Introduction to staggered fermions

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Species doubling

In the continuum

$$S^{-1}(p) = i\not{p}$$

On the lattice (nearest neighbors)

$$S^{-1}(p) = \sum_{\mu} \frac{i}{a}\,\gamma_\mu \sin(ap_\mu)$$

For $a \to 0$, relativistic poles near

$$p = \pi_A \in \left\{(0,0,0,0), (\frac{\pi}{a},0,0,0), \ldots, (\frac{\pi}{a},\frac{\pi}{a},\frac{\pi}{a},\frac{\pi}{a})\right\}$$

$$\uparrow$$

$$\gamma_1 \to -\gamma_1$$

$\Rightarrow$ chirality flips: eight species have Q_A = +1, eight have Q_A = -1

(Karsten & Smit, Nielsen & Ninomiya)

- Reduce species doubling by starting with one-component lattice field:

$$\chi(x)\ , \qquad \overline{\chi}(x)$$

→ only symmetries are (euclidean) space-time symmetries

- Species doubling & hypercubic symmetry:
  16 poles in continuum = 4 (Dirac) × 4 (flavor) ?

- Lattice symmetries:

translations S_μ	→	continuum translations
hypercubic rotations $R_{\kappa\lambda}$	→	$SO(4)$ in continuum
??	→	$SU(4)$ flavor?

- Make "normal" translations follow from S_μ^2 , with

$$S_\mu : \quad \chi(x) \to \zeta_\mu(x)\chi(x+\mu)$$

  $\zeta_\mu(x)$ can only be a phase factor, with $\zeta_\mu(x)\ \zeta_\mu(x+\mu) = 1$

- Get (discrete subgroup of) $SU(4)$ from S_μ → try $S_\mu S_\nu = - S_\nu S_\mu$

→ $\zeta_\mu(x)\ \zeta_\nu(x+\mu) = -\ \zeta_\nu(x)\ \zeta_\mu(x+\nu)$

→ choose $\zeta_\mu(x) = (-1)^{x_{\mu+1}+\cdots+x_4} \equiv e^{i\pi_{\zeta_\mu} x}$ (essentially unique)

- irreps of translation group → momentum space:

$$\begin{aligned} \chi(p) &= \sum_x e^{-ipx}\chi(x) \rightarrow \sum_x e^{-ipx+i\pi_{\zeta_\mu} x}\chi(x+\mu) \\ &= e^{ip_\mu}\chi(p+\pi_{\zeta_\mu}) \end{aligned}$$

→ the 16 fields $\varphi_A(q) = \chi(q+\pi_A)$, $\pi_A \in \{(0,0,0,0),(\pi,0,0,0),\ldots\}$ with $-\pi/2 \leq q_\mu \leq \pi/2$ form a 16-dim representation of the group generated by S_μ, *i.e.*,

$$S_\mu : \qquad \phi_A(q) \rightarrow e^{iq_\mu}(\Xi_\mu)_{AB}\phi_B(q)$$

with $$\{\Xi_\mu, \Xi_\nu\} = 2\delta_{\mu\nu}$$

→ $\Xi_\mu = 1 \times \xi_\mu$ with ξ_μ a set 4×4 of Dirac matrices

$$\xi_A = ie^{-i\frac{\pi}{2}\xi_A} \in \mathrm{SU}(4)$$: "hypercubic" flavor transformations

$$\xi_A \in \{1, \xi_\mu, \frac{i}{2}\xi_{[\mu}\xi_{\nu]}, i\xi_\mu\xi_5, \xi_5\}$$

The ξ_μ generate a 32-element group $\Gamma_4 \subset SU(4)$
$SU(4)$ is the smallest continuous group containing Γ_4:

Γ_4 to enlarge to $SU(4)$ in the continuum limit just like hypercubic rotations enlarge to $SO(4)$

Now we need an action invariant under the lattice group:

$$S = \sum_{x\mu} \frac{1}{2} \overline{\chi}(x) \eta_\mu(x) [\chi(x+\mu) - \chi(x-\mu)] + m \sum_x \overline{\chi}(x) \chi(x)$$

Invariant under shifts S_μ:

$$\zeta_\nu(x) \eta_\mu(x) \zeta_\nu(x+\mu) = \eta_\mu(x+\nu)$$
$$\rightarrow \eta_\mu(x) = (-1)^{x_1 + \cdots + x_{\mu-1}}$$

Define $T_\mu : \quad \chi(x) \to \eta_\mu(x)\chi(x+\mu)$ (not a symmetry!)

in momentum space: $T_\mu : \quad \phi_A(q) \to e^{iq_\mu}(\Gamma_\mu)_{AB}\phi_B(q)$

then $T_\mu T_\nu = -T_\nu T_\mu \;, \quad \mu \neq \nu \quad \to \quad \{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu\nu}$

furthermore $T_\mu S_\nu = S_\nu T_\mu \quad \to \quad [\Gamma_\mu, \Xi_\nu] = 0$

$\to \quad \Gamma_\mu = \gamma_\mu \times 1$ with γ_μ a set 4 × 4 of Dirac matrices

- free action in momentum space:

$$S = \sum_\mu \int_q \frac{1}{2}\overline{\phi}(q)\Gamma_\mu \left(e^{iq_\mu} - e^{-iq_\mu}\right)\phi(q) + m\int_q \overline{\phi}(q)\phi(q)$$

(note: free action invariant under full *SU*(4)!)

- Couple to gauge fields: give χ, $\overline{\chi}$ color index

$$S = \sum_{x\mu} \frac{1}{2}\eta_\mu(x)\overline{\chi}(x)\left[U_\mu(x)\chi(x+\mu) - U_\mu^\dagger(x-\mu)\chi(x-\mu)\right] + m\sum_x \overline{\chi}(x)\chi(x)$$

has full space-time lattice symmetry group, incl. rotations, reflections

Connection with naïve fermions

$$S_{naive} = \sum_{x\mu} \frac{1}{2}\overline{\psi}(x)\gamma_\mu(\psi(x+\mu) - \psi(x-\mu))$$

Define $\psi(x) = \gamma_1^{x_1} \dots \gamma_4^{x_4}\chi(x)$ then

$$S_{naive} = \sum_{x\mu} \frac{1}{2}\overline{\chi}(x)_\alpha \eta_\mu(x)(\chi(x+\mu)_\alpha - \chi(x-\mu)_\alpha)$$

→ drop Dirac index on χ !

ψ transforms in reducible representation of lattice symmetry group:

1 naïve fermion = 4 staggered fermions

Axial symmetry: invariance of S for $m = 0$

$\psi(x) \rightarrow \gamma_5\, \psi(x)$ then $\chi(x) \rightarrow \gamma_5\, \varepsilon(x)\, \chi(x)$

Indeed, $\chi(x) \rightarrow e^{i\alpha\varepsilon(x)}\chi(x)$, $\bar{\chi}(x) \rightarrow \bar{\chi}(x)\, e^{i\alpha\varepsilon(x)}$

is $U(1)_\varepsilon$ axial symmetry of S for $m = 0$ (Kawamoto-Smit)

What is this in the continuum limit?

Note: $\eta_\mu(x)\zeta_\mu(x) = (-1)^{x_\mu} \quad \rightarrow \quad \epsilon(x) = \prod_\mu \eta_\mu(x)\zeta_\mu(x)$

hence $\prod_\mu T_\mu^{-1} S_\mu \chi(x) = \epsilon(x)\chi(x)$

→ in momentum space $\prod_\mu \Gamma_\mu \Xi_\mu = \Gamma_5 \Xi_5$: non-singlet axial symm.!

→ one exact Goldstone boson for $m = 0$, interpolating field $\varepsilon(x)\bar{\chi}(x)\chi(x)$

What about other Goldstone bosons?
Are they automatically massless in the continuum limit,
or is fine tuning à la Wilson required?

→ mass renormalization -- additive or multiplicative? (MG & Smit)

Note that $U(1)_\varepsilon$ does not exclude

$$\sum_\mu \frac{1}{2} m_\mu \bar{\chi}(x)\eta_\mu(x)[U_\mu(x)\chi(x+\mu) + U_\mu^\dagger(x-\mu)\chi(x-\mu)] \rightarrow \sum_\mu m_\mu \bar{\psi}\gamma_\mu\psi$$

Need all lattice symmetries to exclude fine tuning:

$$M = \sum_{A,B} m_{AB} \Gamma_A \Xi_B \ , \qquad A, B = 1, \ldots, 16$$

Rotations: $R_{\kappa\lambda}: \quad \phi_A(q) \to \left(e^{\frac{1}{2}\pi(\frac{1}{2}\Gamma_\kappa \Gamma_\lambda)} e^{\frac{1}{2}\pi(\frac{1}{2}\Xi_\kappa \Xi_\lambda)} \right)_{AB} \phi_B(R^{-1}q)$

(rotates $\bar{\chi}(x)\eta_\mu(x)\chi(x+\mu)$, $\bar{\chi}(x)\zeta_\mu(x)\chi(x+\mu)$ as vectors; "twisted" $SO(4)$)

$$\to \qquad (R_{\kappa\lambda})^2 = \Gamma_\kappa \Gamma_\lambda \Xi_\kappa \Xi_\lambda$$

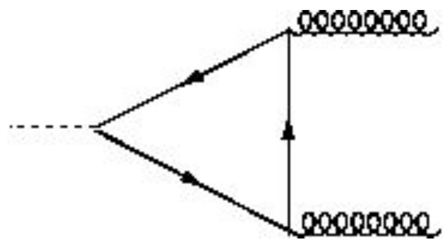
Together with shift symmetry this excludes all Γ_A except 1 and Γ_5 and all Ξ_B except 1.

Then $U(1)_\varepsilon$ excludes 1 and Γ_5 mass terms.

Singlet γ_5 , *i.e.*, $\psi \rightarrow \gamma_5 \psi$? on lattice:

$$\chi(x) \quad \rightarrow \quad T_1 T_2 T_3 T_4 \chi(x)$$
$$= \eta_1(x)\eta_2(x+1)\eta_3(x+1+2)\eta_4(x+1+2+3)\chi(x+1+2+3+4)$$

meson operator: $\bar{\chi}(x) T_1 T_2 T_3 T_4 \chi(x)$ + h.c.

has  $\neq$ 0 in the continuum limit

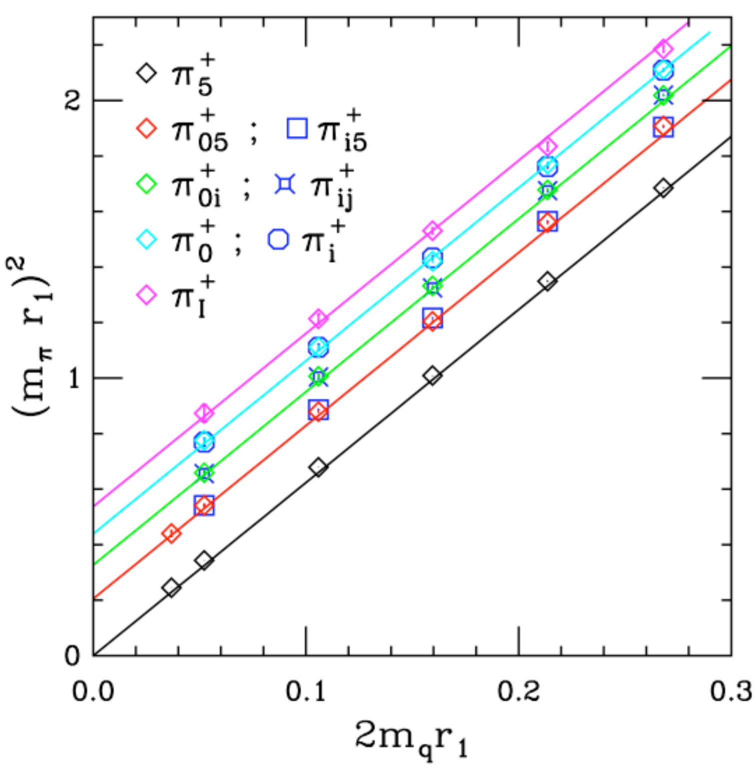
(Sharantchandra, Thun & Weisz)

All other mesons $\varepsilon(x)\bar{\chi}(x) S_A \chi(x)$ with $S_A \neq 1$ have

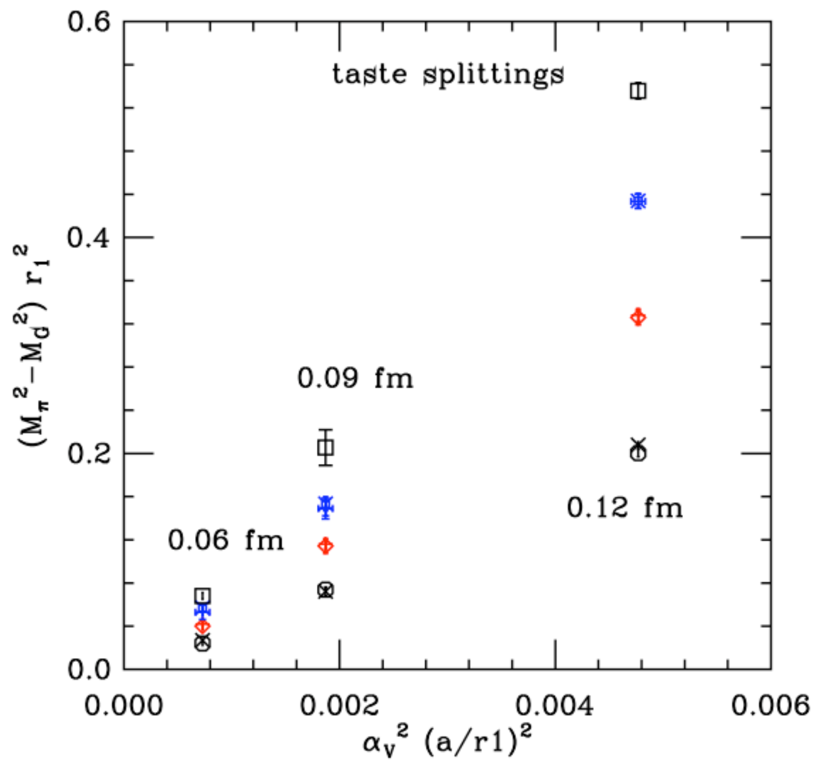
flavor/taste splittings: $$m_A^2 = m\Lambda + c_A a^2 \Lambda^4$$

Pions and taste breaking at $a \neq 0$

(MILC)

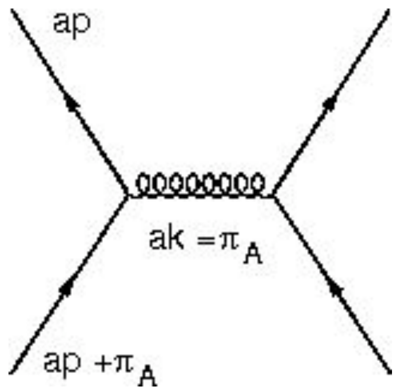


Pion masses as function of quark mass at a = 0.12 fm



Taste splitting among pions as function of a

What causes taste breaking on the lattice?



$$\sim \frac{1}{\sum_\mu \frac{4}{a^2}\sin^2\left(\frac{1}{2}ak_\mu\right)} \approx \frac{a^2}{4n}$$

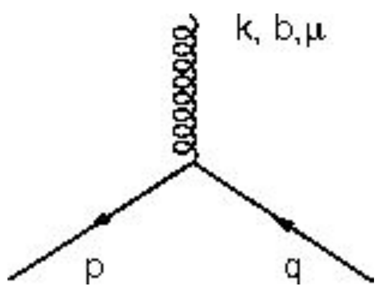
n = # components of π_A equal to π

→ effective 4-fermion taste-breaking operators of order a^2 :

$$a^2(\overline{\psi}_R\xi_\nu\xi_5\psi_L)(\overline{\psi}_R\xi_\nu\xi_5\psi_L) + \text{h.c.} \quad \rightarrow \quad a^2\text{tr}[\xi_\nu\xi_5\Sigma\xi_\nu\xi_5\Sigma] + \text{h.c.}$$

(classify all operators in QCD and in ChPT: Lee & Sharpe, Aubin & Bernard)

→ reduce taste breaking by improving quark-gluon vertex



$$= -ig\,T_b\,\delta(p - q + k + \frac{\pi_{\eta_\mu}}{a})\cos\left(ap_\mu + \frac{1}{2}ak_\mu\right)$$

$ak_\mu = 0 \;:\quad \cos(ap_\mu) \approx 1 - \frac{1}{2}a^2p_\mu^2$ reduce a^2 term by improving

(if $ak_\nu = \pi$ for some $\nu \neq \mu$ the 4-fermion vertex is generated)

Taste basis

(Gliozzi, Kluberg-Stern et al.)

Define a 4-taste Dirac field from the 16 fields $\chi(x = 2y + A)$ living in the hypercube $A_\mu \in \{0, 1\}$:

$$\psi_{\alpha a}(y) = \frac{1}{\sqrt{8}} \sum_A (\gamma_A)_{\alpha a} \chi(2y + A)$$

(free theory; put in Wilson lines along paths to A)

$$\begin{aligned} \rightarrow S &= \frac{1}{2} \sum_{y\mu} \Big(\mathrm{tr}[\overline{\psi}(y)\gamma_\mu(\psi(y+\mu) - \psi(y-\mu))] \\ &\quad - \mathrm{tr}[\overline{\psi}(y)\gamma_5((\psi(y+\mu) + \psi(y-\mu) - 2\psi(y))\gamma_\mu\gamma_5] \Big) \\ &= \frac{1}{2} \sum_{y\mu} \Big(\overline{\psi}(y)\gamma_\mu(\psi(y+\mu) - \psi(y-\mu)) \\ &\quad - \overline{\psi}(y)\gamma_5\xi_5\xi_\mu((\psi(y+\mu) + \psi(y-\mu) - 2\psi(y)) \Big) \end{aligned}$$

with $(\xi_\mu)_{ba}\psi_{\alpha a} = \psi_{\alpha a}(\gamma_\mu)_{ab} = (\gamma_\mu^*)_{ba}\psi_{\alpha a}$

S : naïve kinetic term plus “flavored” (anti-hermitian) Wilson term

Shift symmetry in taste basis:

$$S_\mu \quad : \quad \psi(y) \rightarrow \frac{1}{2}\left[(\xi_\mu + \gamma_5\gamma_\mu\xi_5)\psi(y) + (\xi_\mu - \gamma_5\gamma_\mu\xi_5)\psi(y+\mu)\right]$$

- Need to add gauge fields and keep them on the “fine” lattice in order not to break shift symmetry!
- “Wilson” term is order *a* , but with fixed coefficient
- Easy to construct operators in taste basis, but these are not in irreps of the staggered symmetry group, which live in momentum space

Parity

(MG & Smit)

continuum: $\psi(x,t) \to \gamma_4\, \psi(-x,t)$

lattice: $\chi(x) \to \varepsilon(x)\eta_\mu(x)\zeta_\mu(x)\chi(Ix)$ or $\varphi(q) \to \Gamma_\mu\Gamma_5\Xi_\mu\Xi_5\, \varphi(Iq)$

Combine $I_s = I_1 I_2 I_3$, then $\varphi(q,q_4) \to \Gamma_4\Xi_4\, \varphi(-q,q_4)$

Make into unflavored (tasteless) parity:

$$\chi(x,t) \to S_4 I_1 I_2 I_3 \chi(x) = \zeta_4(x,t)(-1)^{x_1+x_2+x_3}\chi(-x,t+1)$$
$$\phi(q,q_4) \to \Gamma_4 e^{iq_4}\phi(-q,q_4)$$

Not a symmetry of operators on a fixed time slice!

- time-slice operators couple to $\gamma_A\xi_B$ and $\gamma_A\gamma_4\gamma_5\xi_B\xi_4\xi_5$ continuum states
- correlators contain relative $(-1)^t$, *e.g.* $A_+e^{-m_+t} + (-1)^t A_- e^{-m_-t} + \ldots$ ($P = \sigma_t\sigma_s$ with σ_t the eigenvalue of Ξ_4, which $\notin$ time-slice group)

How does shift symmetry work in the Symanzik EFT?

Shift symmetry is lattice translation plus phases, with generators

$$S_\mu \chi(x) = \zeta_\mu(x)\chi(x+\mu)$$

Any representation thus takes the form

$$S_\mu \to e^{ip_\mu}\Xi_\mu \qquad (-\pi/2 < p_\mu < \pi/2)$$

with $$\{\Xi_\mu, \Xi_\nu\} = 2\delta_{\mu\nu}$$

However, any continuum EFT is invariant under continuum translations; which, for distance r , act on any continuum field as

$$\phi(p) \to e^{ip\cdot r}\phi(p)$$

Choose r such that $p{\cdot}r = -p_\mu$

⇒ EFT is invariant under the group Γ_4 generated by the Ξ_μ

Definition of “rooted” staggered fermions:

- separate staggered fields for each physical flavor
  single-site mass terms, masses m_u , m_d , m_s
  each flavor comes in four tastes

- continuum limit: 4 up, 4 down and 4 strange quarks
  with $U(4)_u \times U(4)_d \times U(4)_s$ symmetry (non-deg. masses)

- $\mathrm{Det}(D_{stag}) \sim \mathrm{Det}^4(D_{cont})$ $\Rightarrow$ take $\mathrm{Det}^{1/4}(D_{stag})$

- $\mathrm{Det}(D_{stag}) > 0$ (any m,
  Det only depends on $|m|$ because of $U(1)_\varepsilon$ symm.),
  $\mathrm{Det}(D_{cont}) > 0$ $(m_q > 0)$ $\Rightarrow$ pick positive 4th root
  $m_q \propto |m|$

Questions and answers:

1) Are rooted staggered fermions a regulator like any other, or not?
   No, they are non-local, non-unitary at $a \neq 0$.

2) Can the continuum limit be taken, and is it in the correct universality class?
   Most likely: Yes!

3) But: we work at $a \neq 0$, where the diseases are present!
   $\Rightarrow$ need EFT to parameterize the non-local effects.
   relevant EFT framework: SChPT + "replica trick"

   arguments:
   - SChPT ok for unrooted + decoupling (Bernard)
   - direct derivation from RG framework (Bernard, MG & Shamir)

Continuum limit — more detail:

$$Z_{cont}(J) = \int \mathcal{DU} \exp(-S_g) \prod_{i=1}^{N_f} \mathrm{Det}^{1/4}\Big((D + m_i) \times \mathbf{1}_{taste} + J\Big)$$

- project onto physical Hilbert space by taking $J = \tilde{J} \times \mathbf{1}_{taste}$
  $\Rightarrow$ correct correlation functions for QCD
  with all quark masses positive and any number of flavors!

- no "paradoxes" based on symmetries can arise!

- many unphysical states with non-trivial taste charges
  but can use $SU(4)_{taste}$ to relate (non-anomalous) charges,
  e.g. $\bar{u}\gamma_5 d \xrightarrow[SU(4)_{taste}]{} \bar{u}(\gamma_5 \times \xi)d$

- mixing with gluonic states: must use taste-singlet operators.

(Bernard, MG, Shamir & Sharpe)

Unitarity and the replica trick

Apparent paradox: there are 15 pions per staggered flavor, but the $N_f = 1$ theory made by 4th-rooting should have none

Consider N_f staggered fields, each replicated n_r times:

- perturbation theory:

  number of quarks on a closed loop is $4N_f n_r \Big|_{n_r = 1/4} = N_f$

- pions in $N_f = 1$ theory (Bernard et al. in staggered ChPT):

  intermediate two-pion states in taste-singlet scalar two-point function:

$$(16n_r^2 - 1) \times \mathrm{cut}(2m_\pi) = \begin{cases} 15 \times \mathrm{cut}(2m_\pi)\ , & n_r = 1 \\ 0\ , & n_r = 1/4 \end{cases}$$

zero follows from taste symmetry $\Rightarrow$ positive and negative weights!

1) Non-locality of 4th-rooted staggered fermions:

Assume a ***local*** D exists such that (at $a \neq 0$)

$$\mathrm{Det}^{1/4}\,(D_{stag}) = \mathrm{Det}\,(D)\ \exp(-\,\delta S_{eff}\,/4)\ ,$$

with δS_{eff} local (no long-distance effects). Take fourth power:

$$\mathrm{Det}(D_{stag}) = \mathrm{Det}\,(D_{4t})\ \exp(-\,\delta S_{eff}), \qquad D_{4t} = D \times \mathbf{1}\ ;$$

D_{4t} describes a theory with exact $SU(4)$ taste symmetry.

Compare spectra at $a \neq 0$:

D_{4t}: 15 degenerate pions in adjoint of $SU(4)$

D_{stag}: 15 pions are non-degenerate (only one "exact" pion)

$\Rightarrow$ δS_{eff} knows about long-distance effects! (Bernard, MG & Shamir)

RG taste basis: $D_{taste}^{-1} = \frac{1}{\alpha} + Q D_{stag}^{-1} Q^{\dagger}$ (Shamir)

with (for the free case)

$$Q D_{stag} Q^{\dagger} = \sum_{\mu} \left[i(\gamma_{\mu} \otimes 1) \sin(p_{\mu}) + 2(\gamma_5 \otimes \xi_5 \xi_{\mu}) \sin^2(p_{\mu}/2) \right] + m$$

Q is a unitary matrix connecting one-component and taste bases: rearranges fields on each 2^4 hypercube into 4(spin) $\times$ 4(taste) field; α is of order $1/a$: just adds a contact term. We have

$$\mathrm{Det}(D_{stag}) = \mathrm{Det}((\alpha G)^{-1})\ \mathrm{Det}(D_{taste})$$

$$(\alpha G)^{-1} = \frac{1}{\alpha} D_{stag} + Q Q^{\dagger} = \frac{1}{\alpha} D_{stag} + \mathbf{1}$$

$D_{stag} + \alpha$ is fermion with mass ~$1/a$: short distance contribution

D_{stag} and D_{taste} are completely equivalent.

Note: looks like starting point for RG blocking -- see later

Free theory:

$$D_{taste} = \frac{\sum_\mu i(\gamma_\mu \otimes \mathbf{1})\bar{p}_\mu + (\mathbf{1} \otimes \mathbf{1})\left(m + \frac{1}{\alpha}(\hat{p}^2 + m^2)\right) + \frac{1}{2}\sum_\mu (\gamma_5 \otimes \xi_\mu \xi_5)\hat{p}_\mu^2}{1 + \frac{2m}{\alpha} + \frac{1}{\alpha^2}(\hat{p}^2 + m^2)}$$

$$\bar{p}_\mu \equiv \sin p_\mu \ , \qquad \hat{p}_\mu \equiv 2\sin(p_\mu/2) \ , \qquad \hat{p}^2 \equiv \sum_\mu \hat{p}_\mu^2$$

Note Wilson-like term: taste-invariant part has no doublers!

(Use taste-inv. part as “comparison” theory in RG treatment.)

Non-locality and taste symmetry breaking:

Split
$$D_{taste} = D \otimes \mathbf{1} + \sum_A D_A \otimes \Xi_A$$

then
$$\log \mathrm{Det}(D_{taste}) = 4 \log \mathrm{Det}(D) + \log \mathrm{Det}\left(1 + \sum_A D^{-1} D_A \otimes \Xi_A\right)$$

D and D_A are local, but $\Sigma_A D^{-1} D_A \otimes \Xi_A$ is not!

i.e., taste breaking is local for action, but not for physics.

However, the taste-breaking D_A are irrelevant operators

⇒ conjecture: taste symmetry is restored in continuum limit
⇒ non-localities disappear in continuum limit.
(validity of 4th root is tied to validity of unrooted staggered fermions)

Comments:

- Non-locality comes from breaking of taste symmetry, which implies (*e.g.*) non-degeneracy of (too many) pions:

$$(m_\pi^A)^2 = (m_\pi^{GB})^2 + c^A\, a^2\Lambda^4_{QCD}$$

Two IR effects: quark mass m and splitting $(a\Lambda^2_{QCD})^2$,
(related to splitting $a\Lambda^2_{QCD}$ of IR eigenvalues)

$\Rightarrow$ remove unphysical IR scale first:

- take $a \to 0$ before taking $m \to 0$!

- Other masses split also, but pions lead to the most dramatic effect.

- Non-locality at $a \neq 0$ leads to unitarity violations:

- take $a \to 0$ before continuing to Minkowski space!

2) Continuum limit: an RG framework (Shamir)

Natural framework:

- IR eigenvalues should form taste multiplets, but not UV evs:
  $\Rightarrow$ get rid of UV evs by RG blocking.

- fix coarse spacing $a_c << 1/\Lambda_{QCD}$, take fine spacing $a_f \to 0$:
  gives "perfect action" $\Rightarrow$ same symmetries as continuum

- works for unrooted staggered theory;
  tells us how taste symmetry is restored (scaling with a_f/a_c)

- "bridge" to rooted theory (no direct RG!):
  "reweighted theories":
  lattice theories with exact taste symmetry,
  with same $a_f \to 0$ limit as staggered theory

RG blocking (unrooted!):

Thin out fermion fields, using gaussian kernel ($\psi^{(k)}$ on lattice $a_k = 2^k a_f$)

$$\alpha_k(\overline{\psi}^{(k)} - \overline{\psi}^{(k-1)} Q^{(k)\dagger})(\psi^{(k)} - Q^{(k)}\psi^{(k-1)})$$

Result:

$$Z = \int DU\, DV^{(1)} \dots DV^{(n)} \exp\left(-S_g - \sum_{k=1}^{n} K_g^{(k)} - \sum_{k=1}^{n} S_{eff}^{(k)}\right) \mathrm{Det}(D_n)$$

$$D_k^{-1} = \alpha_k^{-1} + Q^{(k)} D_{k-1}^{-1} Q^{(k)\dagger} \quad \text{or} \quad D_k = \alpha_k - \alpha_k^2 Q^{(k)} G_k Q^{(k)\dagger}$$

$$G_k^{-1} = D_{k-1} + \alpha_k Q^{(k)\dagger} Q^{(k)}$$

- $S^{(k)}_{eff}$ = log Det(G_k) from integrating out UV part of fermions, local: G_k, and thus D_k, are local, because $H_k = (\gamma_5 \otimes \xi_5)\, G_k^{-1} = H_k^\dagger$ has gap ("mobility edge") $\propto \alpha_k \propto 1/a_k$

- "postpone" gauge-field blocking (kernels K_g) (multiple gauge fields $U, \dots, V^{(n)}$)

Reweighting

Split into taste-singlet and taste-breaking part: $D_n = \tilde{D}_n \otimes 1 + \Delta_n$
and interpolate between staggered and taste-invariant theories:

$$Z(t) = \int \prod_k DV^{(k)} \exp\left(-S_g - \sum_{k=1}^{n} K_g^{(k)} - \sum_{k=1}^{n} S_{eff}^{(k)}\right) \mathrm{Det}(\tilde{D}_n \otimes 1 + t\Delta_n)$$

For $t = 0$, this theory has exact $SU(4)$ taste symmetry,
⇒ can take the 4th root and obtain local one-taste theory:

$$Z^{reweigh} = \int \prod_k DV^{(k)} \exp\left(-S_g - \sum_{k=1}^{n} K_g^{(k)} - \frac{1}{4}\sum_{k=1}^{n} S_{eff}^{(k)}\right) \mathrm{Det}(\tilde{D}_n)$$

Claim: for $n \to \infty$, this theory coincides with the non-local theory

$$Z^{root} = \int \prod_k DV^{(k)} \exp\left(-S_g - \sum_{k=1}^{n} K_g^{(k)} - \frac{1}{4}\sum_{k=1}^{n} S_{eff}^{(k)}\right) \mathrm{Det}^{1/4}(D_n)$$

Connect the rooted and reweighted theories:
Assume the scaling relations (up to logs)

$$\|D_n^{-1}\| \lesssim \frac{1}{a_c m_r(a_c)} , \qquad \|\Delta_n\| \lesssim \frac{a_f}{a_c} = \frac{1}{2^n}$$

then for *n* large enough we may expand:

$$\begin{aligned} \left\langle \mathcal{O}^{(n)} \right\rangle_n^{root} &= \left\langle \mathcal{O}^{(n)} \exp\left[\frac{1}{4}\mathrm{tr}\log\left(1 + \Delta_n (\tilde{D}_n \otimes 1)^{-1}\right)\right] \right\rangle^{reweigh} \\ &= \left\langle \mathcal{O}^{(n)} \right\rangle_n^{reweigh} (1 + O(\epsilon_n^2)) \\ \epsilon_n &= \|\Delta_n\| \, \|D_n^{-1}\| \lesssim \frac{1}{2^n} \frac{1}{a_c m_r(a_c)} \end{aligned}$$

For large enough *n*, the expansion of the log is convergent.

Scaling of Δ_n:

- So far, assumed that Δ_n scales like a_f/a_c on an ensemble
- To argue this, use asymptotic freedom, and $\Lambda_{QCD} << 1/a_c << 1/a_f$
  ⇒ scaling as predicted by perturbation theory
  ⇒ Δ_n is a local operator (D_n is), and indeed scales as expected in all theories!

- Summary of argument:
  - Δ_n scales like a_f in unrooted theory (local);
  - thus: Δ_n scales like a_f in 4-taste reweighted theory (local);
  - thus: Δ_n scales like a_f in 1-taste reweighted theory (local);
  - reconstruct rooted theory from 1-taste reweighted

3) SChPT from RG approach (Bernard, MG & Shamir)

For small-enough lattice spacing, a, EFTs like the Symanzik effective theory (SET) and chiral perturbation theory (ChPT) account for lattice artifacts through a systematic expansion in $a\Lambda_{QCD}$

Key assumption: the underlying lattice theory is local

However:
QCD with rooted staggered fermions ($Det^{1/4}(D_{stag})$) is non-local

$\Rightarrow$ can the construction of a SET and staggered ChPT be extended to rooted staggered QCD?

Intuitive idea: consider n_r replicas, then continue $n_r \to 1/4$, but dependence of EFT coefficients on n_r is not known

Even at a = 0.06 fm lattice artifacts (*e.g.*, mass splittings) are significant!

Start from Shamir's RG analysis:

1) Go to taste basis (Q is unitary):

$$D_{taste}^{-1} = \frac{1}{\alpha} + Q D_{stag}^{-1} Q^{\dagger}$$

2) Carry out n RG blocking steps (postpone integration over gauge fields):

$$Z(n_r) = \int \mathcal{D}\mathcal{U} \prod_{k=1}^{n} \mathcal{D}\mathcal{V}^{(k)} \mathbf{B}_n \left(n_r; \mathcal{U}, \mathcal{V}^{(k)}\right) \mathrm{Det}^{n_r} \left(D_{taste,n}\right)$$

Here

n_r is the number of "replicas" -- take integer for now!

$D_{taste,n}$ is staggered Dirac operator after n RG steps;

$\mathcal{V}^{(k)}$ are blocked gauge fields;

$\mathbf{B}_n$ is local (on coarse lattice) Boltzmann weight:

So far standard RG set-up again

Generalized theory:

Replace ($D_{taste,n} = \tilde{D}_{inv,n} \otimes \mathbf{1} + \Delta_n$)

$$\mathrm{Det}^{n_r}\left(D_{taste,n}\right) \to \mathrm{Det}^{n_s}\left(\tilde{D}_{inv,n}\right) \frac{\mathrm{Det}^{n_r}\left(\tilde{D}_{inv,n} \otimes \mathbf{1} + t\Delta_n\right)}{\mathrm{Det}^{n_r}\left(\tilde{D}_{inv,n} \otimes \mathbf{1}\right)}$$

For $t = 1$ and $n_s = 4n_r$ this is staggered theory with n_r replicas ;

For $t = 0$ this is reweighted, local theory with n_s taste-singlet fermions;

For n_r = any positive integer, and any t , this defines a local theory

$\Rightarrow$ assume that SET (and thus ChPT) exist

Now take n_s fixed, not equal to $4n_r$, then SET still exists --

think of SET as expansion in a_f , with coefficients that depend on a_c

(need to assume this works for partially quenched theories)

Important consequences:

Can expand determinant ratio in t :

$$\frac{\mathrm{Det}^{n_r}\left(\tilde{D}_{inv,n}\otimes\mathbf{1}+t\Delta_n\right)}{\mathrm{Det}^{n_r}\left(\tilde{D}_{inv,n}\otimes\mathbf{1}\right)}=\exp\left[n_r\mathrm{Tr}\log\left(1+t(\tilde{D}^{-1}_{inv,n}\otimes\mathbf{1})\Delta_n\right)\right]$$

$\Delta_n \sim a_f \;\Rightarrow$

power of n_r less than power of t less than or equal to power of a_f

Lattice: all correlation functions, expanded to a fixed order in a_f are polynomial in n_r , hence we may continue in n_r to $n_s/4$!!

SET: n_r dependence comes from Symanzik coefficients and loops

$\Rightarrow$ "staggered SET with the replica rule" (set t = 1)

Comments:

- SET is complicated for $t \neq 1$, but don't need explicit form (lattice spacing is a_f, depends also on a_c)

- Only staggered external legs and $t = 1$:
  1) all staggered symmetries apply (shift, $U(1)_\varepsilon$)
     $\Rightarrow$ form of SET is that of Lee & Sharpe (after field redefinitions)
  2) if also $n_r = n_s/4$: lattice spacing is a_f

- Expansion in $\tilde{D}^{-1}_{inv,n}\Delta_n \sim D^{-1}_{cont}(D_{latt} - D_{cont}) \sim a_f p < \frac{a_f}{a_c}$

  hence the generalized theory has no $1/a_f$ divergences (same continuum limit for all t !)

- Transition to ChPT works the same way

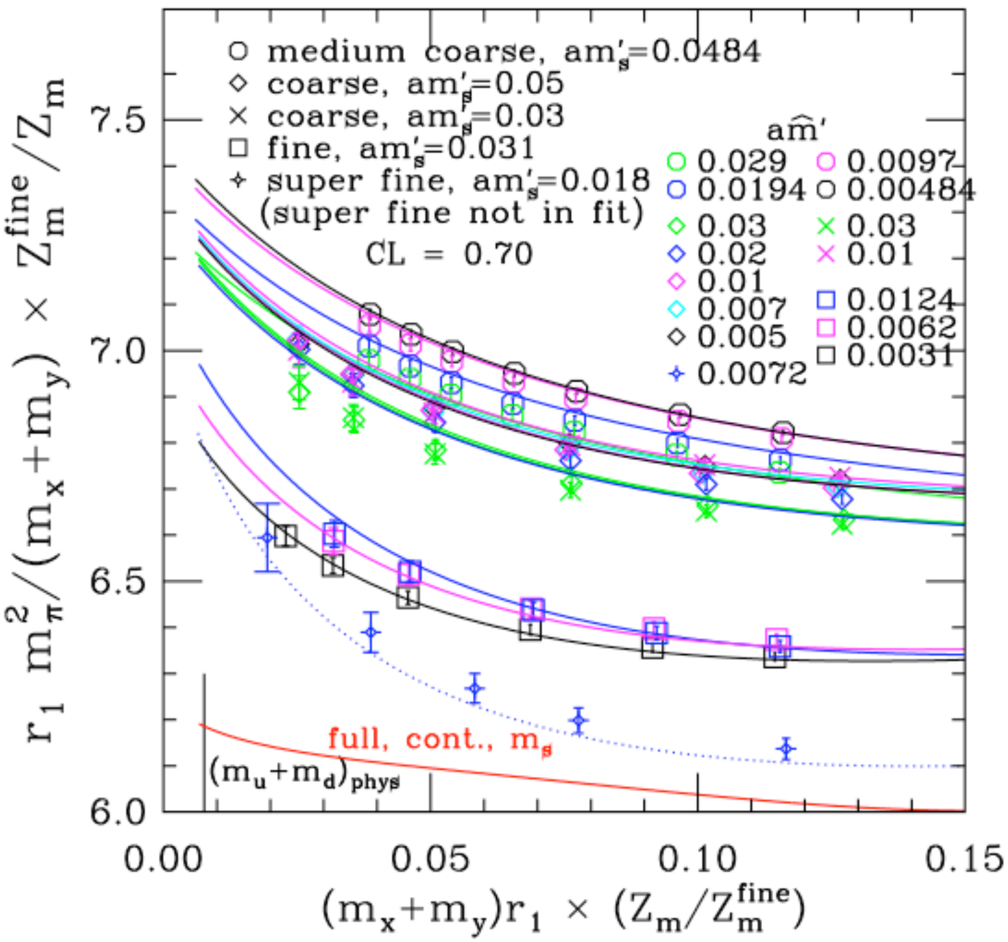
  $\Rightarrow$ "SChPT with the replica rule" (Aubin & Bernard)

- The n_s taste-singlet fermions play the role of the physical flavors; continuum limit ($n \to \infty$) is independent of n_r (and a “perfect” action)

- Turning on t “decorates” or “staggers” these fermions: n_r appears; to any given order in a_f correlation functions are polynomial in n_r thus we may continue n_r to $n_s/4$

- For $n_r = n_s/4$ with n_s not a multiple of four the theory is non-local and this non-locality is reproduced by the EFT: non-local behavior is reproduced by continuing the n_r dependence appearing through loops to $n_s/4$ (see Bernard et al. for example)

$\Rightarrow$ “SET/SChPT with replica rule”



$m_\pi^2/(m_x + m_y) = B + \text{logs}$

dotted line: prediction from other three lattice spacings

red line: predicted continuum curve

Global fit (masses and decay constants) to rooted SChPT (MILC)
930 data points, 28 unconstrained parameters+26 constrained parameters
(All fits with n_r = 1/4 ; fitting n_r from data gives value 0.31(4))

A few interesting results:

f_π = 128.3 (5) (+2.4–3.5) MeV (exp: 130.7 ± 0.1 ± 0.4)
f_K = 156.5 (4) (+1.0–2.7) MeV (exp: 159.8 ± 1.4 ± 0.4)
f_K / f_π = 1.197 (3) (+6–13)

MS-bar masses (at 2 GeV):

m_s = 88 (0) (3) (4) (0) MeV
$(m_u+m_d)/2$ = 3.2 (0) (1) (2) (0) MeV
m_u / m_d = 0.42 (0) (1) (0) (4) (rules out m_u = 0 !)

errors: statistical/systematic/perturbation theory/EM corrections

Final comments:

- There is very good theoretical and numerical evidence that taking the 4th root works, even if the theory at $a \neq 0$ is sick. There is (at present) no argument against!

- Locality and scaling of operators can be tested, numerically and (in principle) in "multi-gauge-field" perturbation theory. Doing this would go long way toward confirming validity of the "rooting trick."

- Can derive EFT valid at $a \neq 0$.

- Spectacular success for mesons; but baryons and (most) weak matrix elements quite difficult. Reason: lack of $SU(4)$ symmetry at $a \neq 0$!

- $\Rightarrow$ use mixed actions: staggered sea + domain-wall valence!