-\mathbf{r}'|) = \left\langle \sum_{i\neq j} \delta_{\alpha,\alpha_i}\delta(\mathbf{r}-\mathbf{r}_i)\delta_{\alpha',\alpha_j}\delta(\mathbf{r}'-\mathbf{r}_j) \right\rangle$$
$$= \langle \hat{n}_\alpha(\mathbf{r})\hat{n}_{\alpha'}(\mathbf{r}')\rangle - \langle \hat{n}_\alpha(\mathbf{r})\rangle \delta_{\alpha,\alpha'}\delta(\mathbf{r}-\mathbf{r}'). \tag{1.7}$$

It is useful to consider also the pair distribution functions

$$g_{\alpha\alpha'}(|\mathbf{r}-\mathbf{r}'|) = \frac{n^{(2)}_{\alpha\alpha'}(|\mathbf{r}-\mathbf{r}'|)}{n_\alpha n_{\alpha'}}, \tag{1.8}$$

the (truncated) pair correlation functions $h_{\alpha\alpha'} = g_{\alpha\alpha'} - 1$, as well as the three-body analogous quantities

$$g^{(3)}_{\alpha\alpha'\alpha''}(\mathbf{r},\mathbf{r}',\mathbf{r}'') = \frac{n^{(3)}_{\alpha\alpha'\alpha''}(\mathbf{r},\mathbf{r}',\mathbf{r}'')}{n_\alpha n_{\alpha'} n_{\alpha''}} \tag{1.9}$$

and the (truncated) three-body correlation function

$$h^{(3)}_{\alpha\alpha'\alpha''}(\mathbf{r},\mathbf{r}',\mathbf{r}'') = g^{(3)}_{\alpha\alpha'\alpha''}(\mathbf{r},\mathbf{r}',\mathbf{r}'') - h_{\alpha\alpha'}(|\mathbf{r}-\mathbf{r}'|) - h_{\alpha'\alpha''}(|\mathbf{r}'-\mathbf{r}''|)$$
$$- h_{\alpha''\alpha}(|\mathbf{r}''-\mathbf{r}|) - 1. \tag{1.10}$$

The paper is organized as follows. In Sect. 2, we use the BGY hierarchy for studying the general mixture of M species of mobile particles embedded in a fixed uniform background. By taking the limit of one of the densities going to zero, we get the case of a guest charge. We retrieve the second moment of the screening cloud around a guest charge immersed in an OCP; we get also the zeroth moment of the excess total number density around a guest charge immersed in a TCP. In Sect. 3, the general formalism for relating the moments of the electric potential to the excess chemical potential of a guest charge is established. The following sections study special cases when explicit calculations are possible: the high-temperature (Debye-Hückel) limit in Sect. 4, the OCP at $\beta e^2 = 2$ in Sect. 5, the TCP in Sect. 6. Section 7 is a Conclusion.

2 Sum Rules for a Guest Charge Immersed in a Coulomb System

We wish to rederive and extend the result of [1] about the 2D OCP in which a point guest charge Ze is immersed. Let the charge density at $\mathbf{r}$ knowing that there is a guest charge Ze at the origin be $\rho(\mathbf{r}|Ze,\mathbf{0})$. In [1], its second moment was shown to be

$$\int \mathrm{d}\mathbf{r} r^2 \rho(\mathbf{r}|Ze,\mathbf{0}) = -\frac{2}{\pi\beta e n}\left[Z\left(1-\frac{\beta e^2}{4}\right) + Z^2\frac{\beta e^2}{4}\right]. \tag{2.1}$$

Our rederivation uses only the BGY hierarchy.

2.1 General Sum Rule for a Mixture with a Background

We start with the mixture of M mobile species, with a fixed uniform background, described in the Introduction. Finally, we shall consider a mixture of only 2 species with respective charges $e_1 = e$ and $e_2 = Ze$; at the end, the density n_2 will be chosen as 0, leaving only one guest charge. But for being able to consider the TCP as well, we start with the more general case of M mobile species. The neutrality constraint is

$$\sum_{\alpha} n_\alpha e_\alpha = -\rho_b. \tag{2.2}$$

The mixture with a background has been studied in three dimensions by Suttorp and van Wonderen [2]. They used the BGY hierarchy and thermodynamical properties of the system for deriving, among other things, a second-moment sum rule, which however involves some thermodynamical functions (the partial derivatives of each density n_α with respect to the background density n_b); there is no explicit expression for these partial derivatives. Fortunately, we found that, in two dimensions, the formalism becomes much simpler and only the BGY hierarchy has to be used (the thermodynamical properties are no longer involved).

The second equation of the BGY hierarchy [3], with $h_{\alpha\alpha'}(r)$ the correlation function between a particle of species α at $\mathbf{r}$ and a particle of species α' at the origin, is

$$\begin{aligned}
\beta^{-1}\nabla h_{\alpha\alpha'}(r) = &-\sum_{\alpha''} n_{\alpha''} \int \mathrm{d}\mathbf{r}''\, h_{\alpha'\alpha''}(r'') e_\alpha e_{\alpha''} \nabla v(|\mathbf{r}-\mathbf{r}''|) \\
&- h_{\alpha\alpha'}(r) e_\alpha e_{\alpha'} \nabla v(r) - e_\alpha e_{\alpha'} \nabla v(r) \\
&- \sum_{\alpha''} n_{\alpha''} \int \mathrm{d}\mathbf{r}''\, h^{(3)}_{\alpha\alpha'\alpha''}(\mathbf{r}, \mathbf{0}, \mathbf{r}'') e_\alpha e_{\alpha''} \nabla v(|\mathbf{r}-\mathbf{r}''|).
\end{aligned} \tag{2.3}$$

The integral in the first term in the rhs of (2.3) is proportional to the electric field at $\mathbf{r}$ due to the charge distribution $h_{\alpha'\alpha''}$ which has a circular symmetry around the origin. Thus, using Newton's theorem, one can rewrite this integral as

$$\int \mathrm{d}\mathbf{r}''\, h_{\alpha'\alpha''}(r'') \nabla v(|\mathbf{r}-\mathbf{r}''|) = \nabla v(r) \int_{r''<r} \mathrm{d}\mathbf{r}''\, h_{\alpha'\alpha''}(r''). \tag{2.4}$$

The integral in the rhs of (2.4) can be written as $\int_{r''<r} \cdots = \int \cdots - \int_{r''>r} \cdots$ and the perfect screening of the charge $e_{\alpha'}$ gives [4]

$$\sum_{\alpha''} e_{\alpha''} n_{\alpha''} \int \mathrm{d}\mathbf{r}''\, h_{\alpha'\alpha''}(r'') = -e_{\alpha'}. \tag{2.5}$$

Therefore (2.3) can be rewritten as

$$\begin{aligned}
\beta^{-1}\nabla h_{\alpha\alpha'}(r) = &\; e_\alpha \sum_{\alpha''} n_{\alpha''} e_{\alpha''} \nabla v(r) \int_{r''>r} \mathrm{d}\mathbf{r}''\, h_{\alpha'\alpha''}(r'') - h_{\alpha\alpha'}(r) e_\alpha e_{\alpha'} \nabla v(r) \\
&- e_\alpha \sum_{\alpha''} n_{\alpha''} \int \mathrm{d}\mathbf{r}''\, h^{(3)}_{\alpha\alpha'\alpha''}(\mathbf{r}, \mathbf{0}, \mathbf{r}'') e_{\alpha''} \nabla v(|\mathbf{r}-\mathbf{r}''|).
\end{aligned} \tag{2.6}$$

In order to make a second moment to appear, we take the scalar product of both sides of (2.6) with $\mathbf{r}$ and integrate on $\mathbf{r}$. Integrating by parts the lhs and performing the integration on $\mathbf{r}$ first in the first term of the rhs, one finds

$$-2\beta^{-1}\int d\mathbf{r}\, h_{\alpha\alpha'}(r) = -\pi e_\alpha \sum_{\alpha''} n_{\alpha''} e_{\alpha''} \int d\mathbf{r}\, r^2 h_{\alpha'\alpha''}(r) + e_\alpha e_{\alpha'} \int d\mathbf{r}\, h_{\alpha\alpha'}(r)$$
$$+ e_\alpha \sum_{\alpha''} n_{\alpha''} e_{\alpha''} \int d\mathbf{r}\, d(\mathbf{r}'' - \mathbf{r})\, h^{(3)}_{\alpha\alpha'\alpha''}(\mathbf{r}, \mathbf{0}, \mathbf{r}'') \frac{(\mathbf{r} - \mathbf{r}'')\cdot \mathbf{r}}{(\mathbf{r} - \mathbf{r}'')^2} \tag{2.7}$$

(in the last term, since $h^{(3)}$ depends only on $\mathbf{r}$ and the difference $\mathbf{r}'' - \mathbf{r}$, we have replaced the integration on $\mathbf{r}''$ by an integration on $\mathbf{r}'' - \mathbf{r}$). An important simplification has occurred in 2D where $\mathbf{r}\cdot\nabla v(r)$ has the constant value -1, while in three dimensions, with the potential $v(r) = 1/r$, one finds $-v(r)$, a result which has led to a more complicated calculation in [2].

Now, we multiply both sides of (2.7) by n_α and sum on α. The term involving $h^{(3)}$ can be simplified by using symmetries under permutations of the variables. Indeed, $h^{(3)}$ has the symmetry property

$$h^{(3)}_{\alpha\alpha'\alpha''}(\mathbf{r}, \mathbf{0}, \mathbf{r}'') = h^{(3)}_{\alpha''\alpha'\alpha}(\mathbf{r}'', \mathbf{0}, \mathbf{r}). \tag{2.8}$$

Thus, interchanging the summation variables α and α'', and the variables $\mathbf{r}$ and $\mathbf{r}''$, we obtain

$$\sum_{\alpha,\alpha''} n_\alpha e_\alpha n_{\alpha''} e_{\alpha''} \int d(\mathbf{r}'' - \mathbf{r})\, h^{(3)}_{\alpha\alpha'\alpha''}(\mathbf{r}, \mathbf{0}, \mathbf{r}'') \frac{(\mathbf{r} - \mathbf{r}'')\cdot \mathbf{r}}{(\mathbf{r} - \mathbf{r}'')^2}$$
$$= \sum_{\alpha,\alpha''} n_\alpha e_\alpha n_{\alpha''} e_{\alpha''} \int d(\mathbf{r}'' - \mathbf{r})\, h^{(3)}_{\alpha\alpha'\alpha''}(\mathbf{r}, \mathbf{0}, \mathbf{r}'') \frac{(\mathbf{r}'' - \mathbf{r})\cdot \mathbf{r}''}{(\mathbf{r} - \mathbf{r}'')^2}$$
$$= \frac{1}{2}\sum_{\alpha,\alpha''} n_\alpha e_\alpha n_{\alpha''} e_{\alpha''} \int d(\mathbf{r}'' - \mathbf{r})\, h^{(3)}_{\alpha\alpha'\alpha''}(\mathbf{r}, \mathbf{0}, \mathbf{r}''), \tag{2.9}$$

where the last line is the half sum of the two first ones.

Using (2.9) in (2.7) gives

$$-2\beta^{-1}\sum_\alpha n_\alpha \int d\mathbf{r}\, h_{\alpha\alpha'}(r) = \pi\rho_b \sum_{\alpha''} n_{\alpha''} e_{\alpha''} \int d\mathbf{r}\, r^2 h_{\alpha'\alpha''}(r) + e_{\alpha'} \sum_\alpha n_\alpha e_\alpha \int d\mathbf{r}\, h_{\alpha\alpha'}(r)$$
$$+ \frac{1}{2}\sum_{\alpha,\alpha''} n_\alpha e_\alpha n_{\alpha''} e_{\alpha''} \int d\mathbf{r}\, d(\mathbf{r}'' - \mathbf{r})\, h^{(3)}_{\alpha\alpha'\alpha''}(\mathbf{r}, \mathbf{0}, \mathbf{r}''). \tag{2.10}$$

For the second term in the rhs of (2.10), perfect screening [4] gives $-e^2_{\alpha'}$. For the last term in the rhs of (2.10), perfect screening gives

$$+\frac{1}{2}\sum_{\alpha,\alpha''} n_\alpha e_\alpha n_{\alpha''} e_{\alpha''} \int d\mathbf{r}\, d(\mathbf{r}'' - \mathbf{r})\, h^{(3)}_{\alpha\alpha'\alpha''}(\mathbf{r}, \mathbf{0}, \mathbf{r}'')$$
$$= -\frac{1}{2}\sum_{\alpha''} n_{\alpha''} e_{\alpha''} (e_{\alpha'} + e_{\alpha''}) \int d\mathbf{r}\, h_{\alpha'\alpha''}(r)$$
$$= \frac{1}{2}\left[e^2_{\alpha'} - \sum_{\alpha''} n_{\alpha''} e^2_{\alpha''} \int d\mathbf{r}\, h_{\alpha'\alpha''}(r) \right]. \tag{2.11}$$

Thus (2.10) becomes the general second-moment sum rule

$$-\beta\pi\rho_b \sum_\alpha n_\alpha e_\alpha \int d\mathbf{r}\, r^2 h_{\alpha'\alpha}(r) = \frac{1}{2}\sum_\alpha n_\alpha (4-\beta e_\alpha^2) \int d\mathbf{r}\, h_{\alpha'\alpha}(r) - \frac{1}{2}\beta e_{\alpha'}^2. \tag{2.12}$$

By multiplying (2.12) by $n_{\alpha'} e_{\alpha'}$ and summing on α', one recovers the usual Stillinger-Lovett sum rule [3]. But (2.12) is a stronger sum rule.

2.2 Guest Charge in a One-Component Plasma

We come to the case of a mixture of two species, with charge $e_1 = e$ and density n_1, charge $e_2 = Ze$ and density n_2, respectively. We choose $\alpha' = 2$ in (2.12). For dealing with one guest charge Ze only, we set $n_2 = 0$, $n_1 = n$, $-\rho_b = ne$; n times the integral of h_{21} is $-Z$, by perfect screening. The sum rule (2.12) becomes

$$\beta\pi n^2 e^2 \int d\mathbf{r}\, r^2 h_{21}(r) = -2\left[Z\left(1 - \frac{\beta e^2}{4}\right) + Z^2 \frac{\beta e^2}{4}\right]. \tag{2.13}$$

Since $\rho(\mathbf{r}|Ze, \mathbf{0}) = neh_{21}(r)$, (2.13) is (2.1).

This result (2.1) can also be retrieved by a different method in the next subsection.

2.3 Another Derivation

(2.1) can be derived in another way if we *assume* that this second moment can be expanded in integer powers of Z.

In the limit of small Z, the term linear in Z in (2.1) can be obtained by linear response theory. Indeed, if we introduce a guest charge Ze, located at the origin, into an OCP, the additional Hamiltonian is

$$\hat{H}' = Ze\hat{\phi}(\mathbf{0}), \tag{2.14}$$

where $\hat{\phi}(\mathbf{0})$ is the microscopic electric potential created by the OCP at the origin. To first order in Z, the charge density at $\mathbf{r}$ is

$$\rho(\mathbf{r}|Ze, \mathbf{0}) = -\beta\langle\hat{\rho}(\mathbf{r}) Ze\hat{\phi}(\mathbf{0})\rangle^{\mathrm{T}} = -Ze\beta \int d\mathbf{r}'\, v(r')\langle\hat{\rho}(\mathbf{r})\hat{\rho}(\mathbf{r}')\rangle^{\mathrm{T}}, \tag{2.15}$$

where $\langle\cdots\rangle^{\mathrm{T}}$ denotes a truncated average. We define the Fourier transforms as

$$\tilde{f}(\mathbf{k}) = \int d\mathbf{r}\, \exp(i\mathbf{k}\cdot\mathbf{r}) f(\mathbf{r}). \tag{2.16}$$

Then, the Fourier transform of $\rho(\mathbf{r}|Ze, \mathbf{0})$ is

$$\tilde{\rho}(\mathbf{k}|Ze) = -\beta Ze \frac{2\pi}{k^2} \tilde{S}(k), \tag{2.17}$$

since the Fourier transform of $v(r)$ is $2\pi/k^2$ and the Fourier transform of the correlation of charge densities is $\tilde{S}(k)$. For small k, $\tilde{S}(k)$ has the expansion [3]

$$\tilde{S}(k) = \frac{k^2}{2\pi\beta} - \frac{(1-\beta e^2/4)k^4}{4\pi^2 n\beta^2 e^2} + \cdots. \tag{2.18}$$

Therefore, we get the zeroth moment

$$\int d\mathbf{r}\,\rho(\mathbf{r}|Ze,\mathbf{0}) = -Ze, \tag{2.19}$$

in agreement with equation (1.20) in [1], and the part linear in Z of the second moment (2.1).

It may be remarked that the k^4 term of (2.18) is related to the compressibility, which is exactly known only for the 2D OCP [5]. Therefore, no extension to 3D, with a closed result, seems possible.

In the opposite case of large Z, the impurity expels the mobile particles from a large region around it, leaving only the background. Essentially, $\rho(\mathbf{r}|Ze,\mathbf{0}) = -ne$ for $r < R$, where R is some large radius, and $\rho(\mathbf{r}|Ze,\mathbf{0}) = 0$ for $r > R$ (there is a transition region [6] of width of the order $n^{-1/2}$, but in the limit of large Z, it gives a correction of lower order in Z). The radius R is determined by the perfect screening condition (2.19) which gives $R^2 = Z/(\pi n)$. The second moment is

$$\int d\mathbf{r}\,r^2\rho(\mathbf{r}|Ze,\mathbf{0}) = -ne\pi R^4/2 = -Z^2\frac{e}{2\pi n}, \quad Z\to\infty, \tag{2.20}$$

which is the Z^2 term of (2.1), and this is the highest-order power of Z in the second moment.

The same argument extended to 3D gives

$$\int d\mathbf{r}\,r^2\rho(\mathbf{r}|Ze,\mathbf{0}) = -(3Z)^{5/3}\frac{e}{5(4\pi n)^{2/3}}, \quad Z\to\infty. \tag{2.21}$$

Therefore, in 3D, the second moment is *not* a polynomial in Z, and no exact formula valid for any Z can be obtained by the present method.

2.4 Guest Charge in a Two-Component Plasma

A sum rule for the TCP can also be obtained from (2.12). Now, we consider a mixture of three species, with charge $e_1 = e$ and density $n_1 = n_+$, charge $e_2 = -e$ and density $n_2 = n_-$, charge $e_3 = Ze$ and density n_3, respectively. There is no background ($\rho_b = 0$). We choose $\alpha' = 3$ in (2.12). Finally, for dealing with one guest charge Ze only, we set $n_3 = 0$, $n_+ = n_-$ (neutrality); the system is stable against collapse if $\beta e^2 < 2$ and $\beta Ze^2 < 2$. We call $n = n_+ + n_-$ the total density of the TCP. In (2.12) appears the quantity

$$n_+h_{31}(r) + n_-h_{32}(r) = n(\mathbf{r}|Ze,\mathbf{0}) - n, \tag{2.22}$$

which is the excess density around the guest charge Ze. Then, (2.12) becomes a sum rule for the zeroth moment of this excess density:

$$\int d\mathbf{r}\,[n(\mathbf{r}|Ze,\mathbf{0}) - n] = Z^2\frac{\beta e^2}{4-\beta e^2}. \tag{2.23}$$

This result is a generalization of the compressibility sum rule [7]

$$\int d\mathbf{r}\,[n(\mathbf{r}|\pm e,\mathbf{0}) - n] = \frac{\partial n}{\partial(\beta p)} - 1 \tag{2.24}$$

with the use of the exact equation of state $\beta p = n(1-\beta e^2/4)$, where p is the pressure.

2.5 Mixture without a Background

In the case $\rho_b = 0$, another derivation of (2.12) is possible starting from the known equation of state [8]

$$\beta p = \sum_\alpha \left(1 - \frac{\beta e_\alpha^2}{4}\right) n_\alpha. \tag{2.25}$$

The M-component plasma may be described in the grand-canonical ensemble, with M chemical potentials μ_α (actually [9], the system turns out to be neutral in the thermodynamic limit, and $M-1$ chemical potentials would suffice for determining the state of the system; but here it is more convenient to use M chemical potentials). The pressure p is given by $\beta p = \lim(1/V)\ln \Xi$, where V is the volume (here area) of the system, Ξ is the grand partition function, and lim is the thermodynamic limit. Taking the partial derivative of (2.25) with respect to $\beta\mu_{\alpha'}$ gives

$$n_{\alpha'} = \sum_\alpha \left(1 - \frac{\beta e_\alpha^2}{4}\right)\left(n_\alpha n_{\alpha'} \int \mathrm{d}\mathbf{r}\, h_{\alpha\alpha'}(r) + n_{\alpha'}\delta_{\alpha,\alpha'}\right), \tag{2.26}$$

which is (2.12) with $\rho_b = 0$.

3 Guest Charge and Potential Fluctuations

Putting a guest particle of charge Ze at the origin $\mathbf{r} = \mathbf{0}$, the original Hamiltonian H_0 of the infinite Coulomb system modifies to $H = H_0 + Ze\hat{\phi}(\mathbf{0})$, where $\hat{\phi}(\mathbf{0})$ is the microscopic electric potential created at the origin by the Coulomb system. The charge density around the guest charge, at point $\mathbf{r}$, is thus expressible as

$$\rho(\mathbf{r}|Ze, \mathbf{0}) = \frac{\langle \hat{\rho}(\mathbf{r}) \exp[-\beta Ze\hat{\phi}(\mathbf{0})]\rangle}{\langle \exp[-\beta Ze\hat{\phi}(\mathbf{0})]\rangle}, \tag{3.1}$$

where $\langle \cdots \rangle$ denotes the thermal average over the homogeneous system with the Hamiltonian H_0.

Let μ_{Ze}^{ex} denotes the excess (i.e., over ideal) chemical potential of the guest charge, i.e. the reversible work which has to be done to bring the guest particle of charge Ze from infinity into the bulk interior of the considered Coulomb plasma. By the coupling parameter technique [7], this chemical potential can be represented in terms of the charge density (3.1) as follows

$$\mu_{Ze}^{\mathrm{ex}} = e \int_0^Z \mathrm{d}Z' \int \mathrm{d}\mathbf{r}\, v(\mathbf{r}) \rho(\mathbf{r}|Z'e, \mathbf{0}). \tag{3.2}$$

With regard to the representation (3.1), μ_{Ze}^{ex} can be expressed as

$$-\beta\mu_{Ze}^{\mathrm{ex}} = \int_0^{-\beta Ze} \mathrm{d}x \frac{\langle \hat{\phi} \exp(x\hat{\phi})\rangle}{\langle \exp(x\hat{\phi})\rangle}. \tag{3.3}$$

Here, since the thermal averages are point-independent, we use the notation $\hat{\phi} \equiv \hat{\phi}(\mathbf{0})$.

Let us recall some basic information about the cumulant expansion. Let $\hat{\phi}$ be a random variable with the probability distribution $P(\hat{\phi})$. The cumulant expansion is defined by

$$\langle \exp(x\hat{\phi}) \rangle = \exp\left(\sum_{l=1}^{\infty} \frac{x^l}{l!} \langle \hat{\phi}^l \rangle_c \right), \tag{3.4}$$

where x is any complex number and $\langle \hat{\phi}^l \rangle_c$ are the cumulants. They are combinations of the standard moments $\langle \hat{\phi}^l \rangle$. Differentiating the equality (3.4) with respect to x gives

$$\frac{\mathrm{d}}{\mathrm{d}x} \sum_{l=0}^{\infty} \frac{x^l}{l!} \langle \hat{\phi}^l \rangle = \exp\left(\sum_{l=1}^{\infty} \frac{x^l}{l!} \langle \hat{\phi}^l \rangle_c \right) \frac{\mathrm{d}}{\mathrm{d}x} \sum_{l=1}^{\infty} \frac{x^l}{l!} \langle \hat{\phi}^l \rangle_c. \tag{3.5}$$

Equating the coefficients of the same power of x in both sides of (3.5) gives the recursion formula

$$\langle \hat{\phi}^l \rangle_c = \langle \hat{\phi}^l \rangle - \sum_{k=1}^{l-1} \binom{l-1}{k-1} \langle \hat{\phi}^k \rangle_c \langle \hat{\phi}^{l-k} \rangle. \tag{3.6}$$

The first cumulants read

$$\begin{aligned} \langle \hat{\phi} \rangle_c &= \langle \hat{\phi} \rangle, \\ \langle \hat{\phi}^2 \rangle_c &= \langle \hat{\phi}^2 \rangle - \langle \hat{\phi} \rangle^2, \\ \langle \hat{\phi}^3 \rangle_c &= \langle \hat{\phi}^3 \rangle - 3\langle \hat{\phi}^2 \rangle \langle \hat{\phi} \rangle + 2\langle \hat{\phi} \rangle^3, \end{aligned} \tag{3.7}$$

etc. In the theory of fluids, the cumulants of type (3.7) are referred to as truncations, and therefore we shall use the notation $\langle \hat{\phi}^l \rangle_c \equiv \langle \hat{\phi}^l \rangle^{\mathrm{T}}$.

Since it holds

$$\frac{\langle \hat{\phi} \exp(x\hat{\phi}) \rangle}{\langle \exp(x\hat{\phi}) \rangle} = \frac{\mathrm{d}}{\mathrm{d}x} \ln \langle \exp(x\hat{\phi}) \rangle, \tag{3.8}$$

the excess chemical potential (3.3) is expressible as

$$-\beta \mu_{Ze}^{\mathrm{ex}} = \ln \langle \exp(-\beta Z e \hat{\phi}) \rangle. \tag{3.9}$$

Based on the recapitulation in the above paragraph, μ_{Ze}^{ex} is expressible either in the form of a cumulant expansion

$$-\beta \mu_{Ze}^{\mathrm{ex}} = \sum_{l=1}^{\infty} \frac{(-\beta Z e)^l}{l!} \langle \hat{\phi}^l \rangle^{\mathrm{T}}, \tag{3.10}$$

or in the form of the standard moment expansion

$$\exp\left(-\beta \mu_{Ze}^{\mathrm{ex}}\right) = \langle \exp(-\beta Z e \hat{\phi}) \rangle \equiv 1 + \sum_{l=1}^{\infty} \frac{(-\beta Z e)^l}{l!} \langle \hat{\phi}^l \rangle. \tag{3.11}$$

It stands to reason that the expansions (3.10) and (3.11) are valid provided all moments exist. We conclude that the knowledge of the excess chemical potential of the guest particle with an arbitrary charge provides the exact information about all moments of the electrostatic potential at a point of the infinite homogeneous Coulomb system.

Going to the infinite system via the thermodynamic limit of a finite system with a disc geometry [10], the fluctuations of the potential at any point become infinite due to the presence of dipoles near the boundary. Here, the potential moments are defined directly for an infinite space, without the presence of a boundary. This corresponds to going to the infinite system via the thermodynamic limit of a finite system, e.g., with periodic boundary conditions, formulated on the surface of a sphere and so on. We thus expect that the average potential at a point is equal to zero and all its moments are finite.

Since in 2D the potential (1.2) is dimensionless, $\hat{\phi}$ has the dimension of the elementary charge e. It is therefore useful to introduce the dimensionless microscopic quantity $\psi = \hat{\phi}/e$ with the probability distribution $P(\psi)$. Setting in (3.11) $\beta Z e^2 = \mathrm{i}k$, one gets

$$\exp\left(-\beta\mu_{Ze}^{\mathrm{ex}}\right)\Big|_{\beta Ze^2=\mathrm{i}k} = \langle \exp(-\mathrm{i}k\psi)\rangle = \int_{-\infty}^{\infty} \mathrm{d}\psi\, \mathrm{e}^{-\mathrm{i}k\psi} P(\psi) \equiv \tilde{P}(k), \tag{3.12}$$

where $\tilde{P}(k)$ is the Fourier component of the ψ-distribution. The original probability distribution $P(\psi)$ can be obtained by the Fourier inversion of this relation

$$P(\psi) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \mathrm{e}^{-\mathrm{i}k\psi} \exp\left(-\beta\mu_{Ze}^{\mathrm{ex}}\right)\Big|_{\beta Ze^2=\mathrm{i}k}. \tag{3.13}$$

All that has been said in this section is valid also for v being the pure Coulomb potential plus any type of short-distance regularization.

4 High-Temperature Limit

The high-temperature (weak-coupling) limit of Coulomb systems is described rigorously by the Debye-Hückel theory [11, 12]. In 2D, the two-body Ursell functions U of charged species are given by [13]

$$U_{\alpha\alpha'}(\mathbf{r},\mathbf{r}') \equiv n_{\alpha\alpha'}^{(2)}(\mathbf{r},\mathbf{r}') - n_\alpha n_{\alpha'} = -e_\alpha n_\alpha e_{\alpha'} n_{\alpha'} \beta K_0(\kappa|\mathbf{r}-\mathbf{r}'|), \tag{4.1}$$

where K_0 is a modified Bessel function [14] and $\kappa = (2\pi\beta \sum_\alpha e_\alpha^2 n_\alpha)^{1/2}$ is the inverse Debye length.

The potential-potential correlation function can be calculated directly from the definition

$$\begin{aligned}\langle\hat{\phi}(\mathbf{0})\hat{\phi}(\mathbf{r})\rangle^{\mathrm{T}} &= \int \mathrm{d}\mathbf{r}_1 v(\mathbf{r}-\mathbf{r}_1) \int \mathrm{d}\mathbf{r}_2\, v(\mathbf{r}_2) \langle\hat{\rho}(\mathbf{r}_1)\hat{\rho}(\mathbf{r}_2)\rangle^{\mathrm{T}} \\ &= \int \mathrm{d}\mathbf{r}_1 v(\mathbf{r}-\mathbf{r}_1) \int \mathrm{d}\mathbf{r}_2\, v(\mathbf{r}_1-\mathbf{r}_2) \langle\hat{\rho}(\mathbf{0})\hat{\rho}(\mathbf{r}_2)\rangle^{\mathrm{T}}.\end{aligned} \tag{4.2}$$

Using for the Coulomb potential the expansion in polar coordinates

$$v(\mathbf{r}_1-\mathbf{r}_2) = -\ln|\mathbf{r}_1-\mathbf{r}_2| = -\ln r_> + \sum_{l=1}^{\infty} \frac{1}{l}\left(\frac{r_<}{r_>}\right)^l \cos l(\theta_1-\theta_2) \tag{4.3}$$

with $r_< = \min\{r_1, r_2\}$ and $r_> = \max\{r_1, r_2\}$, and taking into account the screening sum rule [3]

$$\int \mathrm{d}\mathbf{r}_2\, \langle\hat{\rho}(\mathbf{0})\hat{\rho}(\mathbf{r}_2)\rangle^{\mathrm{T}} = 0, \tag{4.4}$$

the second integral on the rhs of (4.2) can be expressed as

$$\int d\mathbf{r}_2\, v(\mathbf{r}_1-\mathbf{r}_2)\langle\hat{\rho}(\mathbf{0})\hat{\rho}(\mathbf{r}_2)\rangle^{\mathrm{T}} = -\int_{r_1}^{\infty} dr_2\, 2\pi r_2 \ln\left(\frac{r_2}{r_1}\right)\langle\hat{\rho}(\mathbf{0})\hat{\rho}(\mathbf{r}_2)\rangle^{\mathrm{T}}. \tag{4.5}$$

Considering the charge correlation function

$$\begin{aligned}\langle\hat{\rho}(\mathbf{0})\hat{\rho}(\mathbf{r}_2)\rangle^{\mathrm{T}} &= \sum_{\alpha,\alpha'} e_\alpha e_{\alpha'}[U^{(2)}_{\alpha\alpha'}(\mathbf{r}_2) + n_\alpha\delta_{\alpha\alpha'}\delta(\mathbf{r}_2)]\\ &= -\frac{\kappa^4}{(2\pi)^2\beta}K_0(\kappa r_2) + \frac{\kappa^2}{2\pi\beta}\delta(\mathbf{r}_2)\end{aligned} \tag{4.6}$$

in (4.5) implies, after an integration by parts,

$$\int d\mathbf{r}_2\, v(\mathbf{r}_1-\mathbf{r}_2)\langle\hat{\rho}(\mathbf{0})\hat{\rho}(\mathbf{r}_2)\rangle^{\mathrm{T}} = \frac{\kappa^2}{2\pi\beta}K_0(\kappa r_1). \tag{4.7}$$

Inserting this relation into (4.2) and applying once more the expansion (4.3) results into

$$\beta\langle\hat{\phi}(\mathbf{0})\hat{\phi}(\mathbf{r})\rangle^{\mathrm{T}} = -\ln r - K_0(\kappa r). \tag{4.8}$$

This procedure will be repeated, without going into details, also in the cases treated in the next sections.

The result (4.8) has the correct large-distance asymptotic [15]

$$\beta\langle\hat{\phi}(\mathbf{0})\hat{\phi}(\mathbf{r})\rangle^{\mathrm{T}} \underset{r\to\infty}{\sim} -\ln r. \tag{4.9}$$

In the zero-distance limit $r\to 0$, using the expansion

$$K_0(x) = -C - \ln(x/2) + O(x^2\ln x) \tag{4.10}$$

with C being the Euler number, the one-point second-moment fluctuation formula for the potential reads

$$\beta\langle\hat{\phi}^2\rangle^{\mathrm{T}} = C + \ln(\kappa/2). \tag{4.11}$$

One can obtain the last result in an alternative way by considering the charge density induced around the guest charge [1]

$$\rho(\mathbf{r}|Ze,\mathbf{0}) = -Ze\frac{\kappa^2}{2\pi}K_0(\kappa r). \tag{4.12}$$

Then, according to (3.2),

$$\begin{aligned}-\beta\mu^{\mathrm{ex}}_{Ze} &= -\beta e^2\int_0^Z dZ'\, Z'\frac{\kappa^2}{2\pi}\int_0^\infty dr\; 2\pi r\ln r\, K_0(\kappa r)\\ &= \frac{\beta(Ze)^2}{2}[C+\ln(\kappa/2)].\end{aligned} \tag{4.13}$$

With regard to the cumulant expansion (3.10), we recover the previous result (4.11).

From (3.10) and (4.13), all the higher-order truncated moments $\langle\hat{\phi}^l\rangle^{\mathrm{T}}$ with $l\geq 3$ vanish in the Debye-Hückel limit; this indicates a Gaussian distribution for the one-point potential in this limit. We shall return to this problem and present all truncated potential moments, for the TCP, in a high-temperature limit going beyond the Debye-Hückel limit, in Sect. 6.

5 One-Component Plasma at $\beta e^2 = 2$

The 2D OCP is exactly solvable in terms of free-fermions when the dimensionless coupling constant βe^2 has the special value 2 [16, 17]. In the thermodynamic limit, the two-body Ursell function of mobile particles at distance r is

$$U(r) = -n^2 \exp(-\pi n r^2), \tag{5.1}$$

where n is the particle density. All many-body Ursell functions are known at the free-fermion point, too.

The potential-potential correlation function can be calculated in close analogy with the previous steps outlined between (4.2)–(4.8). Substituting the charge correlation function

$$\langle \hat{\rho}(\mathbf{0})\hat{\rho}(\mathbf{r}_2)\rangle^{\mathrm{T}} = -e^2 n^2 \exp(-\pi n r_2^2) + n\delta(\mathbf{r}_2) \tag{5.2}$$

into the relation (4.5) and using an integration by parts, one gets

$$\int \mathrm{d}\mathbf{r}_2\, v(\mathbf{r}_1 - \mathbf{r}_2)\langle \hat{\rho}(\mathbf{0})\hat{\rho}(\mathbf{r}_2)\rangle^{\mathrm{T}} = \frac{e^2 n}{2}\Gamma(0, \pi n r_1^2), \tag{5.3}$$

where

$$\Gamma(x,t) = \int_t^\infty \mathrm{d}s\, s^{x-1}\mathrm{e}^{-s} \tag{5.4}$$

is the incomplete Gamma function. From (4.2), one thus obtains

$$\beta\langle \hat{\phi}(\mathbf{0})\hat{\phi}(\mathbf{r})\rangle^{\mathrm{T}} = -\ln r + \frac{1}{2}[\mathrm{e}^{-\pi n r^2} - (1 + \pi n r^2)\Gamma(0, \pi n r^2)]. \tag{5.5}$$

This result has the correct large-distance asymptotic (4.9). In the zero-distance limit, it yields

$$\beta\langle \hat{\phi}^2\rangle^{\mathrm{T}} = \frac{1}{2}[1 + C + \ln(\pi n)]. \tag{5.6}$$

Note that the large-distance behavior (4.9) is universal, while the zero-distance limit (4.11) or (5.6) depends on the coupling constant βe^2.

All potential moments are available for the present system due to the knowledge of the induced charge density around the guest charge [1, 18]:

$$\rho(\mathbf{r}|Ze, \mathbf{0}) = -en\frac{\Gamma(Z, \pi n r^2)}{\Gamma(Z)}, \quad Z \geq 0. \tag{5.7}$$

By using the relation (3.2), one obtains after some algebra [18]

$$-\beta\mu_{Ze}^{\mathrm{ex}} = \frac{Z^2}{2}[1 + \ln(\pi n)] - \int_0^Z \mathrm{d}Z'\, Z'\psi(1 + Z'), \tag{5.8}$$

where ψ is the psi-function defined by

$$\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x). \tag{5.9}$$

Its Taylor expansion around $x = 1$ reads [14]

$$\psi(1+x) = -C + \sum_{l=2}^{\infty} (-1)^l \zeta(l) x^{l-1}, \tag{5.10}$$

where

$$\zeta(l) = \sum_{k=1}^{\infty} \frac{1}{k^l} \tag{5.11}$$

is the Riemann zeta function. Considering the expansion (5.10) in (5.8) gives

$$-\beta\mu_{Ze}^{\text{ex}} = \frac{Z^2}{2}[1 + C + \ln(\pi n)] + \sum_{l=3}^{\infty} \frac{(-1)^l Z^l}{l} \zeta(l-1). \tag{5.12}$$

The comparison of this expansion with the cumulant expansion (3.10) implies

$$\langle \hat{\phi}^2 \rangle^{\text{T}} = \frac{e^2}{4}[1 + C + \ln(\pi n)], \tag{5.13}$$

$$\langle \hat{\phi}^l \rangle^{\text{T}} = \frac{e^l}{2^l}(l-1)!\zeta(l-1), \quad l \geq 3. \tag{5.14}$$

Note that the second-moment formula (5.13) is identical to the previous one (5.6) derived by the direct calculation from the definition.

6 Two-Component Plasma

6.1 Collapse Point $\beta e^2 = 2$

The 2D TCP of $\pm e$ charges is mappable for the special value of the coupling constant $\beta e^2 = 2$ onto the Thirring model at the free-fermion point [19, 20]. Although this coupling corresponds to the collapse threshold for the pointlike particles, and therefore for a fixed fugacity z the particle density is infinite, the Ursell functions are well defined. Their two-body forms read

$$U_{\pm,\pm}(r) = -\left(\frac{m^2}{2\pi}\right)^2 K_0^2(mr), \qquad U_{\pm,\mp}(r) = \left(\frac{m^2}{2\pi}\right)^2 K_1^2(mr), \tag{6.1}$$

where $m = 2\pi z$. All many-body Ursell functions are also known.

Substituting the charge correlation function

$$\langle \hat{\rho}(\mathbf{0})\hat{\rho}(\mathbf{r}_2) \rangle^{\text{T}} = -2e^2 \left(\frac{m^2}{2\pi}\right)^2 [K_0^2(mr_2) + K_1^2(mr_2)] \tag{6.2}$$

into the relation (4.5) and integrating by parts leads to

$$\int \mathrm{d}\mathbf{r}_2\, v(\mathbf{r}_1 - \mathbf{r}_2) \langle \hat{\rho}(\mathbf{0})\hat{\rho}(\mathbf{r}_2) \rangle^{\text{T}} = e^2 \frac{m^2}{2\pi} K_0^2(mr_1). \tag{6.3}$$

From (4.2), one finds that

$$\beta\langle\hat{\phi}(\mathbf{0})\hat{\phi}(\mathbf{r})\rangle^{\mathrm{T}} = -\ln r + \frac{(mr)^2}{2}[2K_1^2(mr) - K_0^2(mr) - K_0(mr)K_2(mr)]. \tag{6.4}$$

This result has the correct large-distance asymptotic (4.9). In the zero-distance limit, it gives

$$\beta\langle\hat{\phi}^2\rangle^{\mathrm{T}} = 1 + C + \ln(\pi z). \tag{6.5}$$

6.2 Stability Region $0 \le \beta e^2 < 2$

The system of pointlike $\pm e$ charged particles is stable against the collapse of positive-negative pairs of charges provided that the corresponding Boltzmann weight $\exp[\beta e^2 v(\mathbf{r})] = r^{-\beta e^2}$ can be integrated at short 2D distances, i.e. when $\beta e^2 < 2$. The equilibrium statistical mechanics of the neutral TCP is usually studied in the grand canonical ensemble, characterized by the particle fugacities $z_+ = z_- = z$. The full thermodynamics of this system is known [21, 22].

In the stability range of $\beta e^2 < 2$, the grand partition function $\Xi(z)$ of the 2D TCP can be turned via the Hubbard-Stratonovich transformation (see, e.g., Ref. [23]) into

$$\Xi(z) = \frac{\int \mathcal{D}\varphi \exp[-S(z)]}{\int \mathcal{D}\varphi \exp[-S(0)]}, \tag{6.6}$$

where

$$S(z) = \int \mathrm{d}\mathbf{r}\left[\frac{1}{16\pi}(\nabla\varphi)^2 - 2z\cos(b\varphi)\right] \tag{6.7}$$

is the Euclidean action of the $(1+1)$-dimensional sine-Gordon model. Here, $\varphi(\mathbf{r})$ is a real scalar field and $\int \mathcal{D}\varphi$ denotes the functional integration over this field. The sine-Gordon coupling constant b depends on the Coulomb coupling constant via

$$b = \sqrt{\frac{\beta e^2}{4}}. \tag{6.8}$$

The fugacity z is renormalized by the diverging self-energy term $\exp[\beta v(\mathbf{0})/2]$ which disappears from statistical relations under the conformal short-distance normalization of the exponential fields [21, 22]

$$\langle \mathrm{e}^{-\mathrm{i}b\varphi(\mathbf{r})}\mathrm{e}^{-\mathrm{i}b\varphi(\mathbf{r}')}\rangle_{\mathrm{sG}} \sim |\mathbf{r}-\mathbf{r}'|^{-4b^2} \quad \text{as } |\mathbf{r}-\mathbf{r}'| \to 0, \tag{6.9}$$

where $\langle\cdots\rangle_{\mathrm{sG}}$ denotes the average with the sine-Gordon action (6.7). The species densities are expressible in the sine-Gordon format as follows

$$n_\pm = z\langle \mathrm{e}^{\pm \mathrm{i}b\varphi}\rangle_{\mathrm{sG}}. \tag{6.10}$$

The charge neutrality of the system $n_+ = n_- = n/2$ is ensured by the obvious symmetry relation $\langle \mathrm{e}^{\mathrm{i}b\varphi}\rangle_{\mathrm{sG}} = \langle \mathrm{e}^{-\mathrm{i}b\varphi}\rangle_{\mathrm{sG}}$.

The excess chemical potential of the particle species forming the plasma is given by

$$\exp(-\beta\mu_{\pm e}^{\mathrm{ex}}) = \frac{n_\pm}{z} = \langle \mathrm{e}^{\pm \mathrm{i}b\varphi}\rangle_{\mathrm{sG}}. \tag{6.11}$$

It was shown in Ref. [24] that the excess chemical potential of a guest charge Ze immersed in the plasma is expressible in the sine-Gordon format as follows

$$\exp(-\beta\mu_{Ze}^{\rm ex}) = \langle {\rm e}^{{\rm i}Zb\varphi}\rangle_{\rm sG}. \tag{6.12}$$

When $Z=\pm 1$, one recovers the previous result (6.11) valid for the plasma constituents. Due to the symmetry relation $\langle {\rm e}^{{\rm i}a\varphi}\rangle_{\rm sG} = \langle {\rm e}^{-{\rm i}a\varphi}\rangle_{\rm sG}$ valid for any real-valued a, it holds that $\mu_{Ze}^{\rm ex} = \mu_{-Ze}^{\rm ex}$.

The $(1+1)$-dimensional sine-Gordon model is an integrable field theory [25]. Due to a recent progress in the method of the Thermodynamic Bethe ansatz, a general formula for the expectation value of the exponential field $\langle {\rm e}^{-{\rm i}a\phi}\rangle$ was derived by Lukyanov and Zamolodchikov [26]. In the notation of (6.12), $a = Zb$, their formula reads

$$\langle {\rm e}^{{\rm i}Zb\varphi}\rangle_{\rm sG} = \left[\frac{\pi z \Gamma(1-b^2)}{\Gamma(b^2)}\right]^{(Zb)^2/(1-b^2)} \exp[I_b(Z)] \tag{6.13}$$

with

$$I_b(Z) = \int_0^\infty \frac{{\rm d}t}{t}\left[\frac{\sinh^2(2Zb^2t)}{2\sinh(b^2t)\sinh(t)\cosh[(1-b^2)t]} - 2Z^2b^2{\rm e}^{-2t}\right]. \tag{6.14}$$

The interaction Boltzmann factor of the guest charge Ze with an opposite plasma counterion at distance r, $r^{-\beta e^2|Z|}$, is integrable at small 2D distances r if $\beta|Z|e^2 < 2$, i.e. $|Z| < 1/(2b^2)$; this is indeed the condition for the integral (6.14) to be finite, so that the couple of (6.13) and (6.14) passes the collapse test. Finally, using (6.13) and (6.14) in (6.12), one arrives at

$$-\beta\mu_{Ze}^{\rm ex} = Z^2\frac{b^2}{1-b^2}\ln\left[\frac{\pi z\Gamma(1-b^2)}{\Gamma(b^2)}\right] + I_b(Z). \tag{6.15}$$

We have to keep in mind that $b^2 = \beta e^2/4$.

Comparing the cumulant expansion (3.10) with the result (6.15), in which the integral $I_b(Z)$ (6.14) is expanded in powers of Z, one gets the explicit forms of the potential moments:

$$\langle \hat{\phi}^2\rangle^{\rm T} = \frac{e^2}{8b^2(1-b^2)}\ln\left[\frac{\pi z\Gamma(1-b^2)}{\Gamma(b^2)}\right] + \frac{e^2}{4}\int_0^\infty \frac{{\rm d}t}{t}\left[\frac{t^2}{\sinh(b^2t)\sinh(t)\cosh[(1-b^2)t]} - \frac{1}{b^2}{\rm e}^{-2t}\right], \tag{6.16}$$

$$\langle \hat{\phi}^{2l}\rangle^{\rm T} = \frac{e^{2l}}{4}\int_0^\infty {\rm d}t\,\frac{t^{2l-1}}{\sinh(b^2t)\sinh(t)\cosh[(1-b^2)t]}, \quad l = 2,3,\ldots. \tag{6.17}$$

The odd potential moments vanish for the symmetric TCP.

In the high-temperature limit $\beta e^2 \to 0$ ($b^2 \to 0$), (6.15) taken with $z \sim n/2$ reduces to the previous one (4.13); one retrieves the second moment (4.11) and that all higher moments vanish, as it should be. From (6.17), in the limit $b^2 \to 0$, one finds

$$\beta\langle \hat{\phi}^{2l}\rangle^{\rm T} = e^{2(l-1)}8\frac{4^l-2}{4^{2l}}(2l-2)!\zeta(2l-1), \quad l = 2,3,\ldots. \tag{6.18}$$

These expressions go beyond the Debye-Hückel limit of (6.15).

At the collapse point $\beta e^2 = 2$ ($b^2 = 1/2$), the second-moment formula (6.16) reproduces the previous result (6.5) and the higher-order moments (6.17) take forms

$$\langle \hat{\phi}^{2l} \rangle^{\mathrm{T}} = e^{2l} \frac{2}{4^l} (2l-1)! \zeta(2l-1), \quad l = 2, 3, \ldots. \tag{6.19}$$

All potential moments are finite also in the collapse region, up to the Kosterlitz-Thouless critical point $\beta e^2 = 4$ ($b^2 = 1$). We conjecture that, in the case of the hard-core regularization of the Coulomb potential, the obtained result correspond to the limit of a vanishing hard core.

We end up this section by a comment about the possibility of a relationship between the electrostatic potential $\hat{\phi}$ and the sine-Gordon field variable φ. This relationship was suggested in many articles, see, e.g., Ref. [27]. The comparison of (3.11) and (6.12) implies

$$\langle \varphi^{2l} \rangle_{\mathrm{sG}} = (-1)^l (4\beta)^l \langle \hat{\phi}^{2l} \rangle. \tag{6.20}$$

This means that, in view of one-point fluctuations, the fields $\hat{\phi}$ and φ differ from one another only by an irrelevant scaling factor. On the other hand, the large-distance asymptotic of the potential-potential correlations (4.9) is fundamentally different from the one of $\langle \varphi(\mathbf{0}) \varphi(\mathbf{r}) \rangle^{\mathrm{T}}$ The latter two-point correlation function has, like in every massive field theory, a short-range exponential decay as $r \to \infty$. We conclude that the electrostatic-potential interpretation of the sine-Gordon field is not correct.

7 Conclusion

The general study of a mixture of M species of mobile particles, which may be embedded in a uniform background, is simpler in two dimensions; the BGY hierarchy suffices for deriving the general sum rule (2.12) relating the second moments and the zeroth moments of the two-body correlations. Further work should be possible about this mixture.

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