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Supersymmetry on the Lattice

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Abstract. Supersymmetric gauge theories are fascinating for a number of reasons. However, unlike QCD, it was not known until recently how to devise a nonperturbative lattice construction for these theories. This has changed dramatically in recent years, and in these lectures I outline some of the recent progress, which expands our understanding of what is possible in lattice field theories in general.

1 Introduction

Whether or not supersymmetry is discovered to be a symmetry of nature, strongly coupled supersymmetric theories will always be a source of fascination. In these theories one can find explicit examples of many of the basic mechanisms and objects put forward in the early days of gauge theories: confinement, chiral symmetry breaking, magnetic monopoles and dyons, conformal field theories, etc. Especially intriguing are the connections between theories with sixteen supercharges and supergravity and string theory. Unfortunately, with few exceptions, a nonperturbative formulation of these theories on the lattice remained elusive despite many efforts over the years. The problem has been that latticization tends to completely break the supersymmetry, so that no characteristics of the continuum theory are present without excessive fine-tuning[1]. In the past few years, however, there have been significant advances in our understanding, which have led to the construction of a number of interesting supersymmetric lattice theories, including $\mathcal{N}=4$ super Yang-Mills (SYM) in four dimensions. In these lectures I introduce some of these ideas, focusing on the orbifold projection approach which I have contributed to [2–9]. For different approaches and more general reviews, see [10–12].

These lectures necessarily become somewhat technical, while not having the space to be really convincing. However, I have tried to convey the general ideas, as well as an outline of the more technical parts of the subject. If you want more details, I refer you to the literature. However, if you are only interested in the general ideas, I recommend that you read §1-§4 and §8, look at the pictures, and skip the rest.

2 Relevance, Symmetry, and $\mathcal{N}=1$ SYM

How do lattice field theories succeed in describing continuum physics? Clearly, the lattice only looks like smooth spacetime for long wavelength modes, and so it is necessary to understand how the terms in the lattice action affect such "IR" modes. Operators which have a bigger effect on IR modes are called "relevant"; those whose effects get weaker the longer the wavelength

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[1] For some low-dimension supersymmetric field theories, this fine-tuning can be performed analytically [1]

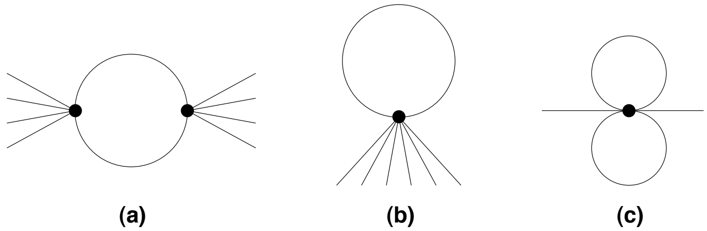

Fig. 1. *Graphs appearing in the theory of eq. (1): (a) Renormalization of* $c_8 \propto (c_6)^2$ *(b) Renormalization of* $c_6 \propto c_8$*; (c) Renormalization of* $c_2 \propto c_6$*.*

are called “irrelevant”; and those whose effects are scale invariant are called “marginal”. At the classical level, the lower the mass dimension of the operator, the more relevant it is. (For an extended discussion, see [13]).

An example of an irrelevant operator in four dimensions is a Fermi interaction, $(\bar{\psi}\psi)^2$ which has dimension 6. As such, it will appear in a Lagrangian (which has dimension 4 in four spacetime dimensions) with a coefficient G with mass dimension -2. A cross section such as $\sigma_{\nu e \to \nu e}$ will then be proportional to G^2 which has dimension -4; however σ is an area with mass dimension -2, and so by dimension counting $\sigma \propto G^2 E^2$, where E is the energy in the scattering process. This cross section gets less important for low E, showing that the $(\bar{\psi}\psi)^2$ operator is irrelevant. Evidently dimension 4 operators are marginal, and dimension < 4 operators are relevant.

Quantum corrections can change the scaling behavior of operators, however. If the interactions are weak, the change in the dimension of an operator will be small. Thus the scaling behavior for relevant and irrelevant operators will not be changed much by weak quantum corrections; however marginal operators will typically be tipped either to the relevant side (asymptotic freedom) or the irrelevant side by any small quantum correction. A special class of theories called conformal field theories are scale invariant even when the quantum corrections are included. An example of a conformal field theory I will mention later on is $\mathcal{N} = 4$ super Yang-Mills (SYM) theory in four dimensions.

The old-fashioned view of field theories was that marginal and relevant operators were good, while irrelevant operators were called “nonrenormalizable” and were bad. The modern view is that irrelevant operators are fine (that is, they are irrelevant), while relevant operators are problematic in that one expects the excitations in a theory with relevant operators to all be heavy. Therefore a theory with relevant operators trying to describe IR physics far below the UV cutoff of the theory is “unnatural” (think: Standard Model with a Higgs mass term, or worse yet, a small cosmological constant).

This concept of naturalness has a semi-precise meaning: that the renormalized couplings in the theory shouldn't be much smaller than the magnitude of the radiative corrections they receive. For example, consider a scalar field theory with a momentum cutoff Λ and the Lagrangian containing the operators

$$\mathcal{L} = ... + c_2 \Lambda^2 \phi^2 + c_4 \phi^4 + \frac{c_6}{\Lambda^2}\phi^6 + \frac{c_8}{\Lambda^4}\phi^8 \ . \tag{1}$$

At weak coupling the $c_{6,8}$ operators are irrelevant (they go away as $\Lambda \to \infty$, relative to physical energy scales), while the c_2 mass term is relevant and the c_4 operator is marginal. Loop graphs re normalize the couplings. For example, consider what happens when we integrate out all the high momentum modes between the cutoff Λ, and a new, lower cutoff Λ'; the c coefficients in the new effective theory will be shifted; for example in Fig.1 we see radiative corrections which can be estimated as

$$\delta c_8 \sim \frac{(c_6)^2}{(4\pi)^2} \ln \Lambda'/\Lambda \ , \quad \delta c_6 \sim \frac{c_8}{(4\pi)^2} \ , \quad \delta c_2 \sim \frac{c_6}{(4\pi)^4} \ . \tag{2}$$

. hen it is quite natural to have all the $c_n \lesssim 1$. However, note that if we want the scalar field to have a mass $\ll \Lambda$, this requires $c_2 \ll 1$, which in turn is only natural is all scalar interactions

$c_4, c_6, c_8 \ldots$ are extremely weak. This is a major source of anxiety about the Standard Model, where the Higgs must have non-negligible gauge and Yukawa couplings, but is supposed to be much lighter than the physics cutoff. One possibility is that new physics appears at the TeV scale, with new particles contributing to graphs such as shown in Fig. 1 and causing them to cancel at least partially amongst each other, as in supersymmetric or Little Higgs theories [14, 15].

The above example of naturalness is too crude, because it omits the crucial role played by symmetries. For example, note that the theory eq. (1) possesses a $\phi \to -\phi$ symmetry. Therefore, there are no graphs in the theory that induce operators with an odd power of ϕ fields. Similarly, if the symmetry is broken explicitly by a small operator $\epsilon\Lambda\phi^3$, then odd power operators may be generated by radiative corrections, but their coefficients must be proportional to powers of the small parameter ϵ; in particular, the ϕ^3 operator itself must be renormalized proportionally to ϵ, and so one says they it is *multiplicatively* renormalized (as opposed to the scalar mass, which is *additively* renormalized since δc_2 is not proportional to c_2). Operators that are multiplicatively renormalized can be naturally small. We would say that the approximate $\phi \to -\phi$ symmetry *protects* the ϕ^3 operator from large additive renormalization.

In the real world, an analogous example is the electron mass, which is renormalized multiplicatively, and can naturally be much lighter than the top quark, for example, being protected by an approximate chiral symmetry. This is the symmetry $\delta\psi = i\gamma_5\psi$ which becomes exact for a free Dirac fermion when its mass vanishes, and which persists when gauge interactions are included. Chiral symmetry effectively makes the fermion mass terms in QED and QCD behave as marginal operators rather than relevant operators, as would naively follow from counting operator dimension.

Another example of the role of symmetry is that of Wilson fermions on the lattice. I won't go into the long and interesting story of chiral symmetry for lattice fermions, and its relation to the fermion doubling problem. However, a comment relevant to the discussion of symmetry is to note that Wilson's lattice fermion action takes the form

$$\bar{\psi}\left(\not{D} - m - ar\triangle\right)\psi \tag{3}$$

where m is the mass, a is the lattice spacing, $\not{D}$ is the lattice, gauge-covariant Dirac operator, and $\triangle$ is the lattice, gauge covariant Laplacian. The Laplacian is called the "Wilson operator" and is designed to eliminate spurious light modes ("doublers"); it only works if the dimensionless coupling $r = \mathcal{O}(1)$. The Wilson term is dimension-5 and therefore is called an "irrelevant" operator; naively it is unimportant in the $a \to 0$ limit. However, note that the mass term and the Wilson term both violate chiral symmetry. It follows that in this theory, the mass term is *not* multiplicatively renormalized, and will receive a shift proportional to r/a ($= ar \times 1/a^2$ from the Wilson term coefficient times a quadratic divergence). Thus the price of Wilson's solution to the doubling problem is that to obtain light fermions in the $a \to 0$ limit, one must fine tune m infinitely well so that all the large radiative corrections cancel and leave behind a light fermion. I like this example because it shows that if one tries to do something unnatural in a theory (e.g., have a light fermion without an approximate chiral symmetry on the lattice), then so-called irrelevant operators become extremely important, generating undesirable relevant operators, and are obstacles to obtaining the desired target theory.

Supersymmetry, a symmetry between bosons and fermions, also has a big effect on radiative corrections. In particular, since exact supersymmetry implies degenerate masses for fermion-boson pairs ψ, ϕ, when the fermion's mass is protected from additive renormalizations by a chiral symmetry, the $\psi \leftrightarrow \phi$ supersymmetry ensures that the ϕ mass must be protected as well. Supersymmetry is one of only two symmetries known which can make light scalar particles be natural. The other is a shift symmetry $\phi \to \phi + f$, where f is an arbitrary constant. This shift symmetry clearly forbids a ϕ^2 mass operator, but it also ensures that ϕ can only couple to other matter derivatively (proportional to $\partial\phi$), and so ϕ interactions vanish at low momentum transfer. Such a particle is called a Goldstone boson, and is associated with a spontaneously broken global symmetry; it cannot carry gauge interactions, which involve non-derivative couplings, without breaking the shift symmetry explicitly by marginal operators; such a breaking of the symmetry allows the scalar mass to receive additive corrections.

So far I have discussed examples of symmetries that are broken explicitly by relevant operators (the $\phi \to -\phi$ and chiral symmetries) and by marginal operators (gauge interactions for a would-be Goldstone boson). I have also discussed breaking of a symmetry by an irrelevant operator when there exists a possible relevant operator that also breaks the symmetry (the case of lattice chiral symmetry broken explicitly by the dimension-5 Wilson term, inducing the dimension-3 mass term). However, one very important category of symmetry breaking is the case where the symmetry breaking occurs in irrelevant operators, where one cannot write down any relevant or marginal symmetry breaking operators. Such is the case, for example, for the baryon number symmetry in the $SU(5)$ Grand Unified Theory (GUT). At low energy the theory is described by the Standard Model, plus higher dimension operators. In particular there are dimension–6 operators of the form $(qqq\ell)/\Lambda^2$, where q and ℓ are quark and lepton fields respectively, and Λ is the GUT scale, about 10^{14} GeV. This operator violates baryon number (B). However, unlike the Wilson fermion example, there is no way to write down a B violating operator in the Standard Model that has lower dimension which is consistent with Lorentz symmetry and the exact $SU(3) \times SU(2) \times U(1)$ gauge symmetry. Thus in this case, B violation really *is* irrelevant. For example, the proton lifetime in this theory is proportional to Λ^4/m_p^5 and baryon violation is very hard to observe, it is so small. (In fact, after great effort, proton decay experiments succeeded in ruling out this particular GUT theory by not observing decay at the predicted rate). Therefore despite the fact that B is not at all a good symmetry at the GUT scale in such theories, it is automatically a very good symmetry at low energy. Turning this around, one can say that the long life of the proton is strongly suggestive that there is no new B violation except possibly at very short distances. We call baryon symmetry in the Standard Model an *accidental symmetry*; in general, an accidental symmetry G occurs at long distances whenever exact symmetries in a theory forbid the existence of relevant and marginal G-violating operators, even though G is not a symmetry at short distances and irrelevant G-violating operators exist.

Accidental symmetry plays a huge role in lattice physics. For example, it explains why Lorentz symmetry emerges naturally in the continuum limit of lattice QCD, even though it is clearly not s symmetry of the underlying lattice interactions. The reason for this is because the combination of the $SU(3)$ gauge symmetry of QCD and the discrete hypercubic symmetry of the lattice action forbid relevant or marginal operators that violate Lorentz symmetry. In contrast, a lattice theory of a vector meson ρ_μ without a gauge symmetry allows the four-boson interaction $\rho_1\rho_2\rho_3\rho_4$ which is invariant under the exact hypercubic lattice symmetry. Since this is a marginal, Lorentz-violating operaotr, it would have to be carefully tuned away by adjusting the bare couplings of the theory in order to attain a Lorentz invariant continuum limit.

In summary,

- the low energy limit of a theory depends on the relevant and marginal interactions of the theory;
- relevant interactions in general are a problem, precluding light states;
- simply omitting unwanted irrelevant operators is insufficient, since in general they receive additive corrections from UV physics, and have to be fine-tuned away;
- symmetries control which relevant and marginal operators can be radiatively generated;
- when a symmetry emerges in the IR which did not exist in the UV theory, we call this an accidental symmetry — it can occur when exact symmetries preclude any possible relevant or marginal operators which break the accidental symmetry;
- accidental symmetries are important for lattice field theory, obviating the need for fine tuning.

3 Supersymmetry

3.1 The supersymmetry algebra

Poincaré symmetry consists of spacetime translations, generated by P_μ, and Lorentz transformations, generated by $\Sigma_{\mu\nu} = -\Sigma_{\nu\mu}$. The algebra has the qualitative structure

$$[P, P] = 0 \ , \quad [P, \Sigma] \sim P \ , \quad [\Sigma, \Sigma] \sim \Sigma \ , \tag{4}$$

where the meaning of the three terms are (i) translations commute with each other; (ii) translations transform under the Lorentz group as a 4-vector; (iii) Lorentz transformations themselves transform as an antisymmetric tensor.

Supersymmetry is a generalization of the Poincaré group, where complex Grassmann generators Q_α, $\bar{Q}_{\dot{\alpha}}$ are added with the (anti-) commutation relations:

$$\{Q,Q\} = 0 \ , \quad [P,Q] = 0 \ , \quad [Q,\Sigma] \sim Q \ , \quad \{Q,\bar{Q}\} \sim P \ . \tag{5}$$

These terms tell us (i) Q is Grassmann; (ii) Q commutes with spacetime translations (and hence the Hamiltonian); (iii) Q transforms under Lorentz transformation as a 2-component Weyl spinor; (iv) two successive supersymmetry transformations yields a translation. From (i) and (ii) it follows that there are pairs of fermion-boson states which are degenerate, and from (iv) we see that in some sense a supersymmetry charge Q is a square root of the Hamiltonian (in the same sense that the Dirac operator is a square root of the Klein-Gordon operator).

3.2 Counting supercharges

The supersymmetry algebra is highly constrained, and in any given number of dimensions there are typically only a few possibilities for how many supercharges can exist. These different solutions are often labeled $\mathcal{N} = 1$, $CN = 2$, etc. What is confusing is that the number of supercharges for $\mathcal{N} = 1$ supersymmetry, for example, is different in different numbers of dimensions. Instead, when discussing supersymmetric theories in dimensions other than four, I will identify a supersymmetric theory by the spacetime dimension d, and *the number of real supercharges*, $\mathcal{Q}$. Thus $\mathcal{N} = 1$ supersymmetry in $d = 4$ has a complex pair Q, $\bar{Q}$ which are each two-component Weyl spinors, giving $\mathcal{Q} = 4$. Similarly, $\mathcal{N} = 4$ supersymmetry in 4D has $\mathcal{Q} = 16$.

3.3 Why study lattice supersymmetry?

Supersymmetry is interesting in its own right. It is also potentially interesting for phenomenology, as the protection it affords scalars from additive renormalization of their masses could have something to do with the mysterious Higgs boson of the Standard Model. And it is worth studying because with the extra symmetry, many interesting results have been obtained for supersymmetric Yang-Mills (SYM) theories, including explicit examples of many mechanisms postulated in the early days of Yang-Mills theories, including spontaneous chiral symmetry breaking, confinement, magnetic monopole condensation, strong coupling - weak coupling duality, massless composite fermions, conformal field theory, and more. In addition, many fascinating connections have been made between between SYM theories and string theory and quantum gravity.

Since there are so many interesting features of SYM theories, especially at strong coupling, it would be very desirable to be able to define these theories nonperturbatively and to study them numerically. As the only method for studying ordinary gauge theories numerically is on the lattice, we are driven to define lattice SYM theories.

3.4 $\mathcal{N} = 1$ supersymmetry in $d = 4$

Let's look at the simplest SYM theory in 4D, which is called pure $\mathcal{N} = 1$ SYM. It consists of gauge bosons v_m (the "gluons", $m = 1, \dots 4$) and a single Weyl fermion λ_α (the "gluino", $\alpha = 1, 2$). The gluino is the supersymmetric partner of the gluon, and like it, transforms as the adjoint representation of the gauge group. (E.g, if the gauge group is $SU(3)$, the gluino is a color octet). Using the two-component fermion notation (see [16]), the Lagrangian for the theory is

$$\mathcal{L} = \bar{\lambda} i \bar{\sigma}^m D_m \lambda - \tfrac{1}{4} v_{mn} v^{mn} \ , \tag{6}$$

where $\bar{\sigma}^m = \{1, -\boldsymbol{\sigma}\}$ ($\boldsymbol{\sigma}$ being the three Pauli matrices and 1 being the unit matrix) and v_{mn} is the gauge field strength. This theory has only one independent coupling constant (the gauge coupling g) and is the most general Lagrangian one could write down without irrelevant operators — with the exception that we have omitted a fermion mass term, $(m\lambda\lambda + h.c.)$. At the classical level, the theory possesses a global $U(1)$ symmetry under which $\lambda \to e^{i\alpha}\lambda$. This does not commute with supersymmetry (because there is no analogous phase rotation of the gluino's partner the gluon) and for obscure historical reasons it is therefore called an R-symmetry. Now this $U(1)$ symmetry is anomalous, and if the gauge group is $SU(N)$, only a Z_{2N} subgroup of the $U(1)$ symmetry is exact in the full quantum theory (see, for example, [17]). Note that a gluino mass term would explicitly violate this Z_{2N} R-symmetry. It is known that gluino condensation occurs in this theory ($\langle\lambda\lambda\rangle \neq 0$), and that the global Z_{2N} symmetry is spontaneously broken to Z_2, giving rise to domain walls, where the strength of the condensate and the domain wall tension can be analytically related.

Can we investigate these properties on the lattice? After all, the theory looks simpler than QCD which has several flavors of quarks with different masses, which is routinely simulated! If you want to preserve supersymmetry on the lattice, then there are several obvious obstacles:

- Supersymmetry requires there to exist a charge Q which satisfies $\{Q, \bar{Q}\} \sim P$, where P is the generator of infinitesimal translations. But on the lattice, there are no symmetries corresponding to infinitesimal translations.
- Conventional lattice formulations put gauge bosons on links and fermions on sites (Wilson) or hypercubes (staggered) or a fifth dimension (Domain Wall Fermions). How could there be a symmetry between objects living at different places on the lattice? And if you put the fermions on links, won't they end up transforming under the Lorentz group improperly (e.g., as vector fields like the gauge bosons)?

Well, it looks like a bad idea to demand supersymmetry in the lattice action...and who needs it? After all, we get Poincaré symmetry from lattice QCD as an accidental symmetry — can't SUSY arise as an accidental symmetry in the IR (continuum limit, where all modes considered have wavelengths long compared to the lattice spacing)? To explore this idea we need to (i) consider how to create a lattice theory where the continuum degrees of freedom consist of a gauge boson and an adjoint Weyl fermion; (ii) itemize all the possible relevant or marginal operators in this theory which could spoil supersymmetry in the continuum limit; and (iii) consider whether there are any exact lattice symmetries which could forbid these undesirable operators.

In the case of $\mathcal{N} = 1$ SYM, this is easy to figure out, since in the continuum theory, the only "bad" relevant operator allowed by gauge plus Lorentz symmetries is a gaugino mass term...and this is forbidden by the Z_{2N} chiral R-symmetry. So that suggests that if we can figure out how to implement a gauge theory with a single adjoint Majorana fermion and a discrete chiral symmetry—then the continuum limit will automatically be the desired $\mathcal{N} = 1$ SYM [2].

Luckily, the problem of how to realize chiral fermions on the lattice has already been solved: the two related techniques are to use domain wall fermions (DWF) [20], or overlap fermions [21, 22]. Neuberger first proposed an overlap fermion solution for $\mathcal{N} = 1$ SYM in [?]. A DWF solution is presented in [3].

The DWF formulation is formulated on a (compact) five-dimensional lattice, with a massive fermion whose mass equals $+m_0$ on half the lattice and $-m_0$ on the other half. The 4D hypersurfaces where the mass changes sign are called "domain walls", and on solving the free Dirac equation, one finds two massless 4D fermion modes, one with $\gamma_5 = +1$ bound to one domain wall, and the other with $\gamma_5 = -1$ bound to the other wall, as shown in Fig. 2. (There is actually a small mass which vanishes exponentially in the fifth dimensional separation between the two domain walls, which I will assume is negligible). Four dimensional gauge fields are introduced á

[2] This scenario whereby supersymmetry could be realized as an accidental symmetry was first proposed in [2], and specifically for lattice simulation in [18]. An alternative pursued heroically is to forgo the chiral symmetry on the lattice and try to fine tune the gaugino mass to zero in the continuum limit, a difficult task in part because at finite lattice spacing the integrand in the path integral has a sign problem; see [19].

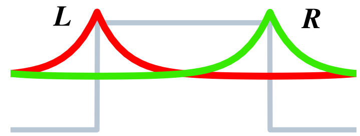

Fig. 2. *The profile of the domain wall fermion mass in the fifth dimension, showing the chiral zero-modes (L,R) bound to the two domain walls where the fermion mass switches sign.*

la Wilson, constant in the fifth dimension, with the fermion transforming as an adjoint under the gauge group. The low energy spectrum therefore looks like a 4D theory consisting of a gauged massless adjoint Dirac fermion and gauge bosons. This 4D (Euclidean) Dirac fermion takes the form $\Psi = \begin{pmatrix}\alpha \\ \bar{\beta}\end{pmatrix}$, $\bar{\Psi} = \begin{pmatrix}\bar{\alpha}^T & \beta^T\end{pmatrix}$, where α and β are the 2-component chiral spinors stuck to the two domain walls respectively. Since the gauge fields are constant in the fifth dimension, they are insensitive to the fact that the spinors α and β reside at different places in the extra dimension. Imposition of the Majorana condition is equivalent to requiring $\Psi = R_5 C \bar{\Psi}^T$, where R_5 is the reflection in the fifth dimension which interchanges the two domain walls, and C is the 4D charge conjugation matrix. To implement a Majorana fermion in the Euclidean path integral then, we just replace $\bar{\Psi}$ everywhere by $\Psi^T R_5^T C^T$, so that the Dirac Lagrangian $\bar{\Psi} \not{D} \Psi$ becomes instead $\Psi^T R_5^T C^T \not{D} \Psi$, and the Dirac determinant $\det \not{D}$ is replaced by the Pfaffian $\mathrm{Pf} R_5^T C^T \not{D}$, which is real and non-negative. This theory was simulated on a computer many years ago [23], but it is long overdue for a new simulation, given the importance of the theory and the huge advances in computer technology since the previous work.

4 Accidental SUSY with Scalars

In the previous section we saw that a gauged adjoint Majorana fermion in four dimensions was automatically supersymmetric provided that the relevant mass term $m\lambda\lambda$ vanished. Since this mass term violates a Z_{2N} chiral symmetry as well as supersymmetry, it follows that in a lattice theory that correctly implements the chiral symmetry, supersymmetry will automatically emerge as an accidental symmetry in the continuum limit, despite that fact that the lattice action is not supersymmetric at all.

Unfortunately, this simple reasoning does not extend readily to other supersymmetric theories, which all contain scalars as well as fermions, and possibly gauge fields. The problem is that supersymmetry is broken by the relevant operator responsible for scalar masses $m^2|\phi|^2$ (among others), which breaks the fermion-boson degeneracy. Following the example of $\mathcal{N}=1$ SYM, we would like to identify some symmetry (*other* than supersymmetry) which is broken by a scalar mass term, and which can be implemented exactly on the lattice. Unfortunately, unlike fermions, there is no chiral symmetry which can be invoked to forbid a scalar mass; the only symmetry that can do that is a shift symmetry $\phi \to \phi + f$, and this shift symmetry is too restrictive, dictating only derivative interactions for the scalar, applicable only to Goldstone bosons. Thus the only useful symmetry that can forbid the dastardly scalar mass term is supersymmetry.

We are apparently left with a paradox: implementing supersymmetry exactly on the lattice seems impossible, and so we would like it to emerge as an accidental symmetry; but in order for supersymmetry to emerge as an accidental symmetry, we are forced to suppress scalar mass renormalization, which requires implementing supersymmetry exactly on the lattice!

The loophole is that perhaps we don't have to find an exact lattice implementation of *all* of the supersymmetry of the target theory, but only realize part of the supersymmetric algebra. After all, the full Poincaré group is not realized on the lattice, but only the finite subgroup generated by finite translations and rotations by $\pi/2$, yet the Poincaré group emerges as an accidental symmetry. It is natural then to ask whether there could exist a "subgroup" of supersymmetry on the lattice? But the answer is no: whereas rotations, for example, are parametrized by a bosonic angle which can be large (e.g., $\pi/2$) supersymmetric transformations are characterized by a Grassmann parameter, which is necessarily infinitesimal, in the same way that there exist classical bosonic fields (such as the electric field) but not classical fermionic fields—bosonic states can have large occupation numbers, but not fermionic states[3].

Instead one must ask whether it is possible to preserve a subalgebra of the full extended supersymmetric algebra

$$\{Q^i_\alpha, Q^j_\beta\} = 0 \ , \quad \{\bar{Q}^i_{\dot\alpha}, \bar{Q}^j_{\dot\beta}\} = 0 \ , \quad \{Q^i_\alpha, \bar{Q}^j_{\dot\alpha}\} = 2P_m \sigma^m_{\alpha\dot\beta}\delta_{ij} \ , \tag{7}$$

where $i,j = 1,\dots,\mathcal{Q}$ run over different supercharges ($\mathcal{Q} = 1,2,4$ for $\mathcal{N} = 1,2,4$ supersymmetry respectively in 4D). As I will show you, the answer in this case is "yes", but it should be far from obvious! In fact it is easy to construct a list of reasons why this approach should fail miserably:

– The same old problem we keep returning to: how can a subalgebra of eq. (7) be chosen given that the P_m, the generator of infinitesimal translations, does not exist on the lattice?
– How can one take part of the algebra without destroying the hypercubic lattice symmetry, and thereby making it impossible to recover Poincaré symmetry in the continuum, let alone supersymmetry?
– Less abstractly, how is it possible to implement scalars, fermions and gauge bosons in a symmetric fashion on the lattice? For example, $\mathcal{N} = 4$ SYM has one gauge field, four Weyl gauginos, and six real scalars in the same supersymmetric multiplet. If we put the gauge fields on links, surely their scalar superpartners have to be on links too! But then the scalars will transform nontrivially under lattice rotations, which means they will can't transform as scalars (rotationally invariant objects) in the continuum limit, right?!
– SYM theories have R-symmetries ($U(1)$, $U(2)$ and $SU(4)$ respectively for $\mathcal{N} = 1,2,4$ theories in 4D; larger symmetries in lower dimensions) which are *chiral* symmetries; how are we to implement chiral fermions in a way that makes them look symmetric with their gauge and scalar partners?!

It looks hopeless without some sort of miracle, and that miracle can be found in a beautiful paper on "deconstruction" by Arkani-Hamed, Cohen and Georgi [24].

5 Deconstruction

5.1 The AKCG model

In reference [24] the authors were not concerned with latticizing supersymmetry; instead they wanted to precise field theoretic way to examine claims about the phenomenology of certain field theories in five dimensions. In order to avoid ill-defined problems with renormalization in five dimensions, they constructed a theory with four continuous dimensions, and a latticized fifth dimension. This can be viewed as a 4D field theory with many "flavors" of fields, corresponding to the discrete values of the fifth coordinate. A diagram of the theory of interest is given in Fig. 3; it is an $\mathcal{N} = 1$ supersymmetric field theory in 4D with gauge group $U(k)^N$ with a single gauge coupling g, where each $U(k)$ factor appears as a node in the picture. The n^{th} node has a vector multiplet associated with it — a gauge field $v^{(n)}_m$ and a gaugino $\lambda^{(n)}$. In addition there are matter fields in the form of chiral supermultiplets Φ_n which appear in the figure as directed links between nodes n and $(n+1)$; they transform as bifundamentals $(\sqcup\!\sqcap, \bar{\sqcup\!\sqcap})$ under the $U(k) \times U(k)$ gauge symmetry associated with those two nodes, and are neutral under the

[3] OK — there are such things as "supergroups" defined with Grassmann generators, but I have never found the concept to be of any practical use for constructing a supersymmetric lattice.

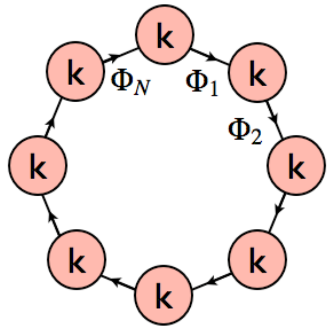

Fig. 3. *A diagram for the AKCG deconstruction model, which is a 4D, $\mathcal{N}=1$ supersymmetric gauge theory. Each node corresponds to an independent $U(k)$ gauge symmetry, with the associated vector supermultiplet V_n. The links represent chiral superfields Φ_n which transform as bifundamentals under the gauge symmetries of the nodes they connect.*

rest of the gauge symmetry; they represent the scalar and fermion component fields $(\phi^{(n)}, \psi^{(n)})$. All the interactions in this model are supersymmetric gauge interactions (which include certain Yukawa and ϕ^4 couplings). Note that since all the fields transform as either adjoints of $U(k)$ or bifundamentals of $U(k)\times U(k)$, they can all be represented as $k\times k$ non-traceless matrices.

So far, this model doesn't look at all like a lattice for a 5D theory; although there are interactions between nearest neighbors the fifth direction, there are no bilinear "hopping terms" corresponding to kinetic energy operators for motion in this extra dimension. However, the authors noted that the theory has a "flat direction" corresponding to

$$\langle\phi^{(n)}\rangle = \frac{1}{a\sqrt{2}}\,\mathbf{1}_k \tag{8}$$

where $\mathbf{1}_k$ represents that $k\times k$ unit matrix, and a is a length scale. By flat direction, I mean that the theory has a degenerate ground state, where the vacuum energy is unaffected by the simultaneous shift of all the scalar link fields $\phi^{(n)}$ as in eq. (8). Furthermore, as I will elaborate on below, AKCG noted that the parameter a behaves like a lattice spacing, and that in the limit

$$N\to\infty\ ,\quad a\to 0\ ,\quad g\to 0\ ,\quad aN\equiv L_5\ (\text{fixed})\ ,\quad g^2/a\equiv g_5^2\ (\text{fixed})\ , \tag{9}$$

the model of Fig. 3 has two amazing properties:

- it possesses $d=5$ Poincaré invariance;
- it possesses $\mathcal{Q}=8$ supercharges, even though the $d=4$ model in Fig. 3 only respected $\mathcal{Q}=4$ exact supersymmetries.

This is exactly the type of phenomenon we were looking for! Both Poincaré symmetry and supersymmetry are enhanced in the continuum limit without any fine tuning of the theory.

I now sketch out how the 5D kinetic terms emerge in the AKCG model in the $a\to 0$ limit, and then discuss how to generalize their procedure to generate true lattices where every spacetime dimension is discretized, a method called "orbifolding".

5.2 Continuum limit of the AKCG model

The the Lagrangian for the AKCG model possesses four types of terms:

1. The Yang-Mills action for the gauge fields $v_m^{(n)}$;

2. Gauge interactions for the adjoint gauginos $\lambda^{(n)}$ and the bifundamental matter fields $\phi^{(n)}$ and $\psi^{(n)}$, the latter involving both $v_m^{(n)}$ and $v_m^{(n-1)}$;
3. Yukawa interactions for the form $\sum_n \mathrm{Tr}\, \lambda^{(n)} \left(\psi^{(n)}\bar{\phi}^{(n)} - \bar{\phi}^{(n-1)}\psi^{(n-1)}\right)$;
4. A ϕ^4 interaction (called the "D-term") proportional to $\sum_n \mathrm{Tr}\left(\phi^{(n+1)}\bar{\phi}^{(n+1)} - \bar{\phi}^{(n)}\phi^{(n)}\right)^2$

It is easy to see then that indeed eq. (8) is a flat direction of the theory, since the D-term vanishes if each field $\phi^{(n)}$ equals the same diagonal matrix. To see how the continuum limit emerges, we expand the ϕ fields about their vacuum value as

$$\phi^{(n)}(x) = \frac{1}{a\sqrt{2}}\,\mathbf{1}_k + \frac{s^{(n)}(x) + iv_5^{(n)}(x)}{\sqrt{2}} \tag{10}$$

where s and v_5 are hermitean matrices. Then, for example, the (4D) kinetic term for ϕ in the AKCG action is

$$\begin{aligned}
&\frac{1}{g^2}\sum_n \int d^4x\ \mathrm{Tr}\,|D_\mu\phi^{(n)}|^2 = \frac{1}{g^2}\sum_n \int d^4x\ \mathrm{Tr}\,|\partial_\mu\phi^{(n)} + iv_\mu^{(n)}\phi^{(n)} - i\phi^{(n)}v_\mu^{(n+1)}|^2 \\
&= \frac{1}{2g^2}\sum_n \int d^4x\ \mathrm{Tr}\,\Big|\left(\partial_\mu s^{(n)} + iv_\mu^{(n)}s^{(n)} - is^{(n)}v_\mu^{(n+1)}\right) \\
&\qquad +i\left(\partial_\mu v_5^{(n)} + iv_\mu^{(n)}v_5^{(n)} - iv_5^{(n)}v_\mu^{(n+1)}\right) + i\left(v_\mu^{(n)} - v_\mu^{(n+1)}\right)/a\Big|^2 \\
&\xrightarrow[a\to 0]{} \frac{1}{2g_5^2}\int d^5x\ \mathrm{Tr}(D_\mu s)^2 - \mathrm{Tr}\, v_{\mu 5}v^{\mu 5}\ ,
\end{aligned} \tag{11}$$

where v_{mn} is the $d = 5$ gauge field strength. Note that the 5D kinetic term for the gauge field has emerged in this limit.

The scalar "D-term" in the AKCG model provided the 5D kinetic term for the field s in the same limit:

$$\frac{1}{2g^2}\sum_n \int d^4x\ \mathrm{Tr}\left(\phi^{(n+1)}\bar{\phi}^{(n+1)} - \bar{\phi}^{(n)}\phi^{(n)}\right) \xrightarrow[a\to 0]{} \frac{1}{2g_5^2}\int d^5x\ \mathrm{Tr}\,(D_5 s)^2\ . \tag{12}$$

Note that this $5D$ term is normalized the same way as the $(D_\mu s)^2$ term in the previous equation, as required by 5D Lorentz invariance.

In the AKCG model, the two Weyl fermions—the gaugino λ and the matter field ψ—combine to form one, 4-component, $d = 5$ fermion

$$\Psi = \begin{pmatrix}\lambda \\ \bar{\psi}\end{pmatrix}\ , \qquad \bar{\Psi} = (\psi \quad \lambda) \tag{13}$$

in the γ-matrix basis

$$\gamma_\mu = \begin{pmatrix} & \bar{\sigma}_\mu \\ \sigma_\mu & \end{pmatrix}\ , \qquad \gamma_5 = \begin{pmatrix}1 & \\ & -1\end{pmatrix} \tag{14}$$

The fifth dimensional part of the fermion kinetic term (and the $\Psi - s$ interaction) arises from the Yukawa interaction in the 4D theory:

$$\begin{aligned}
&\frac{1}{g^2}\sum_n \int d^4x\, i\sqrt{2}\,\mathrm{Tr}\,\lambda^{(n)}\left(\psi^{(n)}\bar{\phi}^{(n)} - \bar{\phi}^{(n-1)}\psi^{(n-1)}\right) + \text{h.c.} \\
&\xrightarrow[a\to 0]{} \frac{1}{2g_5^2}\int d^5x\ \mathrm{Tr}\left(\bar{\Psi}i\gamma_5 D_5\Psi - \bar{\Psi}\gamma_5[s,\Psi]\right)\ .
\end{aligned} \tag{15}$$

It is easy to figure out the limit of the remaining terms. The conclusion is that a 5D supersymmetric gauge theory emerges in the continuum limit, consisting of the scalar s arising as the real part of the link scalar ϕ, the fermion $\Psi, \bar{\Psi}$ arising both from the gauginos λ living at

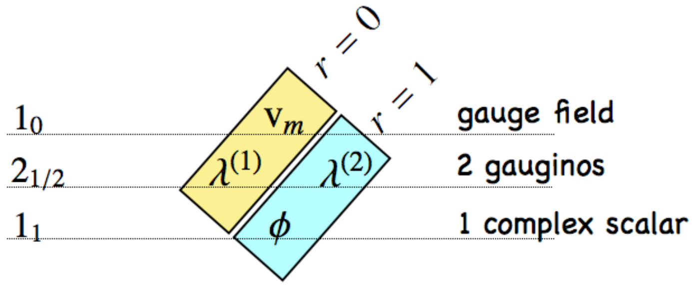

Fig. 4. *The fields for $\mathcal{N} = 2$ SYM theory, along with their $SU(2) \times U(1)$ R-symmetry quantum numbers. The charge $r = (Y - T_3)$ distinguishes which fields become site variables in the AKCG model and which become link variables ($r = 0$ and $r = 1$ respectively).*

the sites of Fig. 3, as well as the link fermions ψ; and the 5D gauge field consisting of the four components of v_μ living on the sites, and v_5 arising as the imaginary part of the link scalar ϕ. It is fascinating to see how these 5D multiplets form by combining both site and link variables. Most importantly for our purposes, recall my claim that this 5D gauge theory possesses $\mathcal{Q} = 8$ supersymmetries, which has somehow emerged in the $a \to 0$ limit from the original $\mathcal{Q} = 4$ theory, without any fine tuning.

6 Lattices from orbifold projection

The mechanism by which enhanced supersymmetry emerges in the continuum limit of the AKCG model is what has been long sought for in a lattice theory — but it is itself still a theory in four continuous dimensions and not on a lattice. To construct a true supersymmetric lattice, we must "reverse engineer" the AKCG model to find general principles for how it is constructed, and then apply those principles to constructing true spacetime lattices.

A simple procedure exists for producing the theory represented by Fig. 3 with N sites and a $U(k)^N$ gauge symmetry. The idea is to start with a "mother theory" which has the following properties:

– it is a $d = 4$ field theory like the AKCG model;
– it possesses the huge gauge group $U(Nk)$;
– it respects the number of supersymmetries of the *target* theory, namely $\mathcal{Q} = 8$.

In other words, it is a $d = 4$, $\mathcal{Q} = 8$ gauge theory with gauge group $U(Nk)$; such a theory is known as an $N = 2$ SYM theory.

What we will then do is project out a Z_N symmetry (which means: identify a Z_N symmetry in the theory, and set to zero all fields which aren't neutral under that symmetry). This projection (called an orbifold projection) breaks the gauge symmetry from $U(Nk) \to U(k)^N$, and it breaks half the supersymmetries of the theory, from $\mathcal{Q} = 8$ to $\mathcal{Q} = 4$. That leaves us with the AKCG model.

To see how this works, consider the field content of an $N = 2$ SYM theory. The gauge multiplet consists of a gauge field v_μ, two Weyl gauginos $\lambda^{(1,2)}$, and a complex scalar ϕ. Note the similarity between this multiplet and the field content appearing in Fig. 3. Each of the fields transforms as the adjoint representation of the gauge group, which in our case is $U(Nk)$; that means we can represent the fields as $Nk \times Nk$ matrices, acted upon by the gauge transformation U as $\phi \to U\phi U^\dagger$ (except for the gauge field, which has the usual inhomogeneous transformation).

But how to define the Z_N symmetry which tells some fields to become site variables and others to become link variables in the AKCG model? The $N = 2$ SYM action possesses an $SU(2) \times U(1)$ R-symmetry, under which the fields transform as shown in Fig. 4. We can find a symmetry which distinguishes between fields destined to become site variables (v_m and $\lambda^{(1)}$) and link variables ($\lambda^{(2)}$ and ϕ) by defining a $U(1)$ charge r which lives in the $SU(2) \times U(1)$

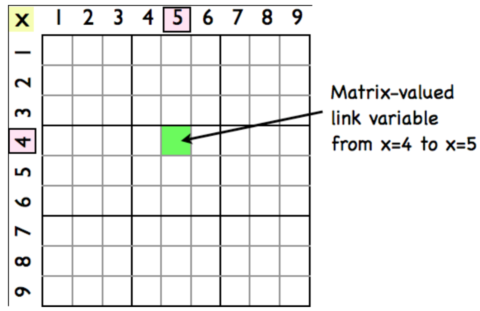

Fig. 5. *Illustration of how a $9k \times 9k$ matrix can represent a 1D lattice with $N = 9$ sites. The highlighted $k \times k$ block represents a $k \times k$ matrix-valued field residing on the directed link from site $x = 4$ to site $x = 5$.*

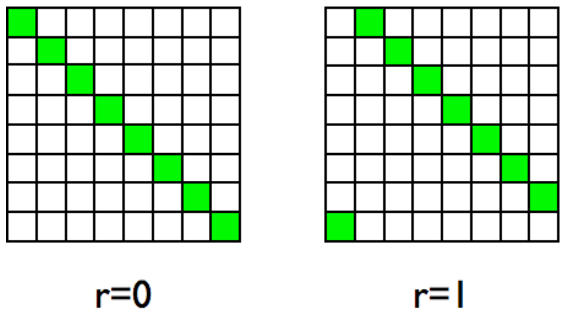

Fig. 6. *The result of the Z_N orbifold projection: For the fields v_m and $\lambda^{(1)}$ with $r = 0$, only the diagonal $k \times k$ blocks survive, and these can be interpreted as site variables, transforming as adjoints under the unbroken $U(k)^N$ gauge symmetry. The $\lambda^{(2)}$ and ϕ fields with $r = 1$ have only the superdiagonal blocks survive; these transform as bifundamentals under the $U(k)^N$ gauge symmetry, and represent the link variables in Fig 3 (with $\lambda^{(2)} \equiv \psi$).*

R-symmetry: $r = Y - T_3$, where Y is the $U(1)$ charge and T_3 is the third $SU(2)$ generator. Then as shown in Fig. 4, site variables have $r = 0$ and link variables have $r = 1$.

Each of the different types of fields of the AKCG model — each of the N "flavors" of $k \times k$ matrices — can be represented as a single sparse $Nk \times Nk$ matrix, as illustrated in Fig. 5. We think of the big $Nk \times Nk$ matrix as being made of N^2 $k \times k$ blocks, each labeled by a row number n_i and a column number n_f; then that block can be thought of as living on a 1D lattice as a link running from site n_i to site n_f. Thus for the site variables ($r = 0$) we want to have an $Nk \times Nk$ matrix with only diagonal $k \times k$ blocks surviving; the link variables ($r = 1$) in Fig. 3 should become sparse $Nk \times Nk$ matrices with nonzero blocks only appearing one row above the diagonal.

We can attain the desired result by defining a Z_N symmetry which combines the r symmetry with a particular $U(Nk)$ transformation:

$$Z_N : \Phi \to \hat{\gamma}\Phi \equiv \omega^r \Omega \Phi \Omega^\dagger \ , \qquad \Omega = \begin{pmatrix} \omega & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^N \end{pmatrix} \ , \qquad \omega = e^{2\pi i/N} \ , \tag{16}$$

where r is the particular r-charge for that field Φ, and each entry in Ω is proportional to a $k \times k$ unit matrix. We then define the orbifold projection operator $\hat{P}\Phi = \frac{1}{N}\sum_{i=p}^{N} \hat{\gamma}^p \Phi$ which annihilates any sub–block in the matrix Φ which is not invariant (this follows from the fact that $[\omega + \omega^2 + \ldots + \omega^N] = 0$). Note that this projection does not commute with the full $U(Nk)$ gauge symmetry of the mother theory and leaves intact only the $U(k)^N$ subgroup which commutes

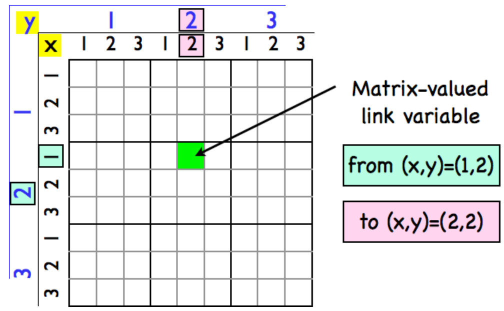

Fig. 7. *Representing a two-dimensional* 3×3 *lattice by a sparse* 9×9 *matrix: each sub-block can be identified as a site or link variable on the 2D lattice.*

with Ω. The result of this projection is shown in Fig. 6. Note that evidently $\hat{P}$ also breaks the $\mathcal{N} = 2$ supersymmetry, since it treats the different members of the gauge multiplet differently. it does, however, preserve an $\mathcal{N} = 1$ supersymmetry, with $\{v_m, \lambda^{(1)}\}$ being an $\mathcal{N} = 1$ vector supermultiplet, and $\{\phi, \lambda^{(2)}\}$ forming an $\mathcal{N} = 1$ chiral matter multiplet.

And the punchline: by plugging the sparsified matrices after projection back into the $\mathcal{N} = 2$ action, one recovers the full action of the AKCG model!

It is straightforward now to generalize our orbifold projection prescription in order to construct true lattices, of varying dimensions. For example, to produce a $d = 2$ lattice, we need to start with a mother theory with a $U(N^2k)$ gauge symmetry, and project out a $Z_N \times Z_N$ symmetry. The idea is that we take the $N^2k \times N^2k$ matrices in the mother theory, divide them into N^2 $NK \times Nk$ blocks, and then divide those into N^2 $k \times k$ sub-blocks. The location of each $k \times k$ sub-block can then be specified by four integers; the interpretation is that this is a link variable going from one site on a 2D lattice (specified by two integers) to another (specified by another two integers); see Fig. 7.

We now have a method for generating supersymmetric lattice actions:

(I) Start with a mother theory which is an SYM with the same number of supercharges $\mathcal{Q}$ as the target theory in the continuum;
(II) This mother theory should be formulated in *zero* dimensions (in other words: it is a matrix model, not a field theory), since we don't want any continuous dimensions, unlike the AKCG model which was formulated in $d = 4$;
(III) For a target theory with d continuous dimensions, make the gauge group of the mother theory $U(N^dk)$, identify the apppropriate Z_N^d symmetry that resides partly in the gauge group and partly in the R-symmetry group of the mother theory, and project it out;
(IV) Travel out along the flat direction in the degenerate vacua as in eq. (9), in order to recover the continuum limit of the target theory.

Oddly enough, this diabolical recipe really works! And in fact, it has recently been claimed that all the different constructions of lattice SYM theories in the literature can be shown to be equivalent to ones obtained through orbifold projection [25].

As with all pacts with the devil there is a price: item (I) and item (III) above are not in general compatible, since a theory with a small number of supercharges will have a small R-symmetry which will not contain a Z_N^d subgroup for large d. Equivalently, since each dimension requires a Z_N projection which breaks half of the remaining supercharges of the mother theory (and since we want the lattice theory to possess at least one unbroken supercharge) we require $\mathcal{Q} \geq 2^{d+1}$. Thus to go to higher dimension d, one needs to consider highly supersymmetric theories with large $\mathcal{Q}$. For $d = 4$, the only supersymmetric lattice that can be constructed via this method must have $\mathcal{Q} \geq 16$, leaving $\mathcal{N} = 4$ SYM theory as the only possibility. This is a very interesting theory though, and there is a greater variety of possibilities for $d < 4$.

7 A Lattice Theory for $(2,2)$ SYM

I now briefly describe the construction of a $d = 2$ lattice for a theory with four supercharges. In two dimensions, supercharges can be specified as "left-handed" or "right-handed", and this theory has two of each, so it is called $(2,2)$ SYM theory. The action for this theory is easy to write down: start with the familiar $\mathcal{N} = 1$ SYM theory in $d = 4$ dimensions (a gauge theory with a massless Weyl adjoint fermion), and erase two of the space dimensions. The gaugino becomes a 2-component Dirac fermion ψ (since γ matrices in $d = 2$ are just Pauli matrices, Dirac spinors have only two components). The four component gauge boson becomes a two component gauge boson plus one complex scalar field s. The gluon and gaugino interactions in the $d = 4$ action become 2D gauge interactions, plus Yukawa and s^4 interactions. The result is the action (in Euclidean spacetime)

$$\mathcal{L} = \frac{1}{g_2^2}\,\mathrm{Tr}\left(\left|D_m s\right|^2 + \bar\psi\, iD_m\gamma_m\psi + \tfrac{1}{4}v_{mn}v_{mn} + i\sqrt{2}(\bar\psi_L[s,\psi_R] + \bar\psi_R[s^\dagger,\psi_L]) + \tfrac{1}{2}[s^\dagger, s\,]^2\right) \quad (17)$$

where $m,n = 1,2$, ψ_R and ψ_L are the right- and left-chiral components of a two-component Dirac field ψ, $D_m = \partial_m + i[v_m,\ \cdot\ ]$ is the covariant derivative, and $v_{mn} = -i[D_m, D_n]$ is the field strength. All fields are rank-k matrices transforming as the adjoint representation of $U(k)$. This is the target theory.

To construct a lattice for this target theory, we need to start with a matrix theory with a $U(N^2k)$ gauge symmetry with $\mathcal{Q} = 4$ supersymmetries. What is a matrix theory you ask? Simple! Start with the same $\mathcal{N} = 1$ SYM theory in $d = 4$, which we know has $\mathcal{Q} = 4$ supersymmetries...and then erase *all* spacetime coordinates from the action (and therefore, all derivatives). The result is a very simple action which will serve as our mother theory:

$$S = \frac{1}{g^2}\left(\frac{1}{4}\,\mathrm{Tr}\, v_{mn}v_{mn} + \mathrm{Tr}\,\bar\psi\,\bar\sigma_m[v_m,\psi]\right)\ , \quad (18)$$

where $m,n = 0,\ldots,3$, ψ and $\bar\psi$ are independent complex two-component spinors, v_m is the 4-vector of gauge potentials, and

$$v_{mn} = i[v_m, v_n]\ ,\quad \sigma_m = \{1, -i\boldsymbol{\sigma}\}\ ,\quad \bar\sigma_m = \{1, i\boldsymbol{\sigma}\}\ , \quad (19)$$

This mother theory possesses $\mathcal{Q} = 4$ supersymmetries, characterized by the transformations

$$\delta v_m = -i\bar\psi\,\bar\sigma_m\kappa + i\bar\kappa\,\bar\sigma_m\psi\ ,\quad \delta\psi = -iv_{mn}\sigma_{mn}\kappa\ ,\quad \delta\bar\psi = iv_{mn}\,\bar\kappa\,\bar\sigma_{mn}\ , \quad (20)$$

where

$$\sigma_{mn} \equiv \tfrac{i}{4}\left(\sigma_m\bar\sigma_n - \sigma_n\bar\sigma_m\right)\ ,\qquad \bar\sigma_{mn} \equiv \tfrac{i}{4}\left(\bar\sigma_m\sigma_n - \bar\sigma_n\sigma_m\right)\ . \quad (21)$$

where κ and $\bar\kappa$ are independent two-component Grassmann parameters.

The R-symmetry of the mother theory is $SO(4)\times U(1) = SU(2)\times SU(2)\times U(1)$. Deriving this result is not very mysterious: the $U(1)$ factor is just the $U(1)$ R-symmetry associated with the 4D $\mathcal{N} = 1$ SYM theory we started with to derive the mother theory (should this be called the grandmother theory?). The $SO(4) = SU(2)\times SU(2)$ factor is nothing but what remains of the (Euclidean) Lorentz symmetry that remains even after all spacetime coordinates are removed from the theory. Therefore v_m transforms as a 4-vector $= (2,2)$ under this $SU(2)\times SU(2)$, while ψ transforms as a $2,1)$ and $\bar\psi$ as a $(1,2)$.

The "daughter theory" we will derive from from this mother theory by orbifolding will be a two-dimensional lattice with N^2 sites and a $U(k)$ symmetry associated with each site (the conventional way to realize a $U(k)$ gauge symmetry). To obtain this daughter theory we must identify the correct $Z_N\times Z_N$ symmetry to project out. The trick is to define two independent analogues of the r-charge from the previous section — I'll call them $\mathbf{r} = \{r_1, r_2\}$ — so that the maximum number of fermions has $\mathbf{r} = \{0,0\}$, since it is possible to show that this number

bosons	r_1	r_2	fermions	r_1	r_2
$z_1 = \frac{v_0 - iv_3}{\sqrt{2}}$	1	0	λ_1	0	0
$\bar{z}_1 = \frac{v_0 + iv_3}{\sqrt{2}}$	-1	0	λ_2	-1	-1
$z_2 = -i\frac{v_1 - iv_2}{\sqrt{2}}$	0	1	$\bar{\lambda}_1$	1	0
$\bar{z}_2 = i\frac{v_1 + iv_2}{\sqrt{2}}$	0	-1	$\bar{\lambda}_2$	0	1

Table 1. *Assignment of the $Z_N \times Z_N$ charges for the variables of the mother theory eq.* (19).

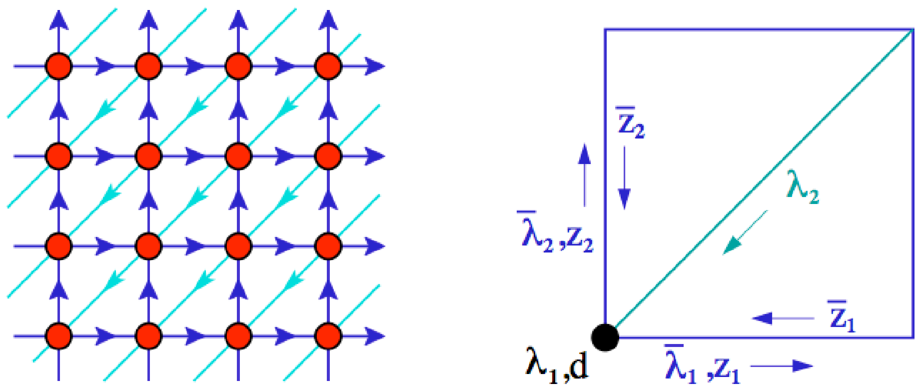

Fig. 8. *The lattice structure and the unit cell for the target theory of eq.* (17), (2, 2) *SYM in two dimensions. The "d" variable is an auxiliary field you can ignore; it proves to be convenient when developing a superfield formulation for the lattice theory.*

equals the number of unbroken supersymmetries. With little work, it is possible to show that a suitable choice yields the charge assignments displayed in Table 1 [6], where we have written the fermion components as

$$\psi = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} , \qquad \bar{\psi} = \begin{pmatrix} \bar{\lambda}_1 & \bar{\lambda}_2 \end{pmatrix} \tag{22}$$

We can then use these r-charges to define a $Z_N \times Z_N$ projection which creates the lattice shown in Fig. 8. Note that there is a very simple correspondence between the $\mathbf{r}$ charges in Table 1, and the location of each variable in the unit cell of the 2D lattice!

I won't give any of the details here, but it is not too difficult to construct the lattice action by substituting the orbifold projected matrices back into the action of the mother theory, eq. (19). One then follows the path of deconstruction, expanding the boson fields as

$$z_i = \frac{1}{a\sqrt{2}} \mathbf{1}_k + \frac{s_i + iv_i}{\sqrt{2}} \tag{23}$$

and taking the continuum limit $a \to 0$ with $g^2 a^2 = g_2^2$ kept fixed. Amazingly enough, one finds the target theory eq. (17) in this limit, with the identification

$$s = \frac{s_1 + is_2}{\sqrt{2}} , \qquad \psi = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} , \qquad \bar{\psi} = \begin{pmatrix} \bar{\lambda}_1 & \bar{\lambda}_2 \end{pmatrix} , \qquad v_m = \begin{pmatrix} v_1 & v_2 \end{pmatrix} . \tag{24}$$

So what about the list of obstructions I gave at the end of §4? How does this theory get around them? For example, the target theory has an exact chiral $U(1)$ R-symmetry; how did this arise in the lattice theory? Did we invent a new type of lattice chiral fermion? Also, I am claiming that the scalar s in the target theory is represented on the lattice by s_1 and s_2 which

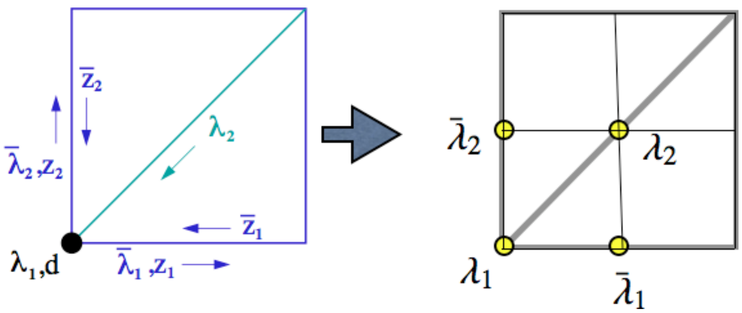

Fig. 9. *The fermions mapped onto a lattice with half the spacing can be recognized as reduced staggered fermions.*

are link variables; this means that even though s_1 and s_2 transform into each other nontrivially under lattice rotations, they must be invariant under rotations in the continuum! Isn't this absurd, since the continuum rotations contain lattice rotations as a subgroup, and an object transforming nontrivially under the latter must transform nontrivially under the former?

To understand what is going on, let us first focus on the quadratic part of the boson action, which looks like:

$$\begin{aligned}
&\frac{1}{2g^2a^2}\sum_{\mathbf{n}}\mathrm{Tr}\Big[(s_{1,\mathbf{n}-\hat{\mathbf{x}}}-s_{1,\mathbf{n}}+s_{2,\mathbf{n}-\hat{\mathbf{y}}}-s_{2,\mathbf{n}})^2\\
&\qquad\qquad+\Big|(s_{1,\mathbf{n}+\hat{\mathbf{y}}}-s_{1,\mathbf{n}}+s_{2,\mathbf{n}}-s_{2,\mathbf{n}+\hat{\mathbf{x}}})-i\,(v_{1,\mathbf{n}+\hat{\mathbf{y}}}-v_{1,\mathbf{n}}-v_{2,\mathbf{n}+\hat{\mathbf{x}}}+v_{2,\mathbf{n}})\Big|^2\Big]\\
&=\frac{1}{2g^2}\sum_{\mathbf{n}}\mathrm{Tr}\left[\sum_{\hat{\boldsymbol{\mu}}}\sum_{i=1,2}\left(\frac{s_{i,\mathbf{n}}-s_{i,\mathbf{n}-\hat{\boldsymbol{\mu}}}}{a}\right)^2+\left(\frac{v_{1,\mathbf{n}+\hat{\mathbf{y}}}-v_{1,\mathbf{n}}}{a}-\frac{v_{2,\mathbf{n}+\hat{\mathbf{x}}}-v_{2,\mathbf{n}}}{a}\right)^2\right]\ ,
\end{aligned}\tag{25}$$

When we take the continuum limit, we get

$$\frac{1}{g_2^2}\int d^2x\,\tfrac{1}{2}\,\mathrm{Tr}\left[(\partial_1 s_1+\partial_2 s_2)^2+(\partial_2 s_1-\partial_1 s_2)^2+(\partial_2 v_1-\partial_1 v_2)^2\right]\ . \tag{26}$$

Note that the first two terms make (s_1, s_2) look like a vector (as you would expect from link variables!) rather than components of scalar: the first term looks like $(\nabla\cdot\mathbf{s})^2$, while the second term looks like $(\nabla\times\mathbf{s})^2$; neither term looks like the scalar kinetic term $\left[(\partial_m s_1)^2+(\partial_m s_2)^2\right]$...yet amazingly enough, when you add the two terms and integrate by parts, that is *exactly* what you get! Not only do we get the correct $SO(2)$ Euclidean "Lorentz" invariance with s_i being invariant, but we get an *independent* internal $SO(2)$ symmetry where the s_i rotate into each other while the derivatives ∂_m don't change. The latter $SO(2)=U(1)$ is just the R-symmetry's action on the scalar s!

If we turn to the quadratic part of the fermion action, we find something more familiar. If one takes our rather unconventional lattice, and superimpose upon it a lattice with spacing $a/2$, the fermions can all be mapped onto sites, as shown in Fig. 9. Examining the lattice action for these fermions in the coordinates of this sublattice, one discovers that the fermions are none other than "reduced staggered fermions" as discussed in [26]. Again, you might wonder how a collection of fermions scattered over different parts of the lattice could reassemble themselves into a continuum spinor; it seems as mysterious as how our link bosons became a complex scalar. However, the mysterious ways of staggered fermions are well understood, and reviewing them in the next section will shed light on what has happened with our scalars. Understanding these features go a long way toward explaining how the obstacles facing lattice supersymmetry have been circumvented by the orbifold projection technique we have been using.

I will finish up this section with a brief discussion about quantum corrections in our lattice theory for $(2,2)$ SYM. Recall that the goal of a supersymmetric lattice action was to prevent unwanted relevant or marginal operators from being radiatively generated which could spoil supersymmetry in the the continuum limit. Since our construction leads to a single site fermion (that is a fermion with $\mathbf{r} = (0,0)$) that implies that the lattice possesses a single supercharge [4]. This single supercharge is enough to protect the lattice theory from unwanted radiatively induced operators which could spoil the supersymmetric continuum limit of the lattice theory, just as we hoped. To show this we can construct the *Symanzik action* for the theory: we expand the z variables about the flat direction $\langle z\rangle = \mathbf{1}_k/a\sqrt{2}$, expand the action for smooth fields in powers of $1/a$, include all operators allowed by the exact symmetries of the lattice, and then consider radiative corrections to the coefficients of these operators, paying special attention to relevant and marginal operators which violate the full $\mathcal{Q} = 4$ supersymmetry of the target theory, and whose coefficients by definition do not vanish in the $a \to 0$ limit. The key is to identify all the operators allowed by the exact lattice symmetries, which include the single supersymmetry. This is most easily done by constructing superfields: we introduce a Grassmann coordinate θ, which has mass dimension $1/2$ (where spacetime coordinates x have mass dimension -1), and define the exact lattice supercharge to be $Q = \partial_\theta$. With this definition of Q, and knowing the action of Q on the lattice variables, it is possible to construct superfields as is done in the more familiar $d = 4$, $\mathcal{N} = 1$ supersymmetry [16]. One finds the following superfields on the unit cell at site $\mathbf{n}$:

$$
\begin{aligned}
Z_1(\mathbf{n}) &= z_1(\mathbf{n}) + \sqrt{2}\theta\bar\lambda_1(\mathbf{n})\ ,\\
Z_2(\mathbf{n}) &= z_2(\mathbf{n}) + \sqrt{2}\theta\bar\lambda_2(\mathbf{n})\ ,\\
\Xi(\mathbf{n}) &= \lambda_2(\mathbf{n}) + 2\left[\bar z_1(\mathbf{n}+\hat{\mathbf{y}})\bar z_2(\mathbf{n}) - \bar z_2(\mathbf{n}+\hat{\mathbf{x}})\bar z_1(\mathbf{n})\right]\theta\ ,\\
\Lambda(\mathbf{n}) &= \lambda_1(\mathbf{n}) - \left[\bar z_1(\mathbf{n}-\hat{\mathbf{x}})z_1(\mathbf{n}-\hat{\mathbf{x}}) - z_1(\mathbf{n})\bar z_1(\mathbf{n})\right.\\
&\qquad \left. + \bar z_2(\mathbf{n}-\hat{\mathbf{y}})z_2(\mathbf{n}-\hat{\mathbf{y}}) - z_2(\mathbf{n})\bar z_2(\mathbf{n}) + id(\mathbf{n})\right]\theta\ .
\end{aligned}
\tag{27}
$$

Since $\int d\theta = \partial_\theta$ the most general supersymmetric action can be written as

$$
\frac{1}{g_2^2}\int d\theta \int d^2x \sum_{\mathcal{O}} C_{\mathcal{O}}\mathcal{O}(x,\theta) \tag{28}
$$

where the $\mathcal{O}$ are local Grassmann operators. Since the action has to be dimensionless, if $\mathcal{O}$ has mass dimension p, it is easy to check that the operator coefficient C_O must have dimension $(7/2-p)$. Now, since the action has a $1/g_2^2$ out front (where g_2 has mass dimension 1), radiative corrections to $C_{\mathcal{O}}$ at ℓ loops will be of the form

$$
\delta C_{\mathcal{O}} \sim c_\ell a^{(p-7/2)}(g_2^2a^2)^\ell\ , \tag{29}
$$

where the c_ℓ are dimensionless coefficients and can only depend on a logarithmically. Since we only care about operator coefficients which do not vanish as $a \to 0$, we need only consider operators and loops satisfying $p \le (7/2-\ell)$. At $\ell = 0$ (tree level) I claim our lattice action gives the correct target theory in the continuum limit. At $\ell = 1$ we need to consider $p \le 5/2$; at $\ell = 2$ we need to consider $p \le 3/2$. it turns out we cannot construct operators with $p \le 1/2$ so that's it! It is then a quick job to convince oneself that there are no bad operators $\mathcal{O}$ with $p = 5/2, 3/2$ which one can construct. Therefore, we we can prove that the supersymmetric lattice does what it was supposed to do: allow one to attain a supersymmetric target theory without fine-tuning. The exact supersymmetry of the lattice was crucial for this to be possible. I refer interested readers to ref. [6] for details of the argument. The analysis for this theory was simplified by the fact that it is "super-renormalizable", namely that each loop correction introduced positive powers of a. Unfortunately, I do not know how to make a general argument about renormalizability for the $d = 4$, $\mathcal{Q} = 16$ theory we can construct, as gauge interactions are marginal in $d = 4$.

[4] This is one of the funny things about supersymmetry in Euclidean spacetime: it is possible to have a theory respecting a single supercharge, which is impossible in Minkowski space. This feature is related to the strange property of fermions continued to Euclidean space, that $\bar\psi$ is not related to the hermitean conjugate of ψ.

8 Fermions and scalars on the SUSY Lattice

Armed with an explicit example of a supersymmetric lattice which overcomes the seemingly insurmountable objections listed in §4, I reexamine here the interplay between supersymmetry, R-symmetry, and Lorentz symmetry. In particular, we saw in the previous section scalar fields which transformed nontrivially under lattice rotations emerged in the continuum as invariant under spacetime rotations, but possessing an *internal* rotational symmetry which becomes the R=symmetry of the target theory. In contrast the gauge bosons transform nontrivially under lattice rotations and emerge in the continuum in the familiar was as spacetime vectors, with no internal symmetry. Thus lattice rotations seem not to be directly related to continuum spatial rotations, but rather should be regarded as some complicated mix of spacetime and internal symmetry rotations. This is an unfamiliar proposition for bosons, but is a well-known phenomenon for staggered fermions. To understand better, we begin by looking at how staggered fermions work, recasting them as Dirac-Kähler fermions, which in turn sheds light on how the orbifold projection approach realizes lattice supersymmetry.

8.1 Staggered fermions

The “naive” action for free Dirac fermions in $d = 2$ is

$$S = \frac{1}{2a} \sum_{\mathbf{n},\mu} \bar{\psi}(\mathbf{n}\gamma_\mu \left(\psi(\mathbf{n}+\hat{\mu}) - \psi(\mathbf{n}-\hat{\mu})\right) \ . \tag{30}$$

This action actually represent $2^d = 4$ Dirac fermions, due to doubling at the corners of the Brouillin zone. But we can define new 2-component spinors χ, $\bar{\chi}$ as

$$\psi(\mathbf{n}) = \gamma_2^{n_2}\gamma_1^{n_1}\chi(\mathbf{n}) \ , \qquad \bar{\psi}(\mathbf{n}) = \bar{\chi}(\mathbf{n})\gamma_1^{n_1}\gamma_2^{n_2} \ , \tag{31}$$

leading to the action

$$S = \frac{1}{2a} \sum_{\mathbf{n},\mu} (-1)^{n_2} \bar{\chi}(\mathbf{n}) \left(\chi(\mathbf{n}+\hat{\mu}) - \chi(\mathbf{n}-\hat{\mu})\right) \ . \tag{32}$$

Note that in this form, there are no gamma matrices in the action. Therefore the two components of χ decouple, and we get two identical copies of the same action. This means we can throw one copy away with impunity and be left with half as many fermions (2 Dirac fermions). So the action is given by eq. (32), where the χ and $\bar{c}hi$ are now one-component fermions living at each site. These are staggered fermions.

It is possible to reduce the number of degrees further, by noting that χ at even (odd) sites only couple to $\bar{\chi}$ at odd (even) sites. Therefore we can eliminate the $\bar{\chi}$ from all the even lattice sites, and the χ from all the odd lattice sites, and since we cut the degrees of freedom by half, we now find a single Dirac fermion in the continuum. These “reduced staggered fermions” are what arose in our supersymmetric lattice, pictured in Fig. 9; all we have to do to make the identification is the renaming

$$\chi(\mathbf{n}) \to \lambda_1(\mathbf{n}) \ , \quad \chi(\mathbf{n}+\hat{\mathbf{x}}+\hat{\mathbf{y}}) \to \lambda_2(\mathbf{n}) \ , \quad \bar{\chi}(\mathbf{n}+\hat{\mathbf{x}}) \to \bar{\lambda}_1(\mathbf{n}) \ , \quad \bar{\chi}(\mathbf{n}+\hat{\mathbf{y}}) \to \bar{\lambda}_2(\mathbf{n}) \ . \tag{33}$$

8.2 Dirac-Kähler fermions

Staggered fermions are mysterious. A much more beautiful formulation of lattice fermions are the Dirac-Kähler fermions [27,28] which are actually equivalent to staggered fermions, but start from a geometric point of view that makes it clear how staggered fermions actually work.

Dirac-Kähler fermions are analogues of p-forms (see ref. [29]. A form in two spacetime dimensions looks like

$$F = \underbrace{f(x)}_{\text{0-form}} + \underbrace{f_\mu(x)\mathrm{d}x_\mu}_{\text{1-form}} + \tfrac{1}{2}\underbrace{f_{[\mu\nu]}(x)\mathrm{d}x_\mu \wedge \mathrm{d}x_\nu}_{\text{2-form}} = f + f_\mu \mathrm{d}x_\mu + f_{12}\mathrm{d}x_1 \wedge \mathrm{d}x_2 \tag{34}$$

where the indices run over the values $1, 2$, $f_{[\mu\nu]} = \epsilon_{\mu\nu} f_{12}$ is an antisymmetric tensor, and the differentials $\mathrm{d}x_\mu$ anti-commute with each other. The p-forms have a natural geometric meaning on a lattice: 0-forms corresponding to site variables, 1-forms to link variables, and $2-forms$ to plaquette variables. The dual form is

$$^*F = f_{12} + \epsilon_{\mu\nu} f_\mu \mathrm{d}x_\nu + \tfrac{1}{2}\epsilon_{\mu\nu} f \mathrm{d}x_\mu \wedge \mathrm{d}x_\nu, \tag{35}$$

which maps the lattice variables onto the dual lattice variables (sites and plaquettes are interchanged, while x- and y-links are interchanged). These definitions can be extended to higher dimensions, where p-forms correspond to p-cells on the lattice. These differential operators also have a natural transcription to the lattice [27].

There are two types of derivatives that can act on forms

$$\begin{aligned} \mathrm{d}F &= \partial_\mu f \mathrm{d}x_\mu + \tfrac{1}{2}\partial_\mu f_\nu \mathrm{d}x_\mu \wedge \mathrm{d}x_\nu \ , \\ \delta F &= {}^*\mathrm{d}^* F = \partial_\mu f_\mu + \epsilon_{\mu\nu}\partial_\mu f_{12} \mathrm{d}x_\nu \ . \end{aligned} \tag{36}$$

d takes p-forms to $(p+1)$-forms, while δ takes p-forms to $(p-1)$-forms.

There is a similarity between the anticommuting differentials $\mathrm{d}x_\mu$ and the Dirac gamma matrices. Suppose you have two, 2-component Dirac fermions, and one arranges them into a 2×2 matrix Ψ. Note that under $SO(2)$ Lorentz transformations, $\Psi \to O\Psi$, while under $SU(2)$ "flavor" transformations, $\Psi \to \Psi U^\dagger$. Since the three gamma matrices $\{1, \gamma_\mu, \sigma_{[\mu\nu]}\}$ (where $\sigma_{\mu\nu} = i\epsilon_{\mu\nu}\gamma_1\gamma_2$) form a complete basis for 2×2 matrices we can expand Ψ as

$$\Psi = \psi \mathbf{1} + \psi_\mu \gamma^\mu + \psi_{12}\gamma_1\gamma_2 \tag{37}$$

The differential operators act on Ψ as

$$\mathrm{d}\Psi = \partial_\mu \psi \gamma_\mu + \epsilon_{\mu\nu}\partial_\mu \psi_\nu \gamma_1\gamma_2 \ , \qquad \delta\Psi = \partial_\mu \psi_\mu \mathbf{1} + \epsilon_{\mu\nu}\partial_\mu \psi_{12}\gamma_\nu \tag{38}$$

With little effort it is possible to show that the Dirac operator acting on Ψ can be simply written in terms of d and δ:

$$\partial\!\!\!/ \Psi = (\mathrm{d} + \delta)\Psi \ . \tag{39}$$

Therefore the Dirac action for these fermions has a natural implementation on the lattice, where ψ gets mapped to sites, ψ_μ to links, and ψ_{12} to plaquettes. For ungauged fermions, the lattice action for Dirac-Kähler fermions is equivalent to that for staggered fermions. It is also possible to "reduce" Dirac-Kähler fermions by imposing a relation between $\bar{\Psi}$ and $^*\Psi$, eliminating half of the degrees of freedom. What results are the fermions we already saw on our supersymmetric lattice in Fig. 9, where λ_1 is equated to ψ and λ_2 on the diagonal link is the plaquette variable ψ_{12}.

The important point about the Dirac-Kähler formulation is not that it is pretty, but that it imbues fermions with a natural geometric interpretation which is easily implemented on the lattice. The key point is that *spinors* do not have a natural geometric interpretation, but *antisymmetric tensors* do. Observe that the components ψ, ψ_μ and $\psi_{[\mu\nu]}$ in eq. (37) transform as tensor representations of the diagonal $SO(2)$ subgroup of $SO(2)_{\text{Lorentz}} \times SU(2)_{\text{flavor}}$, under which $\Psi \to O\Psi O^{-1}$. It is a $\pi/2$ rotation under this diagonal $SO(2)$ that turns an x-link into a y-link on the lattice, therefore. So we see something very interesting emerge: the lattice point group symmetry *cannot* simply be thought of as a subgroup of the Lorentz group of the continuum theory. Instead it is a subgroup that lives in the product of the continuum Lorentz and flavor symmetries (or, in the supersymmetric theories, .Lorentz and R-symmetries); see Fig. 10. Thus

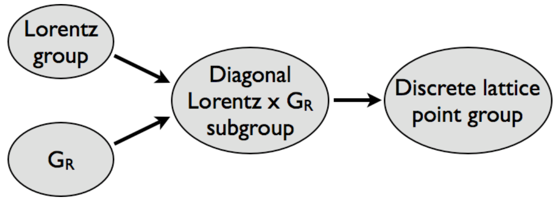

Fig. 10. The lattice point group in supersymmetric lattices cannot be considered to be a subgroup of just the Lorentz group, but rather of the product of Lorentz and R-symmetry group G_R.

a nontrivial representation of the lattice point group will become a nontrivial representation of the product of the Lorentz group and the R-symmetry group; but which representation is determined by the lattice action. Thus bosons which belong to nontrivial lattice representations can become in the continuum Lorentz scalars with nontrivial R-symmetry transformations; or Lorentz vectors which are R-symmetry singlets. The fermions transform in the continuum nontrivially under both Lorentz and R-symmetries.

This Dirac-Kähler analysis can be carried further, and applied to the supercharges themselves: If we rename $\Psi \to Q$ in eq. (37) and consider Q to be a matrix of supercharges, we see that supercharges themselves can be assigned a geometric meaning, and that the ones being preserved in the orbifold projection are the 0-rank tensors, namely those which are mapped to sites. This classification of supercharges as tensors goes under the name "twisted supersymmetry", and has been extensively discussed in the string literature. Twisted supersymmetry has also been a starting point for related formulations of lattice supersymmetry. It is no accident that the orbifold approach and the twisted supersymmetry approach are really the same: recall that to orbifold we had to select a Z_N^d subgroup which resided in the R-symmetry group of the mother theory, which in turn was related to the product of the Lorentz and R-symmetry groups of the target theory. The embedding of the Z_N^d in this product is what allowed us to assign integer $\mathbf{r}$ charges to all of the lattice variables (characteristic of tensors) as opposed to half-integer charges (associated with spinors).

A drawback of using standard staggered fermions for lattice QCD in $d = 4$ is that one is stuck with a minimum of four flavors (unless you are willing to consider a nonlocal lattice theory). We saw the same phenomenon in a different language, when I remarked that the supersymmetric lattices one can construct are constrained by the requirement that in higher dimensions one is forced to consider theories with more supercharges. This is the constraint we discussed earlier, that the orbifold method requires a large R-symmetry (and hence many supercharges) to create a lattice with many dimensions.

9 Other supersymmetric lattices

So what are the supersymmetric lattices we have constructed to date? SYM theories exist in $d \geq 2$ with $\mathcal{Q} = 2, 4, 8, 16$. Since each dimension requires projecting out a Z_N factor, and each projection costs one half of the remaining supersymmetries of the mother theory, and we want at least one unbroken supercharge on the lattice, we can only consider SYM theories with $\mathcal{Q} \geq 2^{d+1}$. That constrains us to

$$\begin{aligned} \mathcal{Q} = 4: &\quad d = 2 \\ \mathcal{Q} = 8: &\quad d = 2, 3 \\ \mathcal{Q} = 16: &\quad d = 2, 3, 4 \ , \end{aligned} \tag{40}$$

and all of these lattice have been constructed. The $\mathcal{Q} = 16$ theories are especially interesting and have especially symmetric lattices, shown in Fig. 11.

The $d = 1$ lattices for $\mathcal{Q} = 16$ SYM give a path integral approach for simulating $\mathcal{Q} = 16$ quantum mechanics. It might be interesting to pursue investigate this theory, since in the limit of large gauge group, such a theory is expected to be equivalent to M-theory and to describe quantum gravity.

In addition to pure SYM theories, a lattice for $(2,2)$ SYM has also been constructed with certain classes of matter fields [9] matter fields.

10 Lattice Supergravity?

We have seen that supersymmetric lattices are possible to construct, and that they have a lot of interesting mathematical structure. For example, the series of well prescribed mathematical steps described in §6 could have been used to discover staggered fermions (if the methods hadn't come along 30 years too late!). One might wonder though whether the power of the analytical approach here could be harnessed to create a lattice for *local* supersymmetry, known as supergravity. It would be pretty nifty if we could construct a lattice theory for quantum gravity without having to hurt our heads on the meaning of geometry and spacetime!

Consider $(2,2)$ supergravity in $d = 2$ dimensions. It's action is derived from $CN = 1$ supergravity in $d = 4$ dimensions by erasing two spacetime dimensions. consists of a graviton, a spin $\frac{3}{2}$ gravitino, the Hilbert action and the Rarita Schwinger action for the kinetic term of the gravitino. It also has lots of auxiliary fields required to make the theory manifestly supersymmetric. The idea we will follow will be to invent "staggered" gravitinos on the lattice. We will then introduce staggered vierbeins, and try to realize one exact supercharge on the lattice, and then hope that the action has enough Lorentz symmetry and supersymmetry to have the desired continuum limit[5].

Does this approach work? No, apparently not! But I think it is interesting, and I also thought that lectures at a school shouldn't only present finished work but should also expose students to the messiness of research in progress.

10.1 Staggered gravitinos

Consider spin 3/2 Majorana fermions in four dimensions. These are self-conjugate Dirac spinors ψ_m where m is a 4-vector index. The Rarita-Schwinger action is given by

$$\epsilon_{mnpq}\psi_m^T C\gamma_n\gamma_5\partial_p\psi_q \ . \tag{41}$$

This possesses a gauge symmetry $\psi_m \to \psi_m + \partial_m\chi$, where χ is an arbitrary Dirac spinor. Following the derivation for staggered fermions, we construct a naive latticization of this action:

$$\frac{1}{2a}\epsilon_{mnpq}\psi_m^T(\mathbf{n})C\gamma_n\gamma_5\left[\psi_q(\mathbf{n}+\hat{p}) - \psi_q(\mathbf{n}-\hat{p})\right] \ . \tag{42}$$

[5] The material in this section is unpublished work done with Michael Endres.

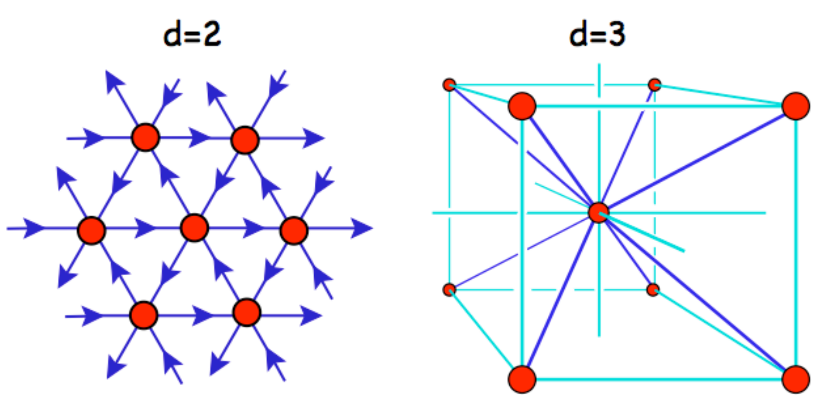

Fig. 11. *The lattices for $\mathcal{Q} = 16$ supersymmetry in $d = 2$ and $d = 3$ dimensions [8].In $d = 4$ the lattice for $\mathcal{N} = 4$ SYM has a $^*A_4$ lattice structure.*

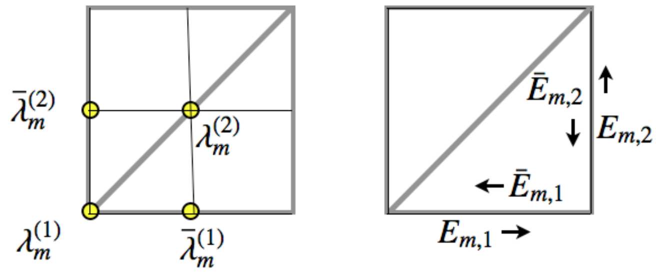

Fig. 12. *Lattice assignments for the gravitino and vierbein for* $(2,2)$ *supergravity*

This lattice action also possesses a gauge symmetry, $\psi_m(\mathbf{n}) \to \psi_m(\mathbf{n}) + (\chi(\mathbf{n}+\hat{m}) - \chi(bfn - \hat{m}))/(2a)$. We now substitute

$$\psi_m(\mathbf{n}) = \gamma_m\,(\gamma_1^{n_1}\cdots\gamma_4^{n_4})\,\lambda_(\mathbf{n}) \tag{43}$$

which is easily shown to eliminate the Dirac structure in the action, leaving us with four identical copies of the action for each spinor component of λ_m. We can therefore choose λ_m to be a one-component fermion (with a four-vector index). The lattice then has one of these four-vector fermions at each site and a simple action involving lattice derivatives with signs that encode the spin 3/2 structure.

In General Relativity the vector index on the gravitino lives in curved spacetime, while the spinor index lives in the tangent space; the way the two talk to each other is through the vierbein e_m^a, where m is a curved space index and a is a tangent space index; the vierbein is related to the metric by $e_{am}e^{an} = g_{mn}$ and to Lorentz symmetry by $e_{am}e_b^m = \eta_{ab}$, where η is the usual flat (Minkowski) space metric. The ease with which one can construct staggered spin 3/2 fermions is encouraging, but the fact that the curved space index does not play any structural role on the lattice is disturbing, even though the action couples the curved space index to the index of lattice derivatives operators.

Ignoring gathering confusion, Endres and I tried to construct a lattice theory for $(2,2)$ supergravity in $d=2$. The gravitino is readily latticized following the staggering procedure, and the lattice assignments are shown in Fig. 12. Pushing on, we latticized the gravitino's supersymmetric partner, the vierbein. Using the structure of our $(2,2)$ lattice construction with matter fields [9] as a guide, as well as the supersymmetry transformations between vierbeins and gravitinos in $(2,2)$ supergravity, we defined

$$e_m^a\sigma_{a\alpha\dot\beta} \equiv \begin{pmatrix} E_{m,1} & E_{m,2} \\ -\bar{E}_{m,2} & \bar{E}_{m,1}\end{pmatrix} \tag{44}$$

and assigned the E fields lattice positions shown in Fig. fig:vierbein. A heartening result is that various objects needed in the supergravity action, such as $e = \det e_m^a$ and $(e_m^a)^{-1}$ are easily constructed as local lattice operators. For example, the determinant e is represented as a "staple" as shown in Fig. 13.

Nevertheless, we hit a brick wall in trying to understand how to formulate the lattice covariant derivative in this theory. After a number attempts to make sense of it, we decided to let the theory hibernate for a while. So I think there remains an open, compelling question here: does lattice supersymmetry give us new insights into lattice supergravity, and therefore about quantum gravity in general?

11 Conclusions

The lattices described in these lectures represent only a small fraction of the continuum supersymmetric theories one would like to study, and it would be interesting to see if somehow the techniques could be extended to include, for example, supersymmetric QCD in four dimensions. Since numbers of quark flavors other than four cannot be represented by staggered fermions, it would be interesting to see if on could somehow implement domain wall fermions in lattice supersymmetry and escape the flavor tyranny of staggered/Dirac-Kähler fermions.

Lattice supersymmetry — and more generally, accidental supersymmetry — has been an obsession of mine for many years (ref. [2] was the first paper I wrote as a graduate student!) After years in the desert, it is delightful to contemplate the intricate structure of the supersymmetric lattices described here and how they evade all the apparently insurmountable obstacles outlined in §4. I still have some hope that these lattices will not only eventually be useful for numerical studies of extended supersymmetric Yang-Mills theories, but also that their reach might be extended and that they might shed light on how to construct at least some restricted class of lattice supergravity theories.

References

1. J.W. Elliott, G.D. Moore, PoS **LAT2005**, 245 (2006), hep-lat/0509032
2. D.B. Kaplan, Phys. Lett. **B136**, 162 (1984)
3. D.B. Kaplan, M. Schmaltz, Chin. J. Phys. **38**, 543 (2000), [http://arXiv.org/abs]hep-lat/0002030
4. N. Arkani-Hamed, A.G. Cohen, D.B. Kaplan, A. Karch, L. Motl (2001), [http://arXiv.org/abs]hep-th/0110146
5. D.B. Kaplan, E. Katz, M. Unsal, JHEP **05**, 037 (2003), hep-lat/0206019
6. A.G. Cohen, D.B. Kaplan, E. Katz, M. Unsal (2003), [http://arXiv.org/abs]hep-lat/0302017
7. A.G. Cohen, D.B. Kaplan, E. Katz, M. Unsal, JHEP **12**, 031 (2003), hep-lat/0307012
8. D.B. Kaplan, M. Unsal, JHEP **09**, 042 (2005), hep-lat/0503039
9. M.G. Endres, D.B. Kaplan, JHEP **10**, 076 (2006), hep-lat/0604012
10. J. Giedt, PoS **LAT2006**, 008 (2006), hep-lat/0701006
11. S. Catterall (2005), hep-lat/0510054
12. S. Catterall, Nucl. Phys. Proc. Suppl. **140**, 751 (2005), hep-lat/0409015
13. D.B. Kaplan (2005), nucl-th/0510023
14. N. Arkani-Hamed et al., JHEP **08**, 021 (2002), hep-ph/0206020
15. M. Schmaltz (????), prepared for Theoretical Advance Study Institute in Elementary Particle Physics (TASI 2004): Physics in D ¿= 4, Boulder, Colorado, 6 Jun - 2 Jul 2004
16. J. Wess, J. Bagger, Princeton, USA: Univ. Pr. (1992) 259 p (????)
17. S.R. Coleman, Subnucl. Ser. **15**, 805 (1979)
18. G. Curci, G. Veneziano, Nucl. Phys. **B292**, 555 (1987)
19. A. Feo (2002), hep-lat/0210015
20. D.B. Kaplan, Phys. Lett. **B288**, 342 (1992), [http://arXiv.org/abs]hep-lat/9206013
21. R. Narayanan, H. Neuberger, Nucl. Phys. **B443**, 305 (1995), [http://arXiv.org/abs]hep-th/9411108
22. H. Neuberger, Phys. Lett. **B417**, 141 (1998), [http://arXiv.org/abs]hep-lat/9707022
23. G.T. Fleming, J.B. Kogut, P.M. Vranas, Phys. Rev. **D64**, 034510 (2001), [http://arXiv.org/abs]hep-lat/0008009

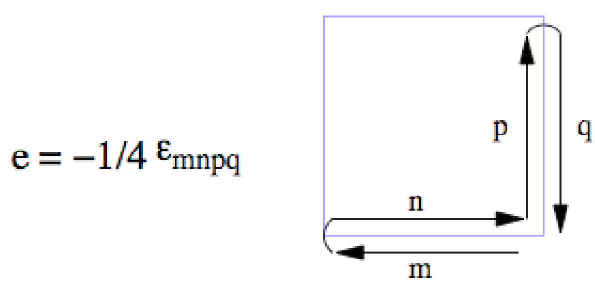

Fig. 13. *A picture of the lattice operator equal to $e = \det e^a_m$ in the continuum: A directed product of the E fields defined of Fig. 12, where the letters represent the curved space indices of the E variable, which are contracted by the ϵ tensor.*

24. N. Arkani-Hamed, A.G. Cohen, H. Georgi, Phys. Rev. Lett. **86**, 4757 (2001), [http://arXiv.org/abs]hep-th/0104005
25. P.H. Damgaard, S. Matsuura (0600), arXiv:0706.3007 [hep-lat]
26. J. Smit, Cambridge Lect. Notes Phys. **15**, 1 (2002)
27. J.M. Rabin, Nucl. Phys. **B201**, 315 (1982)
28. P. Becher, H. Joos, Zeit. Phys. **C15**, 343 (1982)
29. T. Eguchi, P.B. Gilkey, A.J. Hanson, Phys. Rept. **66**, 213 (1980)