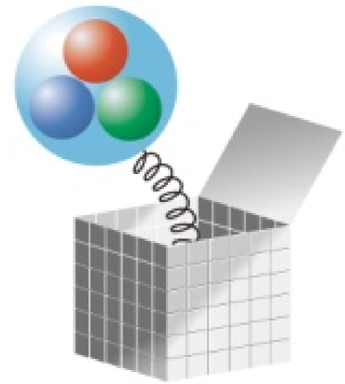


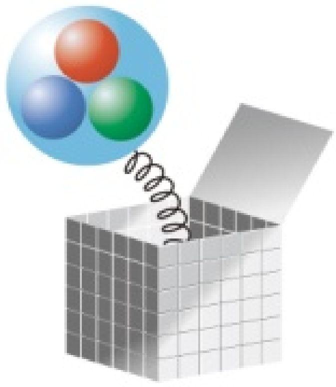
Full QCD with chirally invariant quarks

Shoji Hashimoto (KEK)

@ Block course on “Lattice Simulation of Quantum Fields,” Mar 28, 2008.

Plan

1. Exact chiral symmetry
   - Why necessary/interesting?
   - Properties of the Neuberger operator
   - Implementation
2. Topology issues
   - Index theorem
   - QCD at fixed topological charge
   - Topological susceptibility
3. Early physics results with dynamical overlap fermions
   - Simulation techniques
   - Pion and kaon properties, and more applications



1. Exact chiral symmetry

Why necessary/interesting

Exact chiral symmetry

… is costly, especially for the sea sector.

- Why necessary/interesting?

Theoretically Clean!

Theoretically clean

- Operator mixing
  - Thanks to the continuum-like Ward-Takahashi identities, no unwanted operator mixing appears (see Del Debbio's lecture).
  - To subtract the unnecessary contribution, sometimes one must deal with power divergences, e.g.

$$(\bar{\psi}\psi)^{cont} = Z_S(\bar{\psi}\psi)^{lat} + Z_{mix}\frac{1}{a^3}(1)^{lat}$$

  - Mixing with operators of wrong chirality is also a severe problem, as they may be relatively enhanced.

$$\langle \bar{K} | O_{LL} | K \rangle \propto B_K f_K^2 m_K^2; \quad \langle \bar{K} | O_{PP} | K \rangle \propto \frac{\langle \bar{\psi}\psi \rangle^2}{f_K^2}$$

Theoretically clean

- Operator mixing: more complicated example, K→ππ.
  - Even with exact chiral symmetry, there exists a mixing, which must be subtracted non-perturbatively.

$$Q_6 = \left(\bar{s}_\alpha d_\beta\right)_{V-A} \sum_{q=u,d,s} \left(\bar{q}_\beta q_\alpha\right)_{V+A}$$

$$\leftrightarrow \frac{1}{a^2}\left[(m_s + m_d)\bar{s}d - (m_s - m_d)\bar{s}\gamma_5 d\right]$$

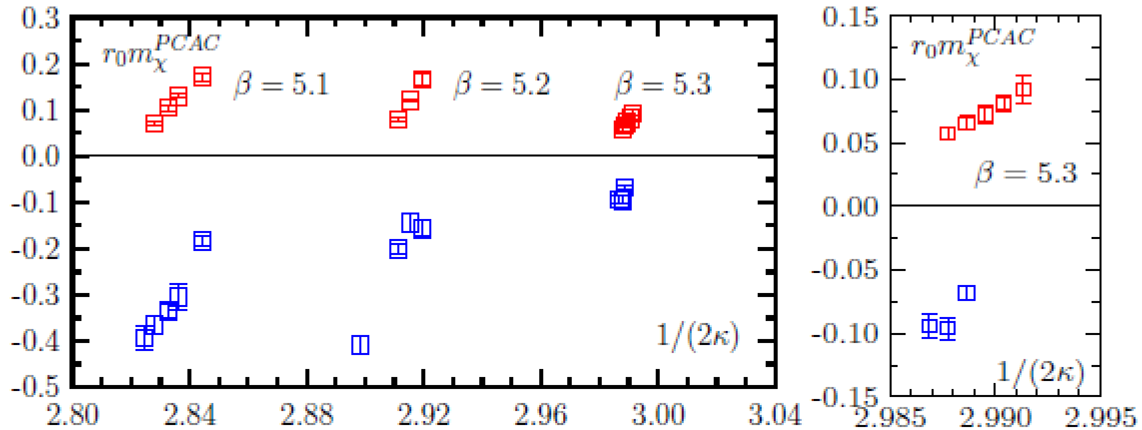
  - Without the chiral symmetry, the problem is harder, i.e. mixing with

$$\leftrightarrow \frac{1}{a^3}\bar{s}d$$

Theoretically clean

- Existence of the chiral limit
  - Without the exact chiral symmetry, the uniqueness or even the existence of the "chiral limit" may be lost.
    - What is *the chiral extrapolation*? $m_q \to 0$, $m_\pi^2 \to 0$, …
    - There may be a wall before reaching there (1st order phase transition). Situation is better for improved gauge actions; may still show up near the chiral limit.

Farchioni et al., PLB624, 324 (2005).



Theoretically clean

- Chiral effective theory
  - Constructed assuming the spontaneously broken chiral symmetry. Is it valid when chiral symmetry is explicitly violated?
  - Continuum extrapolation before chiral extrapolation.
  - Or, Combined continuum/chiral extrapolation
    - Under some assumptions introduce additional terms (e.g. proportional to a^2) to ChPT. More parameters to be determined.
    - Wilson ChPT, staggered ChPT, twisted mass ChPT, …

Neuberger operator

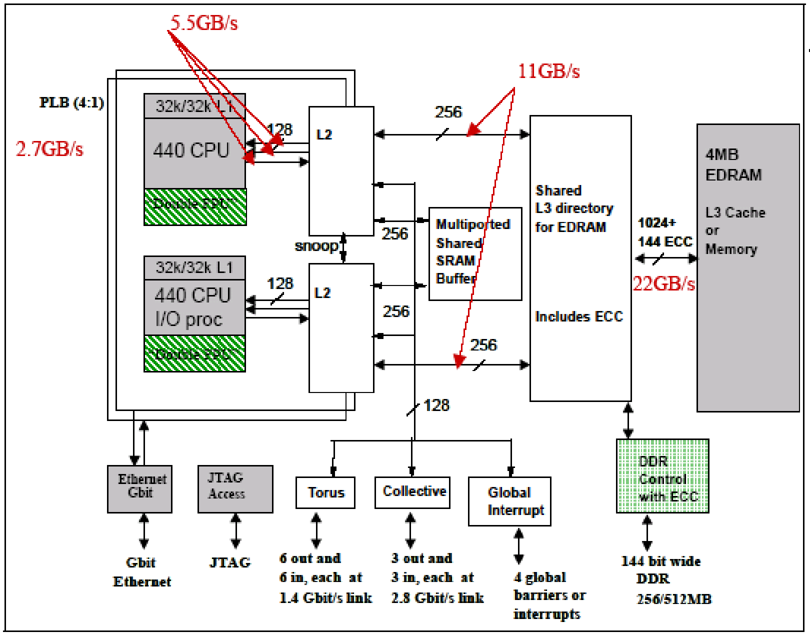
- Fortunately, the lattice fermion with exact chiral symmetry is known = overlap fermion.

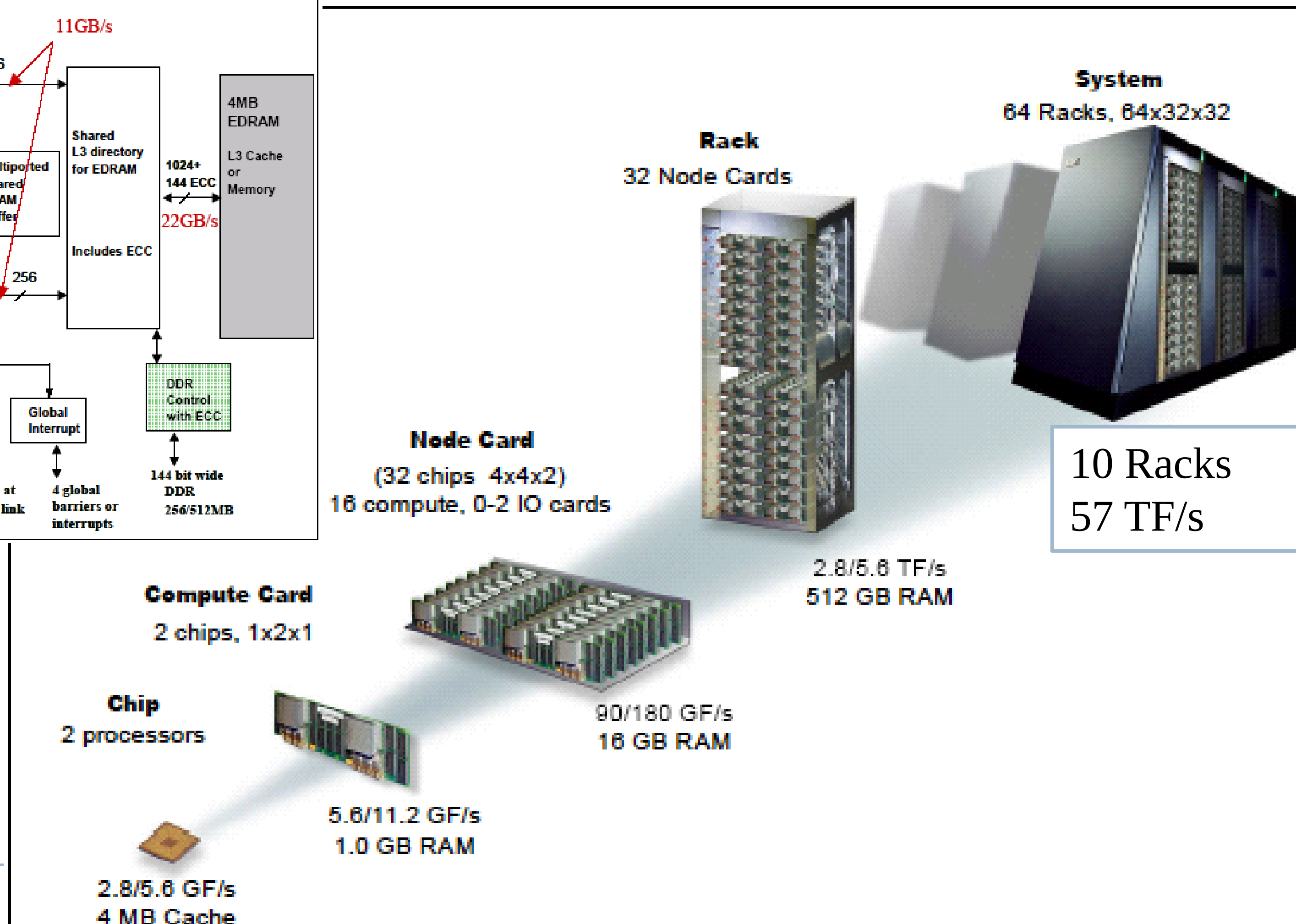
- Why don't you use it?
  - Computational cost (~ x100 more)
  - Yes. But, history proved the Moore's law = exponential growth of computer power (x10 per every 5 years).
  - Wait for 10 years, or start now!

JLQCD collaboration

- Members:
  - KEK: S. Hashimoto, H. Ikeda, T. Kaneko, H. Matsufuru, J. Noaki, E. Shintani, N. Yamada
  - Kyoto: T. Onogi, H. Ohki
  - Tsukuba: S. Aoki, N. Ishizuka, K. Kanaya, Y. Kuramashi, Y. Taniguchi, A. Ukawa, T. Yoshie
  - Hiroshima: K. Ishikawa, M. Okawa
  - NBI: H. Fukaya
- Also, with TWQCD (T.W. Chiu, T.H. Hsieh, K. Ogawa)
- Machines at KEK (since 2006)
  - SR11000 (2.15 Tflops)
  - BlueGene/L (57.3 Tflops)

IBM BlueGene/L





10 Racks
57 TF/s

Properties and implementation of the overlap fermion

Overlap fermion

- Neuberger's overlap fermion (1998)

$$D = \frac{1}{a}\left[1 + \frac{X}{\sqrt{X^\dagger X}}\right],\ X = aD_W - 1$$
$$= \frac{1}{a}\left[1 + \gamma_5\,\mathrm{sgn}(aH_W)\right],\ aH_W = \gamma_5(aD_W - 1)$$

  - Exact chiral symmetry through the Ginsparg-Wilson relation.

$$\delta\psi = \gamma_5\left(1 - \frac{1}{2}aD\right)\psi,\ \delta\overline{\psi} = \overline{\psi}\left(1 - \frac{1}{2}aD\right)\gamma_5$$

$$D\gamma_5 + \gamma_5 D = aD\gamma_5 D$$

  - Continuum-like Ward-Takahashi identities
  - Index theorem (relation to topology) is satisfied
- Numerical cost depends on
  - Spectrum of H_W
  - Approximation of sgn(H_W)

(Simple) exercises

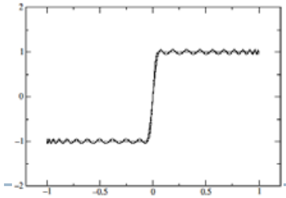
If you have not done it before,

- Show that

1. The lattice action is invariant under the modified chiral transformation when the Ginsparg-Wilson relation is satisfied.
2. The overlap-Dirac operator D satisfies the Ginsparg-Wilson relation.
3. The overlap operator approaches the continuum Dirac operator as $a \to 0$. Do the all calculation keeping relative order a terms. You may use a relation

$$D_W = \not{D}_c - \frac{1}{2} a D_c^2 + O(a^2)$$

Sign function



- Approximation of the sign function is needed. May use tanh, for instance, but not very precise.
- Instead, consider an approx for a given interval

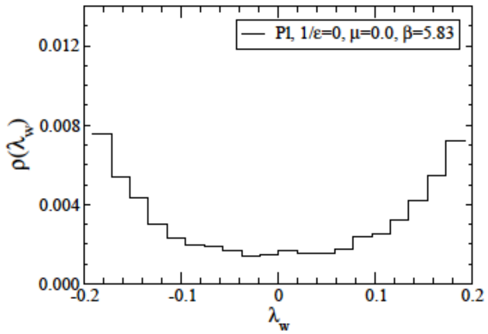
$$|\lambda_W| \in [\lambda_{W,th}, \lambda_{W,\max}]$$

  - Chebyshev: minmax polynomial approximation
  - Zolotarev: minmax rational approximation

$$\varepsilon[x] = x\left(p_0 + \sum_{l=1}^{N_{pole}} \frac{p_l}{x^2 + q_l} \right)$$

- Problem arises if there are near-zero modes.
  - In fact, their density is non-zero at any finite β (Edwards, Heller, Narayanan, 1998)

$$\rho(\lambda_W) = \frac{1}{V}\left\langle \sum_i \delta(\lambda_i - \lambda_W) \right\rangle$$



  - Comes from dislocations (local lumps of gauge field)
  - Need to subtract before approximate

Things to do

- For a given gauge configuration…
  1. Calculate the near-zero modes of H_W, i.e. their eigenvalues and eigenvectors using Lanczos algorithm for instance.

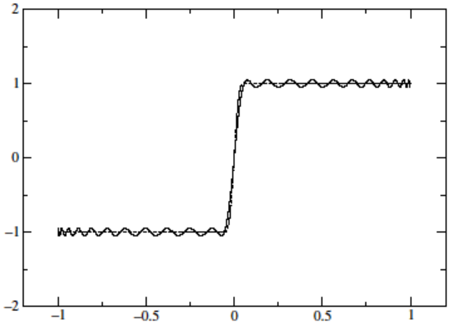
$$H_W = \sum_{\lambda_W \le \lambda_{W,th}} \lambda_W v_{\lambda_W} \otimes v_{\lambda_W}^\dagger + (1-P)H_W; \quad P = \sum_{\lambda_W \le \lambda_{W,th}} v_{\lambda_W} \otimes v_{\lambda_W}^\dagger$$

  2. Sign function is trivial for the near-zero modes; the rest must be approximated.

$$\mathrm{sgn}(H_W) = \sum_{\lambda_W \le \lambda_{W,th}} \mathrm{sgn}(\lambda_W) v_{\lambda_W} \otimes v_{\lambda_W}^\dagger + \varepsilon\left[(1-P)H_W\right]$$

  3. Use a rational approximation (Zolotarev)

$$\varepsilon[x] = x\left(p_0 + \sum_{l=1}^{N_{pole}} \frac{p_l}{x^2 + q_l} \right)$$



Multi-shift solver

- Rational approximation involves N_{pole} inversions.

$$\varepsilon[H_W] = x\left(p_0 + \sum_{l=1}^{N_{pole}} \frac{p_l}{H_W^2 + q_l} \right)$$

  - Can be calculated at the same time using the multi-shift CG solver.
    - They share the Krylov subspace. Therefore, the generation of the orthogonal vectors in the Krylov subspace must be done only once; their coefficients must be calculated for each inversion.

  - Faster than the (Chebyshev) polynomial approximation to achieve a given accuracy: $10^{-(7-8)}$ in our case.
    - Accuracy improves exponentially for larger N_{pole}; increase of the numerical cost is mild.

Near-zero mode suppression

- Near-zero modes are unphysical (associated with a local lump = lattice artifact)
- Make sense to design a lattice action to suppress them
  - Fat links are effective to suppress the dislocations, but not completely.
  - Introduce unphysical (heavy negative mass) Wilson fermions (Vranas, JLQCD, 2006)

$$\det H_W(-m_0)^2$$

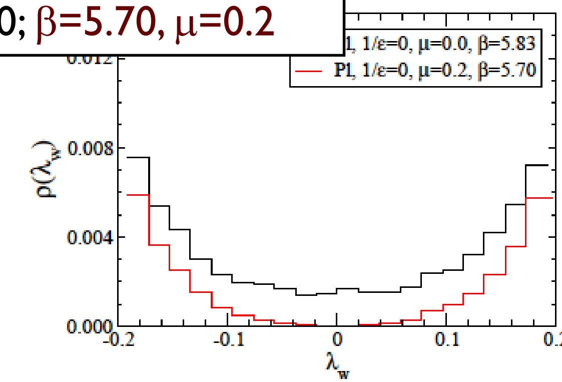
m_0 must be chosen the same as that in the overlap kernel. Must be in between 0 and 2.

- We chose a form

$$\det\left[\frac{H_W(-m_0)^2}{H_W(-m_0)^2+\mu^2}\right]$$

to minimize the effect in the UV region.

Plaquette gauge,
β=5.83, μ=0; β=5.70, μ=0.2



Completely wash-out the near-zero modes. Overlap is much faster.

Locality

- Lattice Dirac operator must be local in order that a local theory is obtained in the continuum limit.
  - Locality is not obvious for the overlap operator due to 1/√.

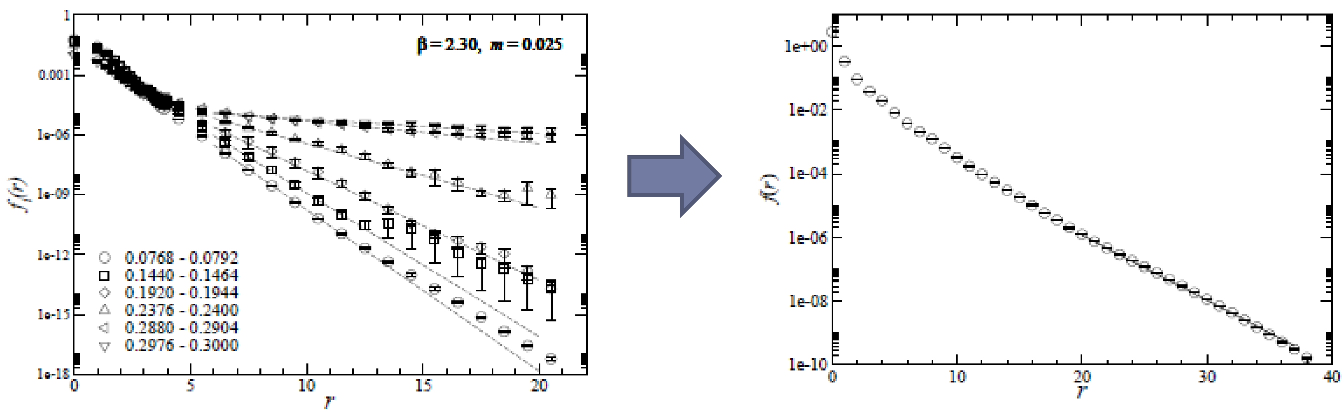
$$D = \frac{1}{a}\left[1 + \frac{X}{\sqrt{X^\dagger X}}\right],\ X = aD_W - 1$$

  - Locality in the sense that |D|<exp(-μx), with μ a number of order 1/*a*, may be satisfied.
  - "Proof" is known for smooth enough gauge canfigurations (Hernandez, Jansen, Luscher (1999)).
  - No mathematical proof in more realistic situations where there is non-zero density of the near-zero modes.

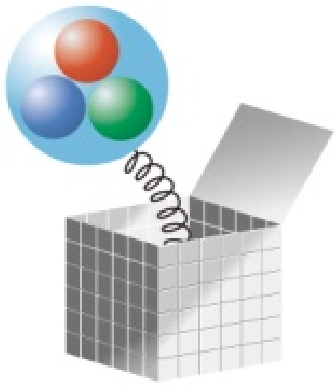
    ⇒ Okay if near-zero modes are always localized.

Locality

- Maybe analyzed by looking at individual eigenmodes of H_W.
  - Near-zero modes are more localized. Higher modes are extended. There is a critical value above which the modes are extended = "mobility edge" (Golterman, Shamir (2003)).



  - An important lessen: do not use the overlap fermion in the Aoki phase (where the near-zero modes are extended).



2. Topology issues

U(1) anomaly

- Fermion measure is not invariant under the axial U(1) rotation.

$$\prod_n d\psi_n d\bar{\psi}_n \to J^{-2}\prod_n d\psi_n d\bar{\psi}_n; \ln J = i\alpha\int d^4x \sum_k \bar{u}_k(x)\gamma_5\left(1-\frac{a}{2}D\right)u_k(x)$$

- Eigenvalues of the Dirac operator
  - Make a pair with their complex conjugate, except for the zero modes.

$$Du_k(x) = \lambda_k u_k(x); D\gamma_5 u_k(x) = \lambda_k^* \gamma_5 u_k(x)$$

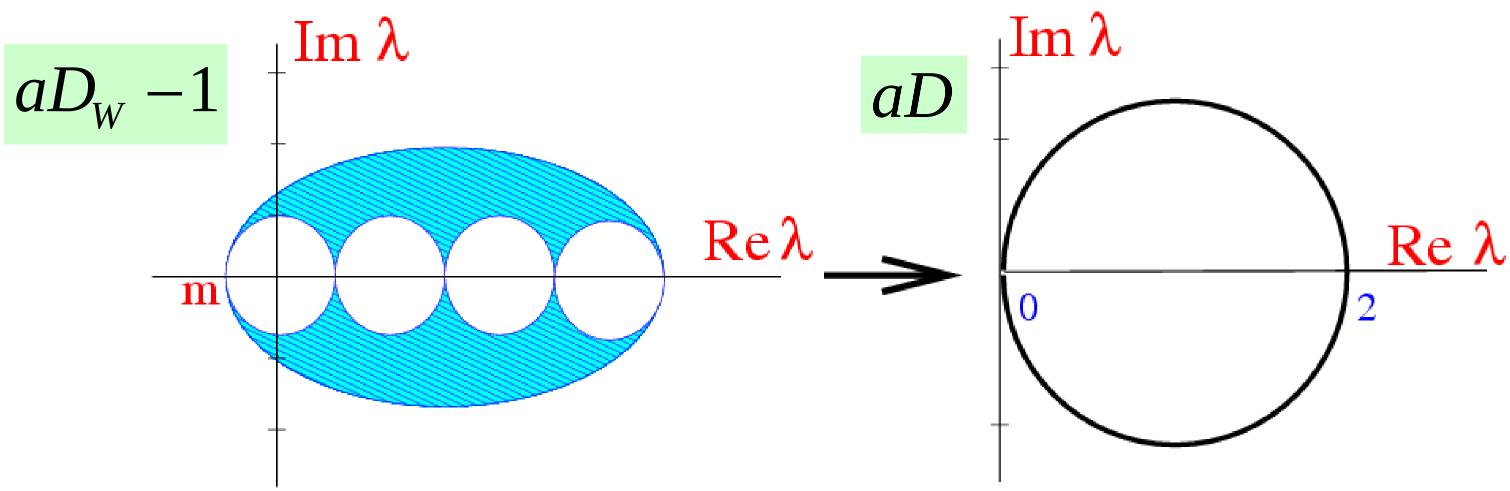
- Lattice version of the index theorem (Atiyah-Singer)

$$\int d^4x \sum_k \bar{u}_k(x)\gamma_5\left(1-\frac{a}{2}D\right)u_k(x) = n_R - n_L = q \quad \left(= \frac{1}{16\pi^2}\int d^4x \,\mathrm{Tr}\, F_{\mu\nu}\tilde{F}_{\mu\nu}\right)$$

Correctly represents the chiral zero-modes

Overlap = projection

$$D = \frac{1}{a}\left[1 + \frac{X}{\sqrt{X^\dagger X}}\right], \; X = aD_W - 1$$



$$(aD - 1)(aD^\dagger - 1) = 1$$

- Eigenvalues are paired: $\lambda \leftrightarrow \lambda^*$
- Real eigenvalues are exceptions.

Topological charge

- Usually defined using some cooling techniques; topological charge is an integer, only approximately.
- The overlap-Dirac operator provides another way of defining topological charge on the lattice = counting the number of zero-modes.
  - Unambiguous definition.
  - Topology change occurs when an eigenvalue of H_W crosses zero.
  - Near-zero modes of H_W indicate dislocation, for which the topology cannot be defined.

Frozen topology

- Suppress the dislocations, e.g. by adding extra Wilson fermions

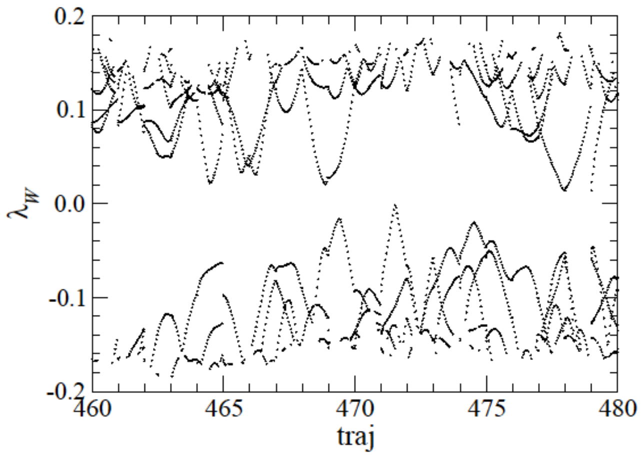
$$\det H_W(-m_0)^2$$

- Zero probability to have an exact zero-mode

↕

- No chance to tunnel between different topological sectors.

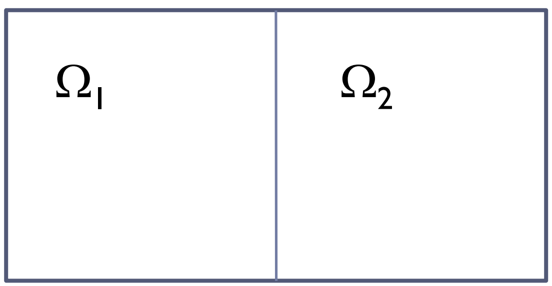
- If the MD-type algorithm is used, the global topology never changes.
  - Provided that the step size is small enough.



Property of the continuum QCD: common for all lattice formulations as the continuum limit is approached.

Cluster decomposition

- In QCD, the real vacuum has a certain distribution of the topological charge = the θ vacuum.
  - Required to satisfy the cluster decomposition property: topology distribution must satisfy



$$f(Q_1 + Q_2) = f(Q_1)\, f(Q_2)$$

$$\Rightarrow \quad f(Q) = e^{i\theta Q}$$

- Can one reproduce the physics of the θ vacuum from the fixed topology simulations?
  - Sum-up the topology! Or, not?

Sum-up the topology!

- Partition function of the vacuum $Z(\theta) = \exp\left[-VE(\theta)\right]$
  - Vacuum energy density E(θ) $E(\theta) = \frac{\chi_t}{2}\theta^2 + \frac{c_4}{12}\theta^4 + \ldots$
  - Partition function for a fixed Q

$$Z_Q = \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\theta Z(\theta) e^{i\theta Q} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\theta \exp\left[-VF(\theta)\right];\ F(\theta) = E(\theta) - i\theta Q/V$$

  - Using a saddle point expansion around (θ_c=iQ/V), one can evaluate the θ integral to obtain

$$Z_Q = \frac{1}{\sqrt{2\pi\chi_t V}}\exp\left[-\frac{Q^2}{2\chi_t V}\right]\left[1 - \frac{c_4}{8V\chi_t} + O\left(\frac{1}{(\chi_t V)^2}, \frac{Q^2}{(\chi_t V)^2}\right)\right]$$

  - Then, the original partition function can be recovered, if one knows χ_t, c_4, etc., as $Z(\theta) = \sum_Q Z_Q e^{-i\theta Q}$

Or, not?

- Fixing topology = Finite volume effect

Aoki, Fukaya, SH, Onogi, PRD76, 054508 (2007)

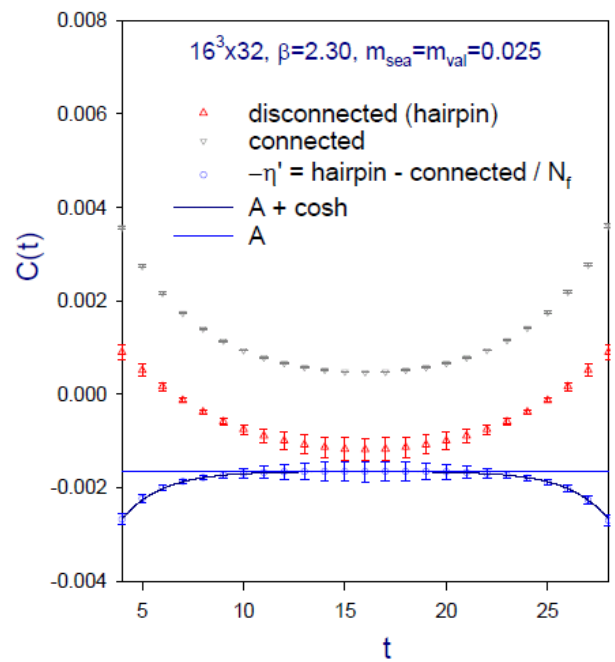
  - When the volume is large enough, the global topology is irrelevant.
  - Topological charge fluctuate locally, according to χ_t, topological susceptibility.
  - Physics of the θ-vacuum can be recovered by a similar saddle-point analysis, e.g. Some Green's function:

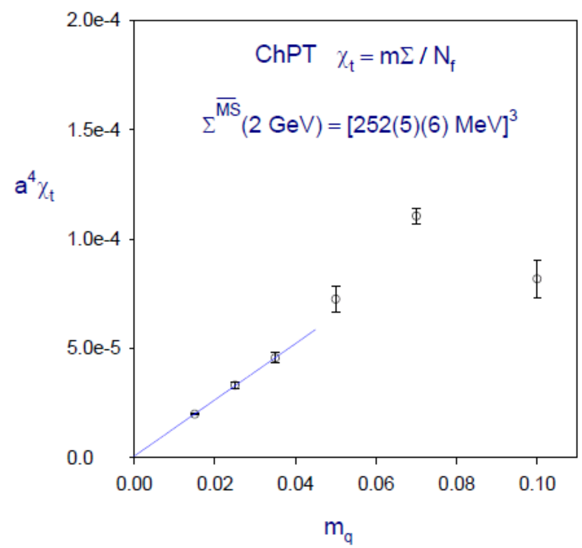
$$G_Q^{\text{even}} = G(0) + G^{(2)}(0)\frac{1}{2\chi_t V}\left[1 - \frac{Q^2}{\chi_t V} - \frac{c_4}{2\chi_t^2 V}\right] + G^{(4)}(0)\frac{1}{8\chi_t^2 V^2} + O(V^{-3}),$$

Topological susceptibility $\chi_t = \frac{\langle Q^2 \rangle}{V}$

- Applying the same formula for the flavor-singlet PS density, χ_t can be extracted.

$$\lim_{x\to\infty} \langle mP(x)mP(0)\rangle_Q = -\frac{1}{V}\left(\chi_t - \frac{Q^2}{V} + O(1/V)\right) + O(e^{-m_{\eta'}x})$$

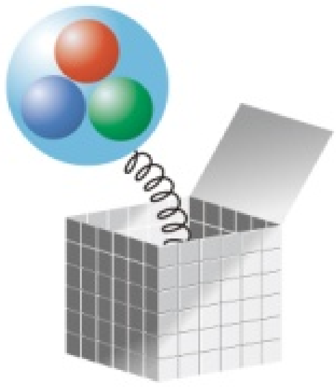




JLQCD and TWQCD (2007)

$$\chi_t = \frac{m\Sigma}{N_f}$$

Local topological fluctuation is indeed active as expected.



3. Early physics results

Simulation techniques; then m_π and f_π

Dynamical overlap

Recent attempts:

- Fodor-Katz-Szabo (2003)
  - Reflection/refraction trick
- Cundy et al. (2004)
  - Many algorithmic improvements
- DeGrand-Schaefer (2005)
  - Fat-link
  - Some physics results

Our work:

Aoki et al., arXiv:0803.3197 [hep-lat]

- Fixed topology: no reflection/refraction required.
- Large scale simulation with $L \approx 2$ fm, $m_q \sim m_s/6$.
- Mass preconditioning (Hasenbusch) + multi-time step

Broad physics program:

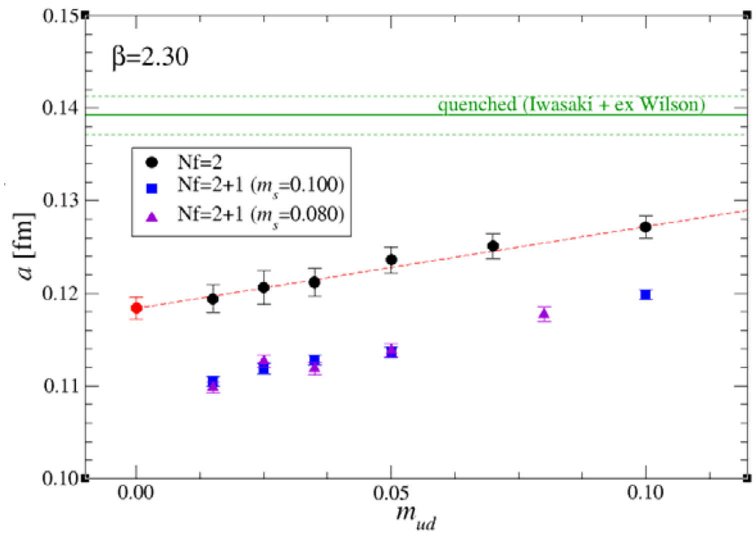
- Pion/kaon physics
- ε-regime

Parameters

Nf=2 runs (finished)

many physics analysis on-going

- β=2.30 (Iwasaki), a=0.12 fm, 16^3x32
- 6 sea quark masses m_q = 0.015 … 0.100, covering $m_s/6 \sim m_s$
- 10,000 HMC traj.
  - ~4,000 with 4D solver
  - ~6,000 with 5D solver
- Q=0 sector only, except Q=–2, –4 runs at m_q=0.050



- β=2.30 (Iwasaki), a=0.11 fm, 16^3x48
- 5 ud quark masses, covering $m_s/6 \sim m_s$
- x 2 s quark masses
- 2,500 HMC traj.
  - Using 5D solver
- Q=0 sector only

Measurement techniques

Measurements at every 20 traj ⇒

500 conf / m_{sea}

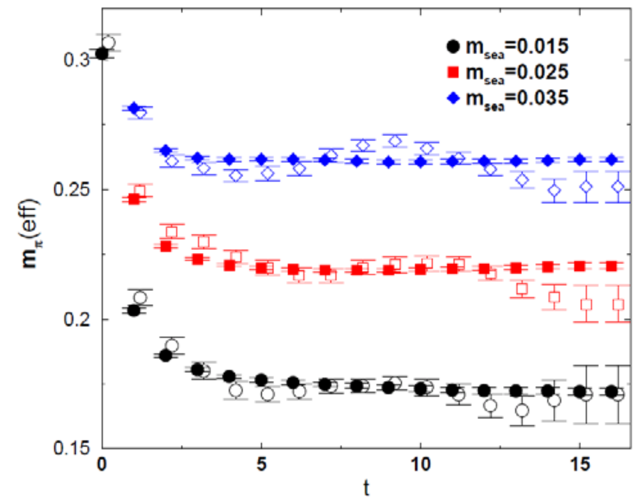
- Improved measurements
  - 50 pairs of low modes calculated and stored.
  - Used for low mode preconditioning (deflation)

    ⇒ (multi-mass) solver is then x8 faster
  - Low mode averaging

    (and all-to-all)

$$D_m^{-1}(x,y) = \sum_{k=1}^{N} \frac{u_k(x)u_k^\dagger(y)}{\lambda_k + m} + D_m^{(h)-1}(x,y)$$

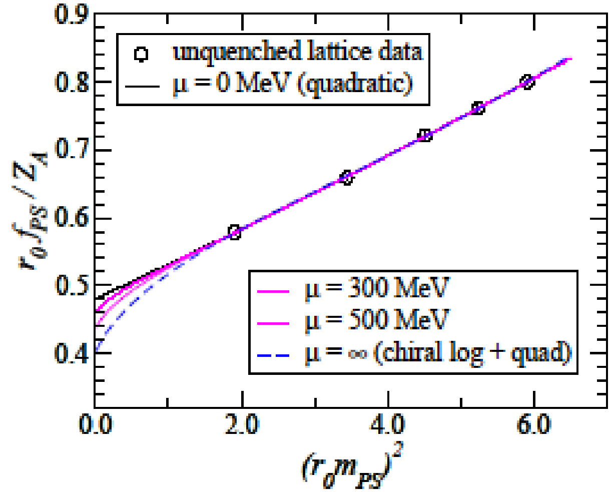
$$C(x,y) = C^{ll}(x,y) + C^{hh}(x,y) + C^{hl}(x,y) + C^{lh}(x,y)$$



Test of ChPT

- Need a critical test of ChPT, before using it in the analysis of other quentities.
- With exact chiral symmetry, we don't have to worry about the explicit breaking terms.
  - Not ambiguous even at finite lattice spacing.

JLQCD (2002)
With clover



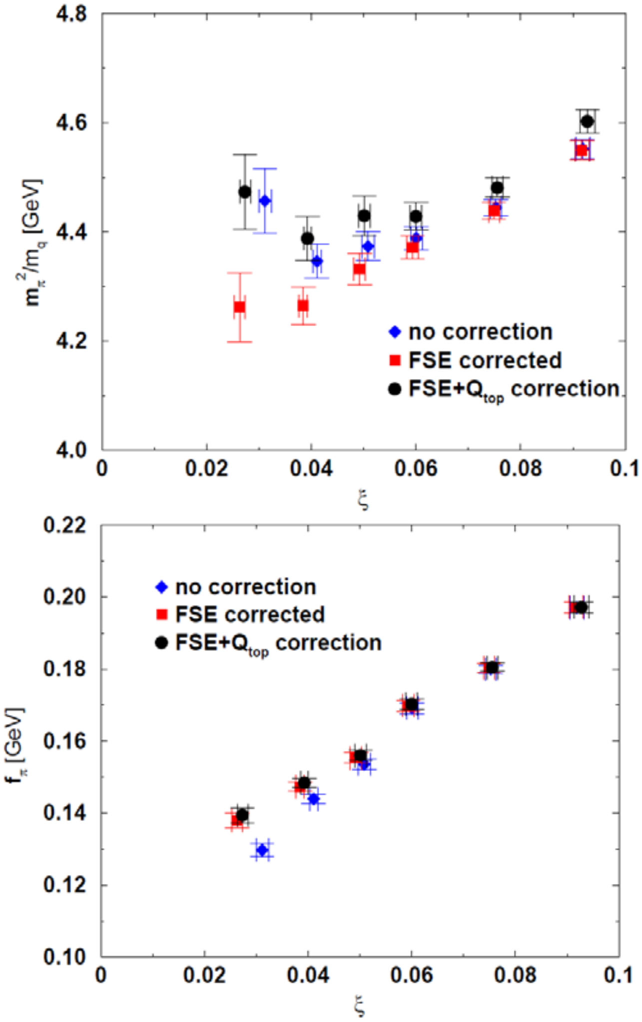
- Use m_π and f_π:
  - Simplest quantities, numerically easy to calculate to a good precision.
  - Other quantities will follow.

$$\frac{f_\pi}{f} = 1 - \frac{N_f}{2}\frac{m_\pi^2}{(4\pi f)^2}\ln\frac{m_\pi^2}{\mu^2 + m_\pi^2} + \cdots$$

Finite size effect

At a price of … finite volume lattice

- At L~1.9 fm (smallest $m_\pi L$~3), finite size effect is not negligible; corrected using
  - χPT at NNLO
    - Colangelo-Durr-Haefeli, NPB721, 136 (2005))
  - Fixed topology
    - Aoki et al., arXiv:0707.0396 with NLO χPT and measured χ_t



NNLO analysis

NNLO χPT predicts the mass dependence as

$$\frac{m_\pi^2}{m_q} = 2B_0\left[1+\xi \ln\xi+\frac{7}{2}(\xi \ln\xi)^2+\left(\frac{2L_4}{f}-\frac{4}{3}(\tilde{L}+16)\right)\xi^2 \ln\xi\right] + L_3(\xi-9\xi^2 \ln\xi)+K_1\xi^2$$

$$f_\pi = f\left[1-2\xi \ln\xi+5(\xi \ln\xi)^2-\frac{3}{2}\left(\tilde{L}+\frac{53}{2}\right)\xi^2 \ln\xi\right] + L_4(\xi-10\xi^2 \ln\xi)+K_2\xi^2$$

simultaneous fit

input: $\tilde{L} = 7\ln\left(\frac{\Lambda_1}{4\pi f}\right)^2+8\ln\left(\frac{\Lambda_2}{4\pi f}\right)^2$ from phenomenology

$$\xi = \left(\frac{m_\pi}{4\pi f_\pi}\right)^2$$

NNLO analysis

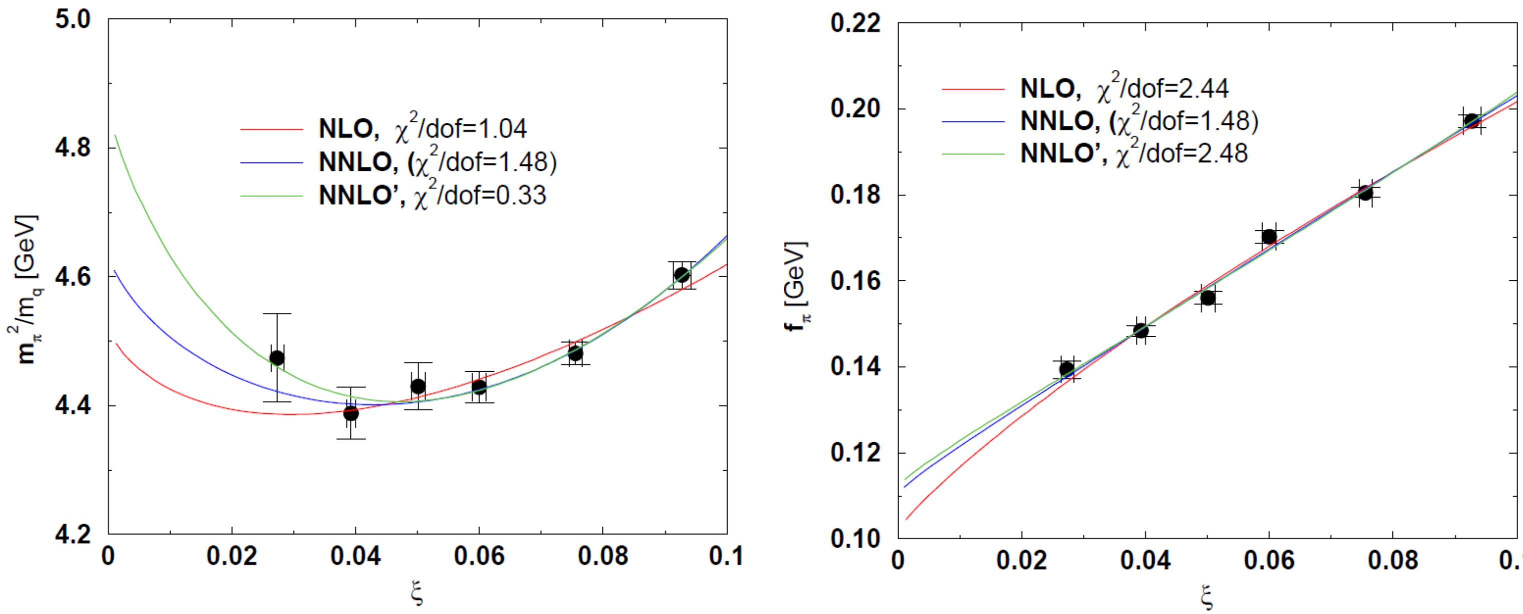
Also, NNLO'

$$\frac{m_\pi^2}{m_q} = 2B_0\left(1+\xi \ln\xi + \frac{7}{2}(\xi\ln\xi)^2\right) + L_3\xi + K_1'\xi^2$$

$$f_\pi = f\left(1-2\xi\ln\xi+5(\xi\ln\xi)^2\right)+L_4\xi+K_2'\xi^2$$

$$\xi = \left(\frac{m_\pi}{4\pi f_\pi}\right)^2$$

Noaki at Lattice 2007



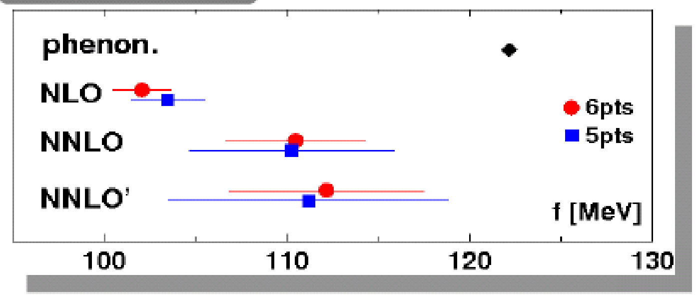
Data slightly favor NNLO; not clear from these plots alone.

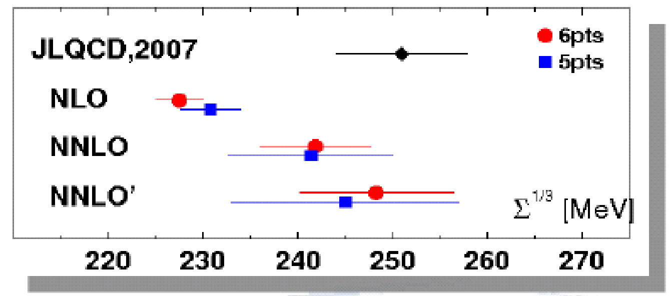
LECs

Noaki at Lattice 2007

$f = 122.2\,\mathrm{MeV}$ Gasser & Leutwyler,1984

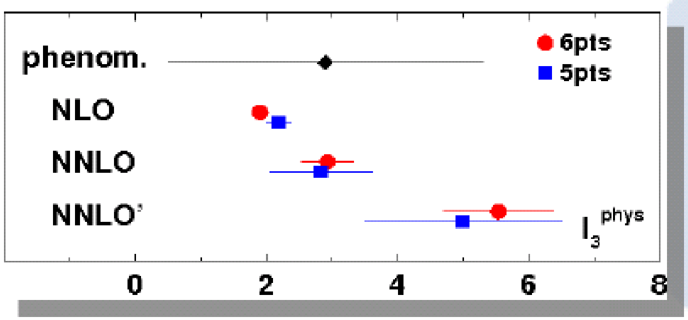
$\Sigma = B_0 \cdot f^2 / 2$

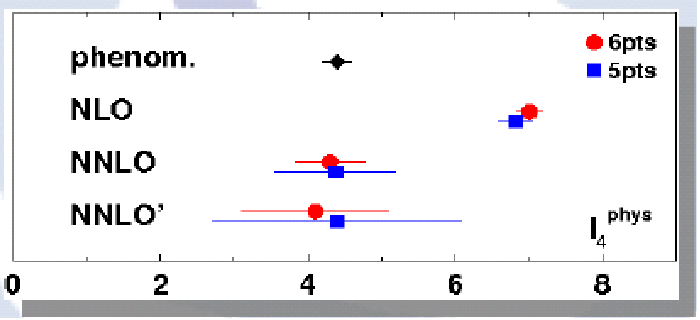




$$l_3^{\mathrm{phys}} = \ln\left(\frac{\Lambda_3}{m_\pi^{\mathrm{phys}}}\right)^2 = \ln\left(\frac{4\pi f}{m_\pi^{\mathrm{phys}}}\right)^2 - \frac{L_3}{2B_0}$$

$$l_4^{\mathrm{phys}} = \ln\left(\frac{\Lambda_4}{m_\pi^{\mathrm{phys}}}\right)^2 = \ln\left(\frac{4\pi f}{m_\pi^{\mathrm{phys}}}\right)^2 + \frac{L_4}{2f}$$

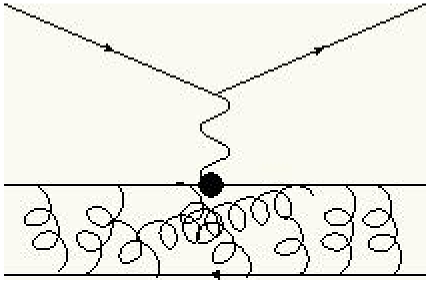




Inconsistency at NLO; okay after including NNLO.

Other physics measurements

Pion form factor



- The simplest form factor

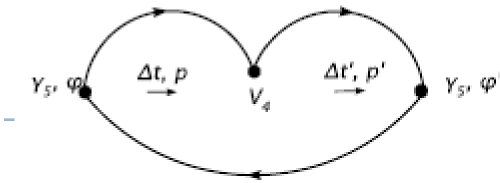
$$\langle \pi(p')|V_\mu|\pi(p)\rangle = i(p_\mu + p_\mu')F_V(q^2), \quad q_\mu \equiv p_\mu' - p_\mu$$

  - Momentum transfer q_μ by a virtual photon. Space-like ($q^2<0$) in the $\pi e \to \pi e$ process.
  - Vector form factor $F_V(q^2)$ normalized as $F_V(0)=1$, because of the vector current conservation.

$$F_V(q^2) = 1 + \frac{1}{6}\langle r^2\rangle_V^\pi q^2 + O(q^4),$$

  - Vector (or EM) charge radius $\langle r^2\rangle_V^\pi$ is defined through the slope at $q^2=0$.

All-to-all



To improve the signal

- Usually, the quark propagator is calculated with a fixed initial point (one-to-all)
- Average over initial point (or momentum config) will improve statistics; possible with all-to-all

$$D^{-1}(x,y) = \sum_{k=1}^{N_{ev}} \frac{1}{\lambda^{(k)}} u^{(k)}(x) u^{(k)\dagger}(y) + \sum_{d=1}^{N_d} \left[D_{high}^{-1} \eta^{(d)} \right](x) \eta^{(d)}(y)$$

Low mode contribution

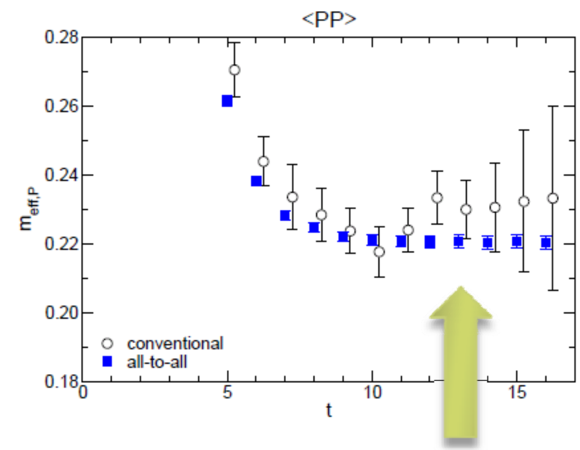
High mode propagation
From the random noice

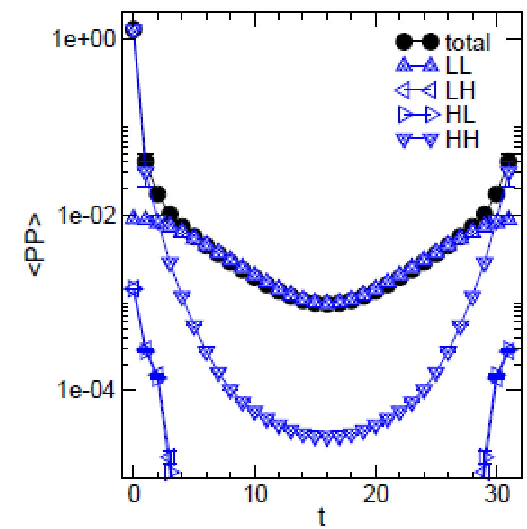
Random noise

An example: two-point func

Dramatic improvement of the signal, thanks to the averaging over source points

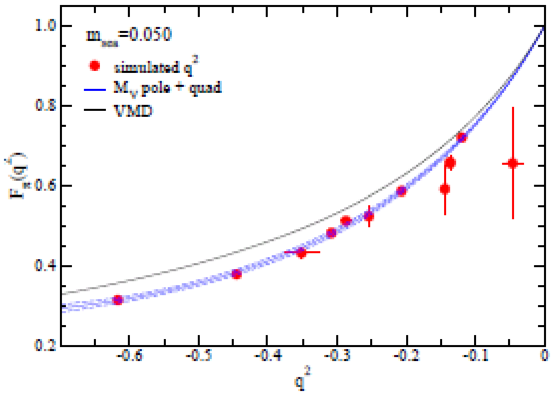
- Similar to the low mode averaging; but all-to-all can be used for any n-point func.
- PP correlator is dominated by the low-modes



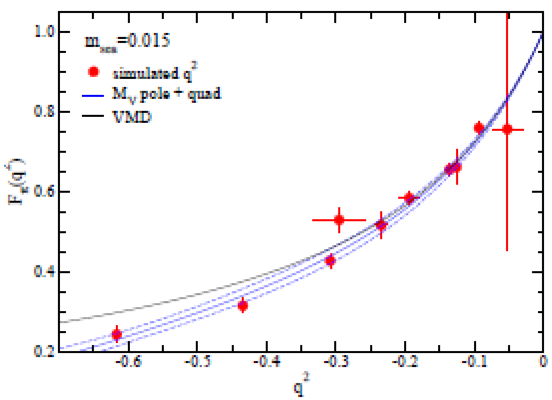


Form factor results

$m_q \sim m_s/2$



$m_q \sim m_s/6$



- All-to-all ⇒ many momentum combinations
  - (1,0,0) → (0,1,0), etc. in units of $2\pi/L$.

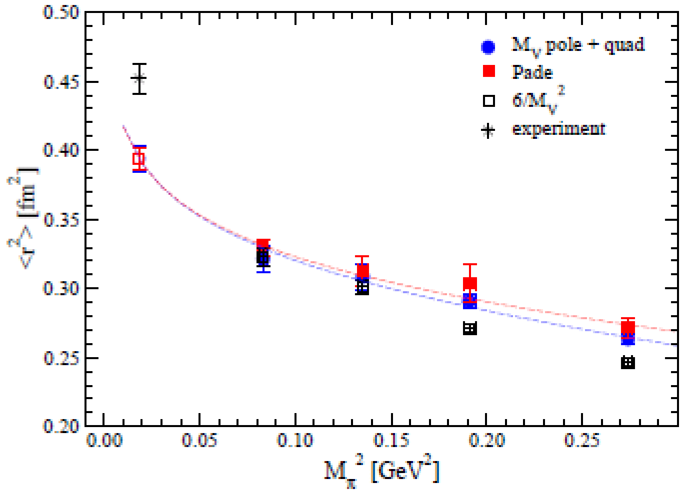
- q^2 dependence well approximated by a vector meson pole + corrections

$$F_\pi(q^2) = \frac{1}{1 - q^2/m_V^2} + c_1 q^2 + \ldots$$

with m_V obtained at the same quark mass.

Chiral extrapolation

- Lattice data
  - Mass dependence very similar to VMD, but the difference is visible.
  - χlog may become significant beyond the region of lattice data.



$$\langle r^2 \rangle_V^\pi = 0.388(9)(12)\ \mathrm{fm}^2$$

$$\langle r^2 \rangle_V^\pi = -\frac{1}{(4\pi f_\pi)^2}\left[\ln\frac{m_\pi^2}{\mu^2} + 12(4\pi)^2 L_9 + O(m_\pi^2)\right]$$

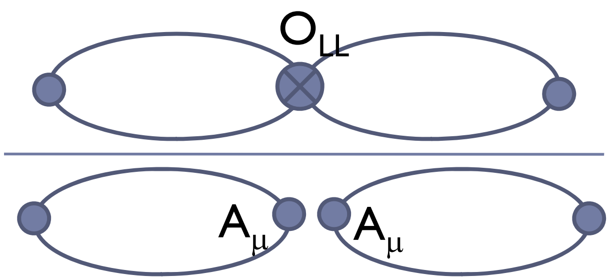
Lower than the exp number, even after the chiral enhancement.

B_K

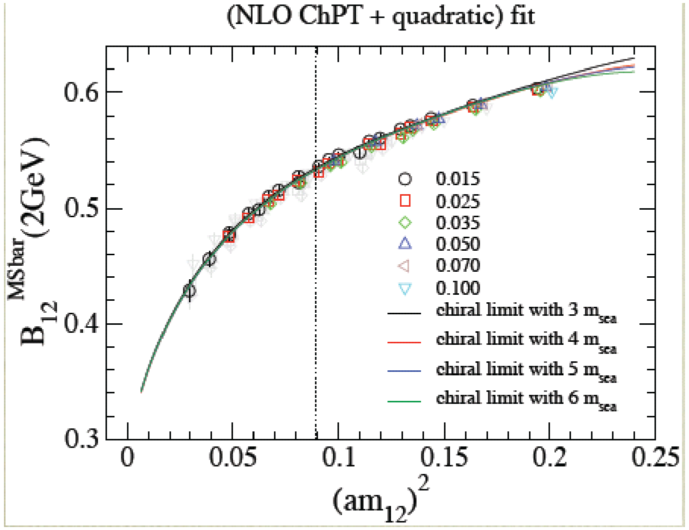
First (unquenched) lattice calculation with exact chiral symmetry:

JLQCD collab, arXiv:0801.4186 [hep-lat].

$$\langle \bar{K}^0 | O_{LL}(\mu) | K^0 \rangle = \frac{8}{3} B_K(\mu) f_K^2 m_K^2$$



- No problem of operator mixing; otherwise, mixes with O_{LR}, for instance. Enhanced by its wrong chiral behavior.
- Another test of chiral log. Here the data follows the NLO ChPT.



$$B_P = B_P^\chi \left[1 - \frac{6m_P^2}{(4\pi f)^2} \ln \frac{m_P^2}{\mu^2} + b m_P^2 + O(m_P^4) \right]$$

Two-point functions

- A new application: two-point functions in the momentum space.

$$\langle J_\mu J_\nu \rangle = \left(g_{\mu\nu}q^2 - q_\mu q_\nu\right)\Pi_J^{(1)} - q_\mu q_\nu \Pi_J^{(0)}$$
$$= \int_0^\infty \frac{ds}{s-q^2+i\varepsilon}\left[\left(g_{\mu\nu}s^2 - s_\mu s_\nu\right)\mathrm{Im}\,\Pi_J^{(1)} - s_\mu s_\nu\,\mathrm{Im}\,\Pi_J^{(0)}\right]$$

- Weinberg sum rules:

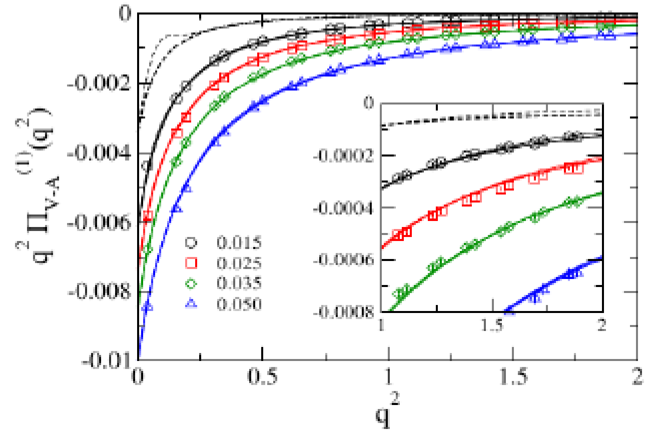
$$f_\pi^2 = -\lim_{Q^2\to 0} Q^2\left[\Pi_V^{(1+0)}(Q^2) - \Pi_A^{(1+0)}(Q^2)\right],$$
$$S = -\lim_{Q^2\to 0}\frac{\partial}{\partial Q^2} Q^2\left[\Pi_V^{(1+0)}(Q^2) - \Pi_A^{(1+0)}(Q^2)\right]$$

Chiral symmetry is essential.

- Pion mass difference

Das et al. (1967)

$$\Delta m_\pi^2 = -\frac{3\alpha_{\mathrm{EM}}}{4\pi f_\pi^2} \times \int_0^\infty dQ^2\; Q^2\left[\Pi_V^{(1+0)}(Q^2) - \Pi_A^{(1+0)}(Q^2)\right]$$



ε-regime

Entering the ε-regime:

- Pion is nearly massless.
- Compton wavelength is longer than the lattice extent $m_\pi L \leq 1$
- Finite momentum mode is suppressed.

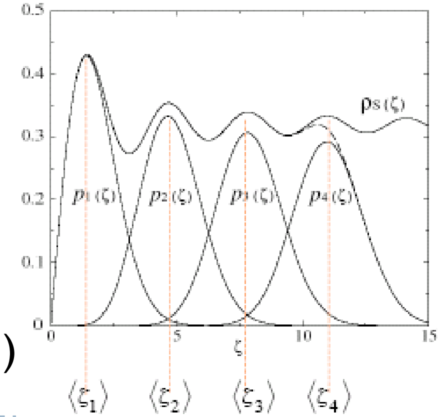
$$L_{\chi PT} = \Sigma \mathrm{Tr}\left[M(U+U^\dagger)\right]$$

- Dependence on topological charge
- Info on the Dirac operator eigenvalue spectrum through

$$Z_{QCD} = Z_{\chi PT}$$

(Leutwyler-Smilga, 1992)

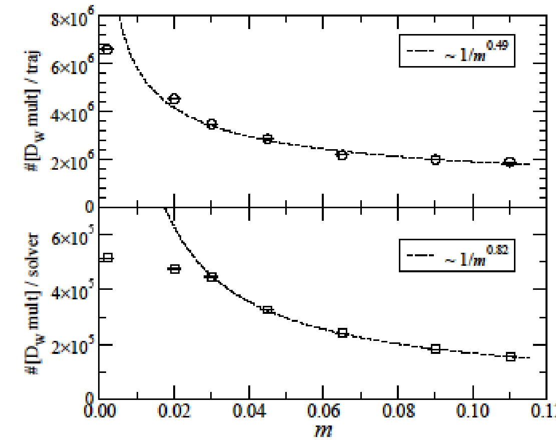
- More detailed analytical info through Chiral Random Matrix Theory (χRMT)

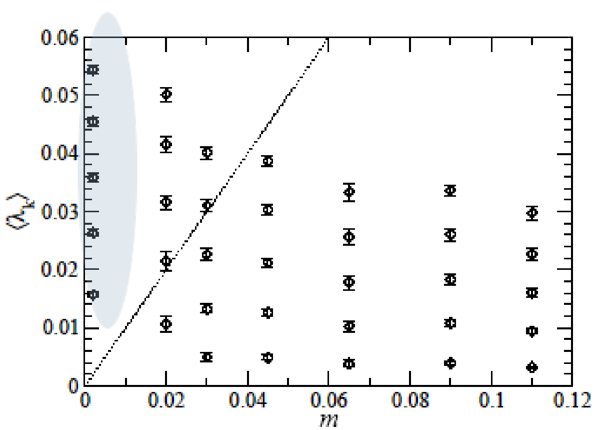


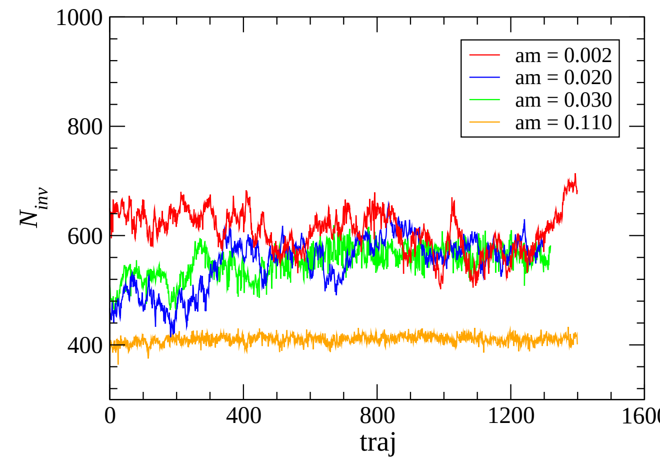
(Damgaard, Nishigaki, 2001)

Simulation in the ε-regime

- **Harder, but not prohibitive**
  - Cost grows rather mildly.
  - Condition number is governed by the first eigenvalue, not m
  - First eigenvalue is lifted by the fermion determinant $\prod_k \left(|\lambda_k|^2 + m^2\right)$
  - Auto-correlation is longer.





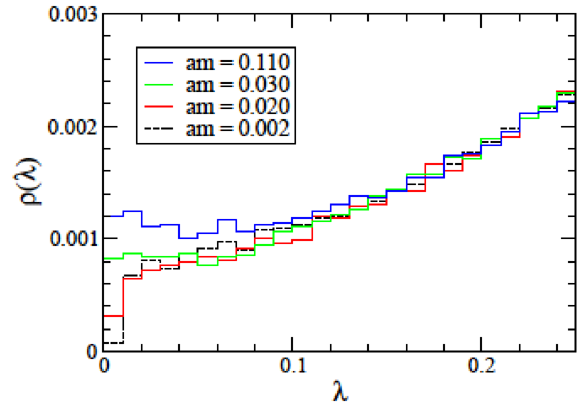


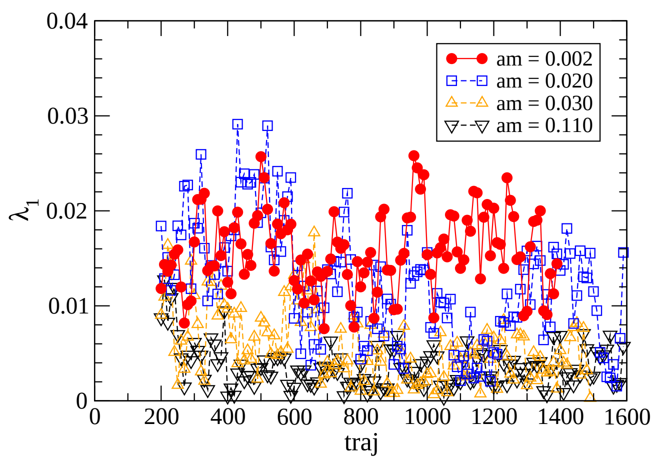
Eigenvalue spectrum

Simulation parameters:

- β=2.35, a = 0.11 fm, 16^3x32, 6 m_q + 1
- m_q = 3 MeV reached. $m_\pi L \approx 1$
- 4,600 HMC traj.
- 50 pairs of eigenvalues

Spectral densiry





s-Casher relation

$$-\langle \overline{\psi}\psi \rangle = \pi \rho(0)$$

hiral symmetry restored in the assless limit (fixed V)

Comparison with χRMT

Fukaya et al., PRL98, 172001 (2007)

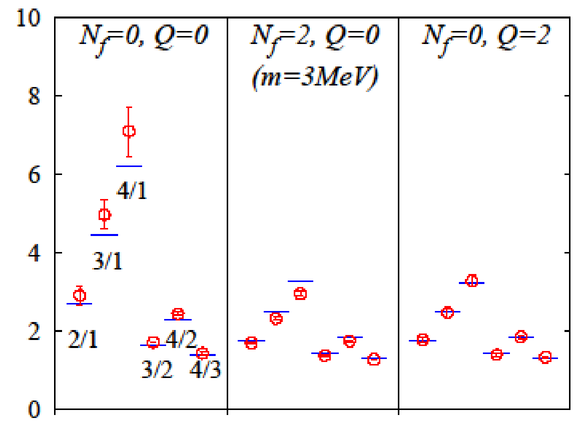
χRMT

- Equivalent to χPT at the LO in the ε-expansion.
- Predicts eigenvalues of D in unit of $\lambda\Sigma V$.
  - $\langle\lambda_1\rangle\Sigma V = 4.30$
  - $\langle\lambda_2\rangle\Sigma V = 7.62$
  - …

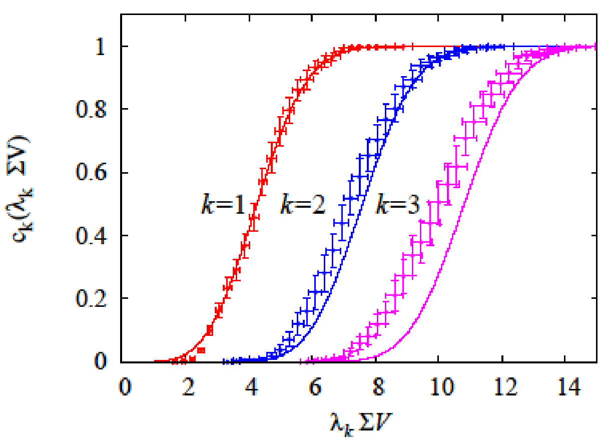
  for N_f=2, Q=0.
- Σ may be extracted from average eigenvalues.

$$\Sigma^{\overline{MS}}(2\,\mathrm{GeV}) = \left[251(7)(11)\ \mathrm{MeV}\right]^3$$

eigenvalue ratios



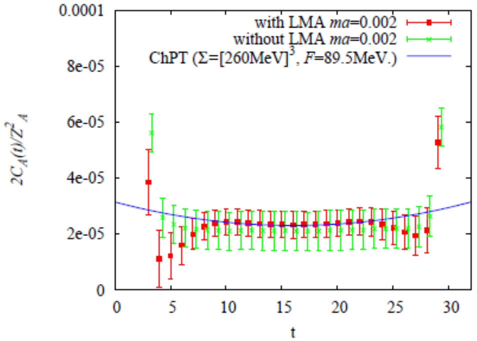
cumulative density



Other observables in the ε-regime

- Meson correlators (Damgaard et al., 2002)
  - in the ε-regime
  - can extract f_π
  - NLO calculation possible/available.

Fukaya et al., arXiv:0711.4965v3 [hep-lat]



$$F = 87.3(5.6)\ \mathrm{MeV}$$

$$\Sigma(2\ \mathrm{GeV}) = [239.8(4.0)\ \mathrm{MeV}]^3$$

- 3-pt functions (Hernandez-Laine, 2006)
  - Several B parameters
- Eigenvalue correlations (Damgaard et al., 2006)
  - with imaginary chemical potential
  - can extract f_π

Wide variety of applications, all without chiral extrapolations.

And, more to come

- More applications, for which the exact chiral symmetry is essential or at least useful. Partial list includes
  - Pion scalar form factor
  - Nucleon sigma term, strange quark content
  - Strong coupling constant, gluon condensate

Details will appear at Lattice 2008.

Summary

- Motivation for using chirally invariant fermions should be obvious.
  - Operator mixing, power divergence, chiral perturbation theory, topology, ε-regime, …

  Only question is its practical feasibility.
- Dynamical overlap fermion simulation is feasible with O(10 Tflops) machine.
  - In addition to several clever algorithms, fixing the topological charge is the key. Will become necessary for other fermion formulations too.
  - Many physics applications to emerge.