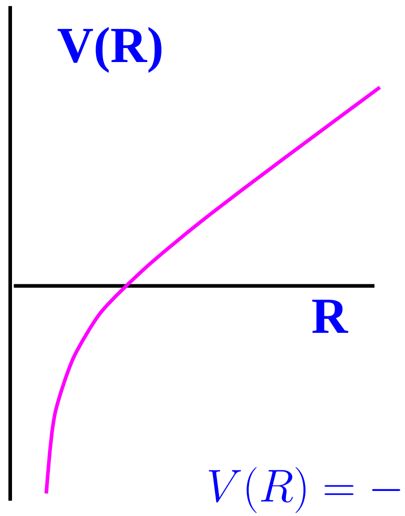
Lattice QCD
at finite temperature and chemical potential

non perturbative phenomena in the hadron phase :

Confinement

- quarks and gluons are permanently confined in hadrons
- heavy quarks : potential models



$$V(R) = -\frac{\alpha_s(R)}{R} + \sigma R$$

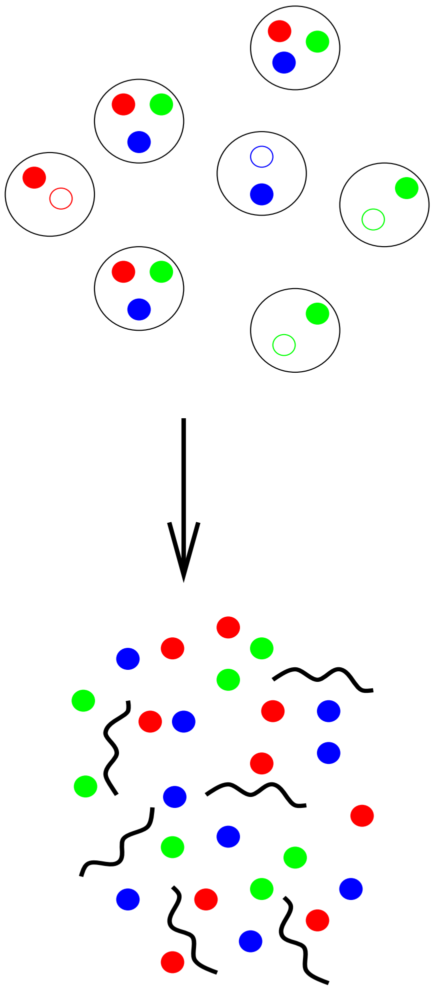
spontaneous breaking of chiral symmetry

- for $m_f = 0$: $\mathcal{L}_{QCD}$ invariant under

$$q_R \longrightarrow U_R\, q_R \qquad U_R \;\epsilon\; SU_R(N_F)$$
$$q_L \longrightarrow U_L\, q_L \qquad U_L \;\epsilon\; SU_L(N_F)$$

- $m_u, m_d \leq$ 10 MeV $\ll m_{proton} \simeq$ 1000 MeV
  $m_s \simeq$ 100 MeV
  $m_c \simeq$ 1.3 GeV
  $m_b \simeq$ 4.5 GeV
  $m_t \simeq$ 175 GeV
- vacuum invariant
  $\Rightarrow$ parity doubling in the mass spectrum or
- spontaneous breaking
  $\Rightarrow$ Goldstone particles $m_\pi \simeq$ 140 MeV
  $\Rightarrow$ chiral condensate $\langle \bar{q}q \rangle \neq 0$

at high temperatures and/or density : transition to a new state of matter



nuclear matter

$T < T_c$

$T_c \cong 1/\text{fm} \cong 200$ MeV

quark gluon plasma

$T > T_c$

asymptotically

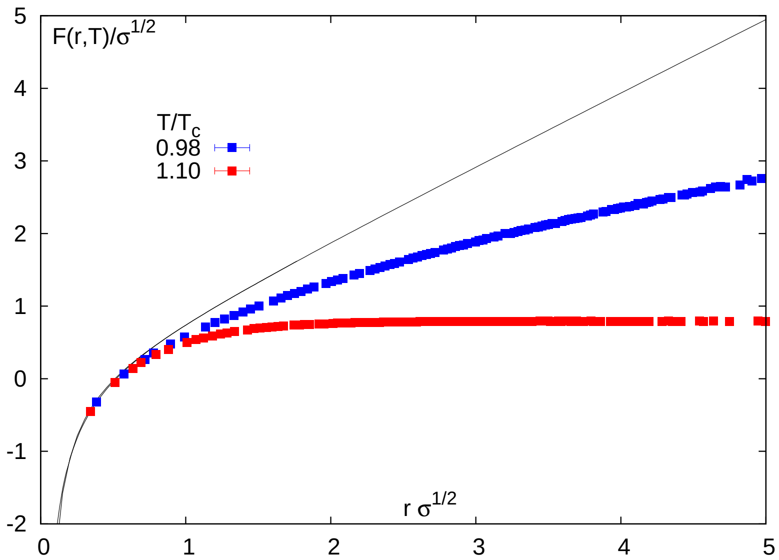
$$g(T) \longrightarrow 0$$

$\rightsquigarrow$ **Deconfinement**

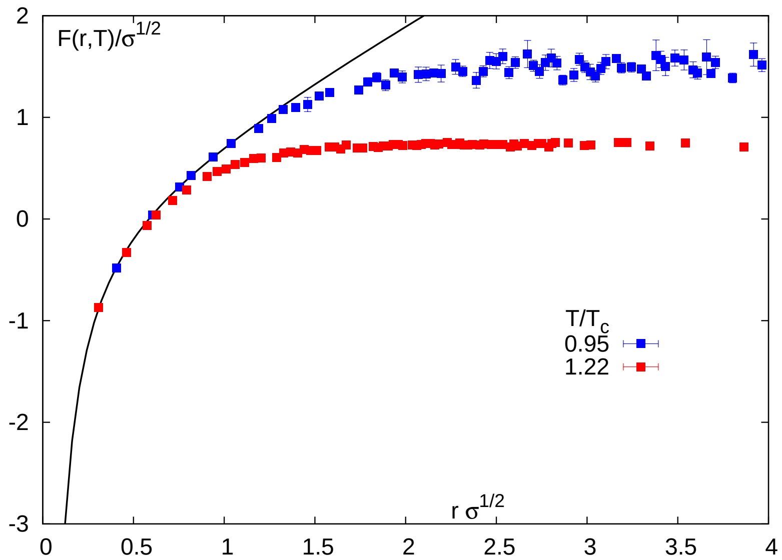
$\rightsquigarrow$ **chiral symmetry restoration** indeed, lattice simulations provide evidence for

confinement → deconfinement

quenched:

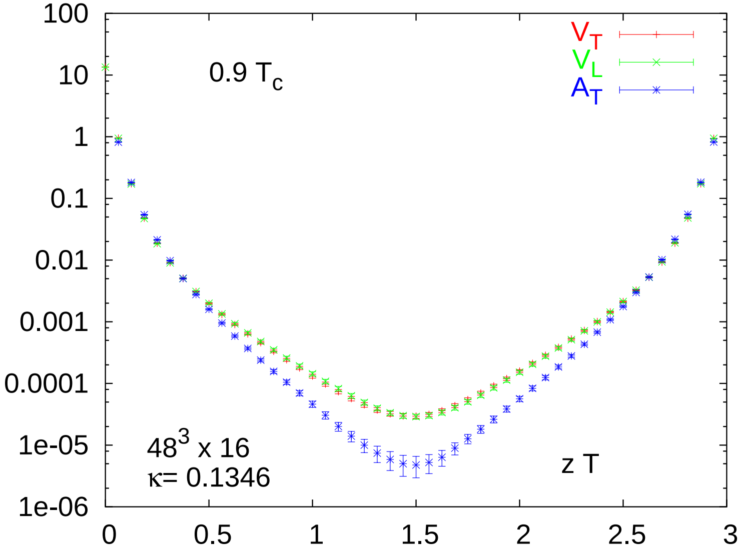


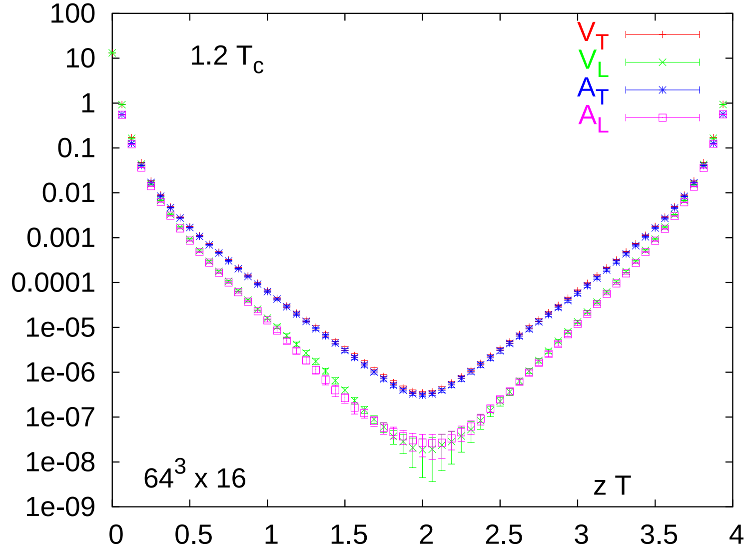
full QCD:



↑ string breaking

Chiral symmetry restoration $SU_V(N_F) \to SU_L(N_F) \times SU_R(N_F)$





$T < T_c$: $V_T = V_L \neq A_T = A_L$

$T > T_c$: $V_T = A_T \neq V_L = A_L$

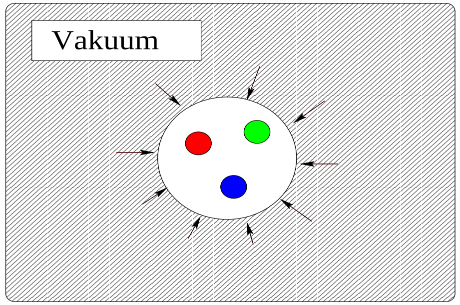
- at $T > T_c$, chiral symmetry restoration: $V = A$
- at $T \neq 0$, for spatial correlations: rotational $SO(3) \to SO(2) \times Z(2)$

$$\Rightarrow \quad V_T \neq V_L, \quad A_T \neq A_L \quad \text{possible}$$

expected properties - qualitatively :

bag model

vacuum pressure $p = B$



ideal gas (Stefan Boltzmann)

hadron - view

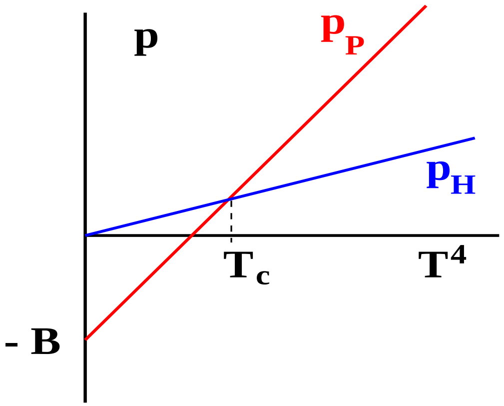
$$p = d_H \frac{\pi^2}{90} T^4$$
$$d_H = \#\,\text{d.o.f.} = 3 \qquad (\pi^\pm, \pi^0)$$

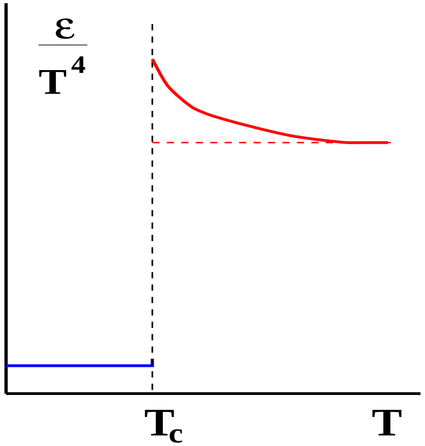
quark - gluon - view

$$p = d_P \frac{\pi^2}{90} T^4 - B$$
$$d_P = 2 \times 8 + 21/2 N_F \qquad (G, q)$$

$F/V = -p$ minimal

$\Rightarrow$ phase transition





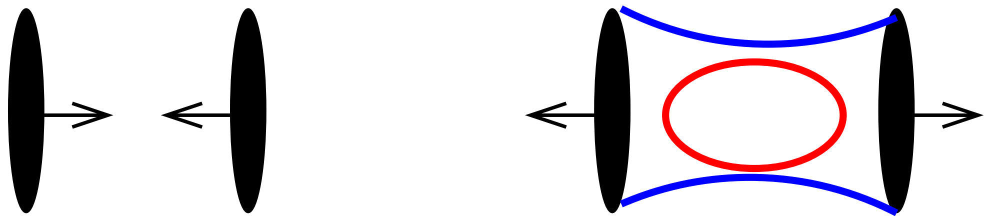
$$\frac{\epsilon_H}{T^4} = d_H \frac{\pi^2}{30}$$
$$\frac{\epsilon_P}{T^4} = d_P \frac{\pi^2}{30} + \frac{B}{T^4}$$

- $p_H(T_c) = p_P(T_c) \quad \Leftrightarrow \quad T_c = \left(\frac{90}{\pi^2(d_P - d_H)}\right)^{1/4} B^{1/4} \simeq 0.7 B^{1/14} \simeq 140\ \mathrm{MeV}$
- $\epsilon_P(T_c) \simeq 15\, T_c^4 \simeq 1\, \mathrm{GeV}/fm^3$
- $t_c \simeq \frac{2.4}{\sqrt{d(T_c)}} \left(\frac{1\,\mathrm{MeV}}{T_c}\right)^2 sec \simeq 10^{-5} sec$

  $\longrightarrow$ early universe
- $\epsilon_{\mathrm{plasma}}(T_c) \simeq 4\, \epsilon_{\mathrm{neutron}} \simeq 1\, \mathrm{GeV}/fm^3$

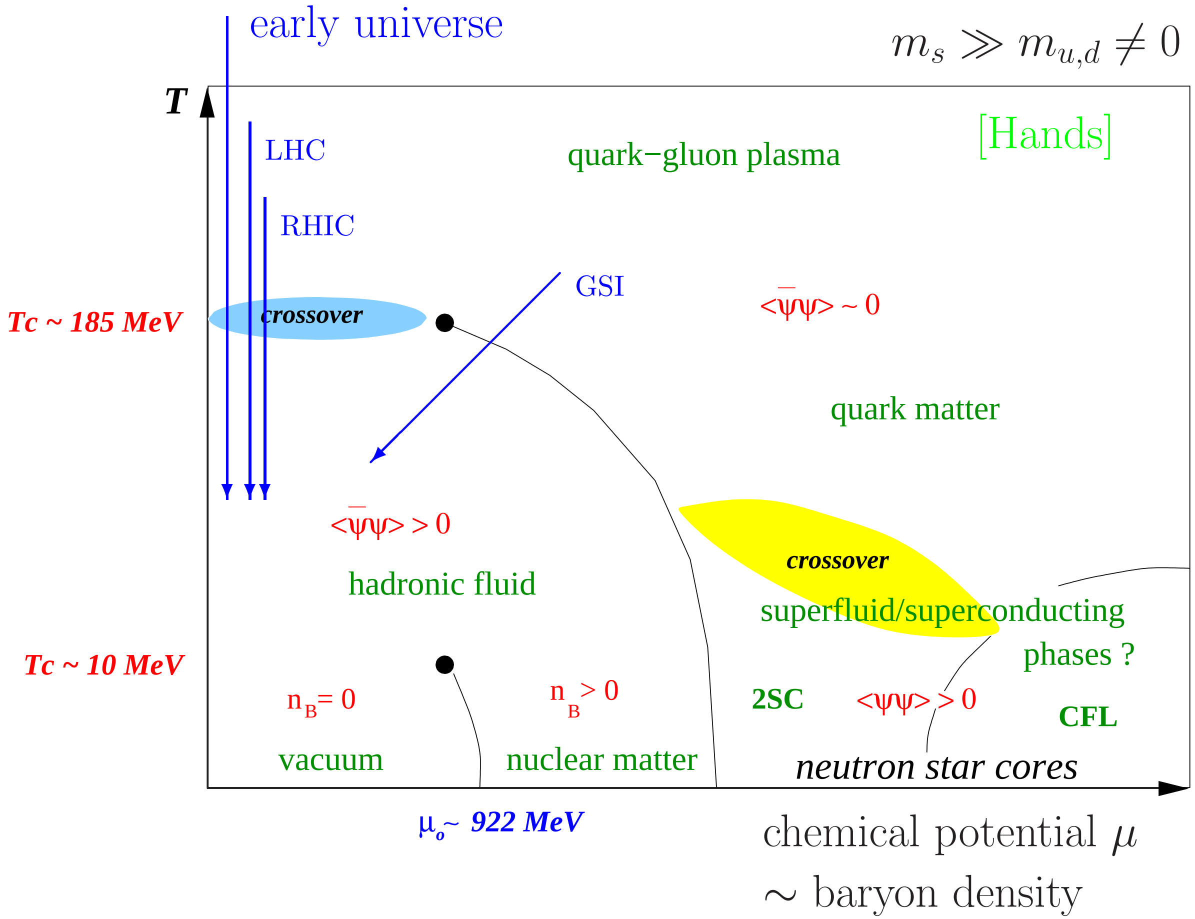
  $\longrightarrow$ neutron stars
- $\longrightarrow$ heavy ion collisions (RHIC,LHC)

  $\epsilon \simeq 0.1 A^{1/3} \left(\frac{dN}{dy}\right)_{pp} [\mathrm{GeV}/fm^3]$



$$\left.\begin{array}{r} A \simeq 200 \\ \ln\sqrt{s} \sim (dN/dy)_{pp} \simeq 4 \end{array}\right\} \Rightarrow \epsilon \simeq 2.4\, \mathrm{GeV}/fm^3$$

expected properties of the phase diagram at $\mu \neq 0$:



in detail dependent on
masses of light flavors

$m_{u,d} \ll m_s \quad N_F = 2$

$m_{u,d} < m_s \quad N_F = 2 + 1$

$m_{u,d} \simeq m_s \quad N_F = 3$

[see e.g. Rajagopal, Wilczek]

$\star$ **for RHIC, LHC, universe: μ small**

Questions to theory :

- phase diagram
  - critical temperature T_c
  - nature of the transition: order, critical exponents
- Equation of State $\epsilon(T), p(T)$
  - critical energy density
  - latent heat
- properties of the plasma phase
  - existence of hadronic excitations
  - masses, widths
  - screening lengths
  - strangeness content
  - response functions
  - ...
- properties of the hadron phase at $T \lesssim T_c$
  - "approach to chiral symmetry" $\longrightarrow$ hadron masses and widths
  - "approach to deconfinement" $\longrightarrow$ quark potentials
  - ...

$\longrightarrow$ **genuine non-perturbative methods required**

Quantum Statistics in equilibrium :

partition function $Z = \mathrm{tr}\left\{e^{-\hat{H}/T}\right\}$

→ Feynman path integral

$$Z(T,V) = \int \mathcal{D}\phi(\vec{x},\tau)\,\exp\left\{-\int_0^{1/T} d\tau \int_0^V d^3\vec{x}\,\mathcal{L}_E[\phi(\vec{x},\tau)]\right\}$$

- integral over all configurations $\phi(\vec{x},\tau)$
- weighted by Boltzmann factor $\exp(-S_E)$
- euclidean "time" $\tau = it$
- (anti-) periodic boundary conditions in τ

apply standard thermodynamic relations, e.g.

energy density $\epsilon = \frac{T^2}{V}\frac{\partial \ln Z}{\partial T}\Big|_V$

specific heat $c_V = \frac{1}{VT^2}\frac{\partial^2 \ln Z}{\partial(1/T)^2}\Big|_V$

in general

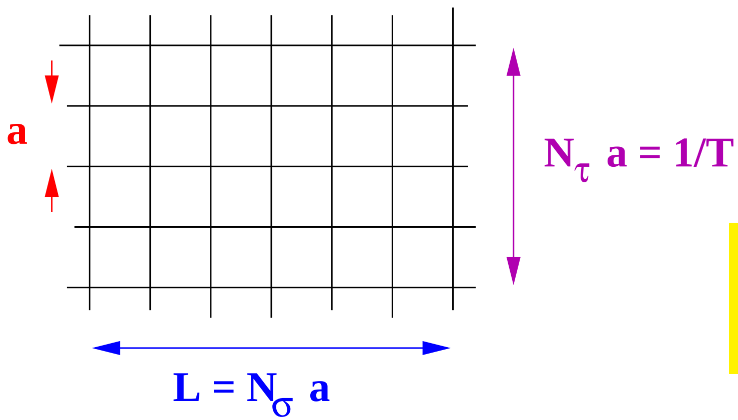
$$\langle \mathcal{O} \rangle = \frac{1}{Z}\mathrm{tr}\left\{\hat{\mathcal{O}}e^{-\hat{H}/T}\right\} = \frac{1}{Z}\int \mathcal{D}\phi\,\mathcal{O}[\phi]e^{-S_E[\phi]}$$

also : starting point of perturbation theory i.e. expansion in coupling strength g

numerical treatment of QCD $\Rightarrow$ discretize (Euclidean) space-time

$\Rightarrow$ **lattice**

$N_\sigma^3 \times N_\tau$



$$Z(T,V) = \int \prod_{i=1}^{N_\tau N_\sigma^3} d\phi(x_i) \exp\{-S[\phi(x_i)]\}$$

finite yet high-dimensional path integral

$\longrightarrow$ **Monte Carlo**

- thermodynamic limit, IR - cut-off effects $\quad LT = \frac{N_\sigma}{N_\tau} \longrightarrow \infty$
- continuum limit, UV - cut-off effects $\quad aT = \frac{1}{N_\tau} \longrightarrow 0$
- chiral limit $\quad m \longrightarrow m_{\text{phys}} \simeq 0$

  numerical effort $\sim (1/m)^p$

Localisation of the phase transition

- **order parameter**

1.) **Polyakov loop** $L(\vec{x}) = \frac{1}{3} \mathrm{tr} \prod_\tau U_\tau(\vec{x}, \tau)$

- sensitive on $Z(3)$ symmetry (pure gauge theory only)
- measures free energy of an isolated quark $\langle L \rangle \sim e^{-F_{\mathrm{quark}}/T}$

hadron phase	$F_{\mathrm{quark}} \longrightarrow \infty$
	$\langle L \rangle = 0$
plasma phase	F_{quark} finite
	$\langle L \rangle \neq 0$

2.) **chiral condensate**

- sensitive on chiral symmetry ($m \longrightarrow 0$)

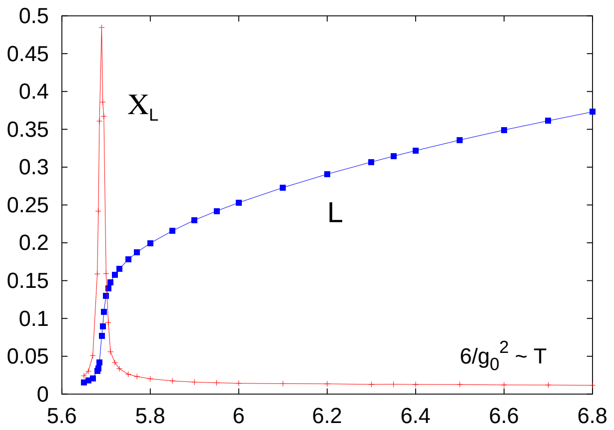
hadron phase	$\langle \bar{q}q \rangle \neq 0$
plasma phase	$\langle \bar{q}q \rangle = 0$

- **susceptibilities**

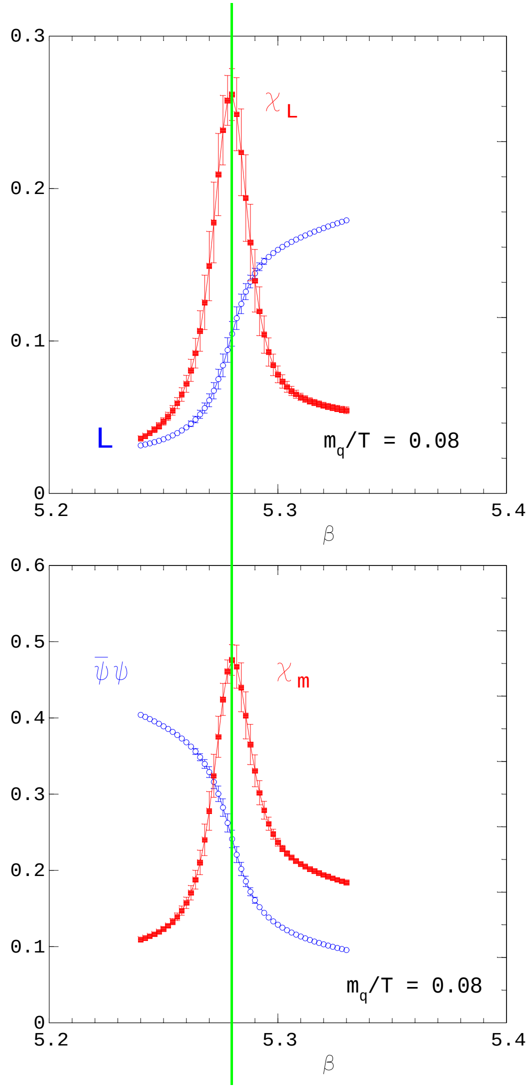
e.g. chiral susceptibility
- measures fluctuations

$$\chi_m \sim \frac{\partial^2 \ln Z}{\partial m^2} \sim \langle\, (\overline{q}q)^2 \,\rangle - \langle\, \overline{q}q \,\rangle^2$$

quenched QCD



full QCD ($N_F = 2$)



Polyakov loop

chiral condensate

$$g_{crit}^{Polyakov} = g_{crit}^{chiral}$$

$\beta = 6/g^2 \sim T$

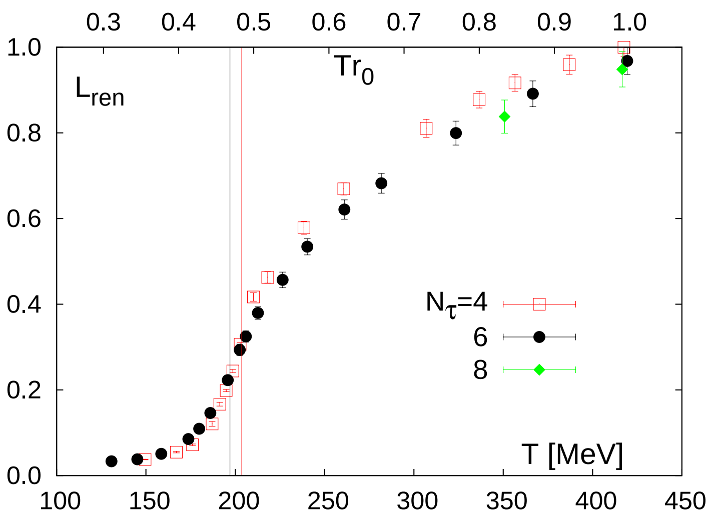
$$T = \frac{1}{N_\tau a(g)}$$

in perturbation theory (lowest order)

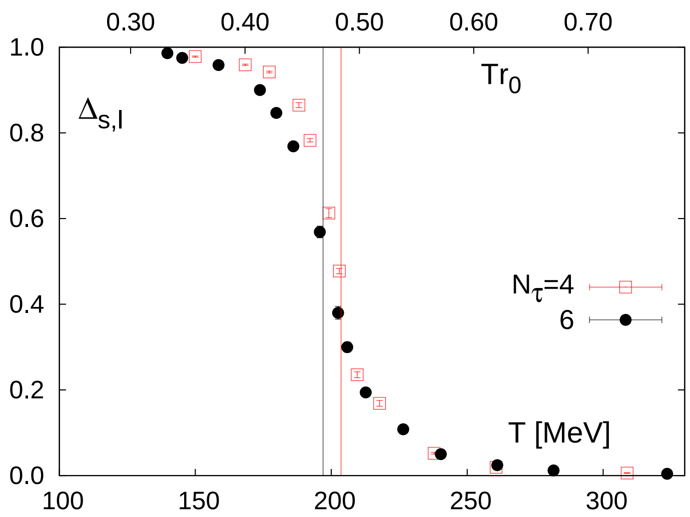
$$a = \frac{1}{\Lambda} \exp\left\{ -\frac{24\pi^2}{33 - 2N_F} \frac{1}{g^2} \right\}$$

from a recent simulation at $m_K = 500$ MeV, $m_\pi \simeq 220$ MeV

Polykov loop



chiral condensate



$$\frac{\langle \bar{l}l\rangle_T - \frac{\hat{m}_l}{\hat{m}_s}\langle \bar{s}s\rangle_T}{\langle \bar{l}l\rangle_0 - \frac{\hat{m}_l}{\hat{m}_s}\langle \bar{s}s\rangle_0}$$

⋆ no significant temperature difference between

- confinement → deconfinement transition
- chiral symmetry restoration

critical temperature T_c

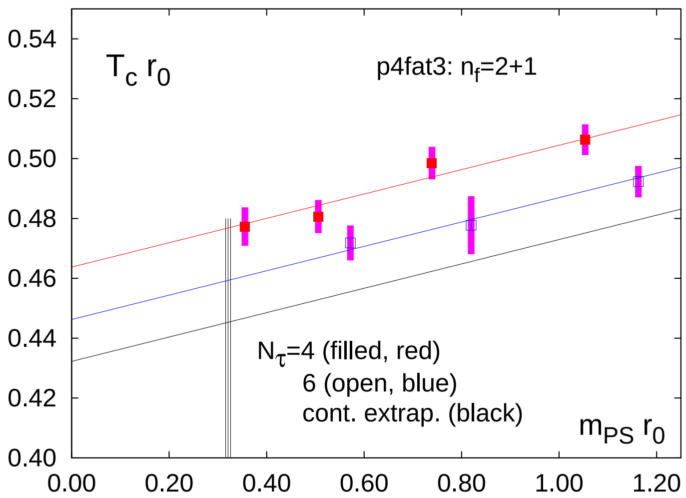
$$T_c = \frac{1}{N_\tau\, a(g_c)}$$

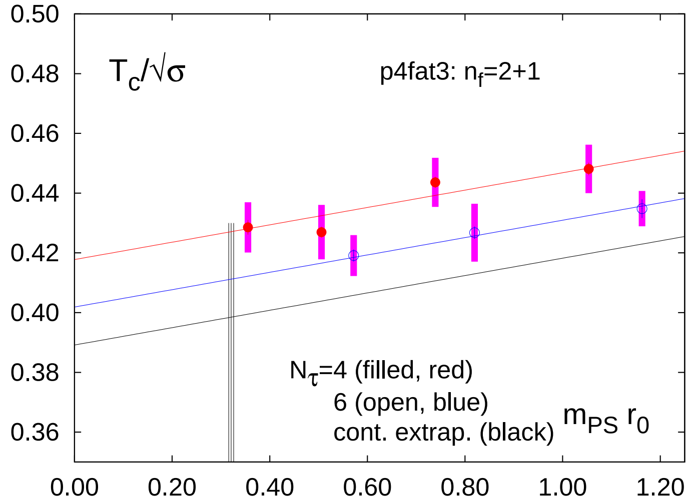
at $T = 0$, same (bare) coupling g_c, measure e.g. string tension σ $\Rightarrow$: $\sqrt{\sigma}\, a(g_c) = \mathit{number}$

$\Rightarrow$ dimension less ratio

$$\frac{T_c}{\sqrt{\sigma}} = \frac{1}{N_\tau\, a(g_c)} \star \frac{a(g_c)}{\mathit{number}} = \frac{1}{N_\tau \star \mathit{number}}$$

$\sqrt{\sigma}$ only weakly affected by quark mass





combined continuum/chiral extrapolation $(T_c\, r_0)_{m_l, N_\tau} = T_c\, r_0 + A(m_{PS}\, r_0)^d + B/N_\tau^2$

phys. point $T_c r_0 = 0.456(7)^{+3}_{-1}$ $T_c/\sqrt{\sigma} = 0.408(7)^{+3}_{-1}$

with new $T = 0$ MILC (lattice) results for $r_0 = 0.469(7)$fm obtain: $T_c = 192(5)(4)$MeV

preliminary $N_\tau = 8$ results favor lower end of error bar

nature of the phase transition (at $\mu = 0$)

expected

- $\Phi(\tau) = \Sigma_n \exp\{i\omega_n \tau\}\ \Phi(\omega_n)$ with Matsubara frequencies $\omega_n = \begin{cases} 2\pi T n & \text{bosons} \\ \pi T(2n+1) & \text{fermions} \end{cases}$
  $\longrightarrow$ for high temperatures static boson-modes only
  $\longrightarrow$ three-dimensional effective theory
- long range correlations
  $\longrightarrow$ global symmetries count, microscopic details don't (universality)
  $\longrightarrow$ here: chiral symmetries, σ models

$\Rightarrow N_F = 2$ [Wilczek,Pisarski]

- if phase transition continuous (2nd order), then $SU_R(2) \otimes SU_L(2) \simeq O(4)$
- if $U_A(1)$ effectively restored (no non-trivial topological configurations at T_c^{chiral}), then phase transition discontinuous (1st order).

$\Rightarrow N_F = 3$ [Wilczek,Pisarski]

- phase transition discontinuous $\longrightarrow$ even at $m \leq m_c \neq 0$
- at critical end point m_c: $Z(2)$ Ising universality [Gavin,Gocksch,Pisarski]

$\Rightarrow N_F = 2 + 1$

- depending on quark masses $m_{u,d}, m_s$

critical behavior

in the vicinity of a phase transition: correlation length $\longrightarrow \infty$

$\Rightarrow$ scaling behavior of the free energy density

$$f(t,m,L) = b^{-d}\, f(\,b^{y_t}\,t, b^{y_h}\,m, L/b\,) \qquad \text{with reduced temperature } t = \frac{|T-T_c|}{T_c}$$

$\Rightarrow$ scaling laws, e.g.

$$\langle M \rangle \sim m^{1/\delta}$$

$$\chi_m \sim L^{\gamma/\nu}$$

$$B_4 = \frac{\langle (\delta M)^4 \rangle}{\langle (\delta M)^2 \rangle^2} \qquad \text{(here: } \delta M = M - \langle M \rangle, M \simeq \overline{q}q\text{)}$$

with critical exponents $\delta, \gamma, \nu, \ldots$ und Binder-cumulant B_4 universal

	Z(2)	O(2)	O(4)
γ/ν	1.963(3)	1.962(5)	1.975(4)
B_4	1.604(1)	1.242(2)	1.092(3)

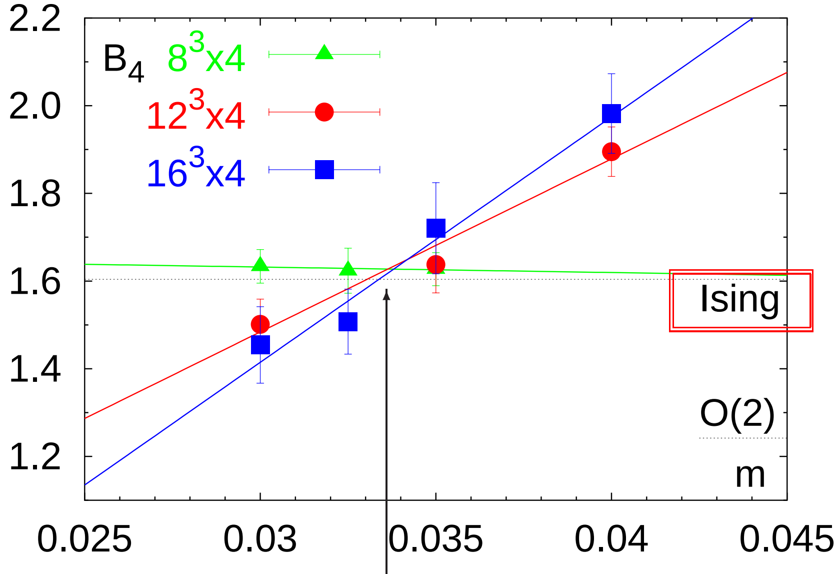
possible, but tough to do with necessary precision

$N_F = 3$

Binder cumulant B_4

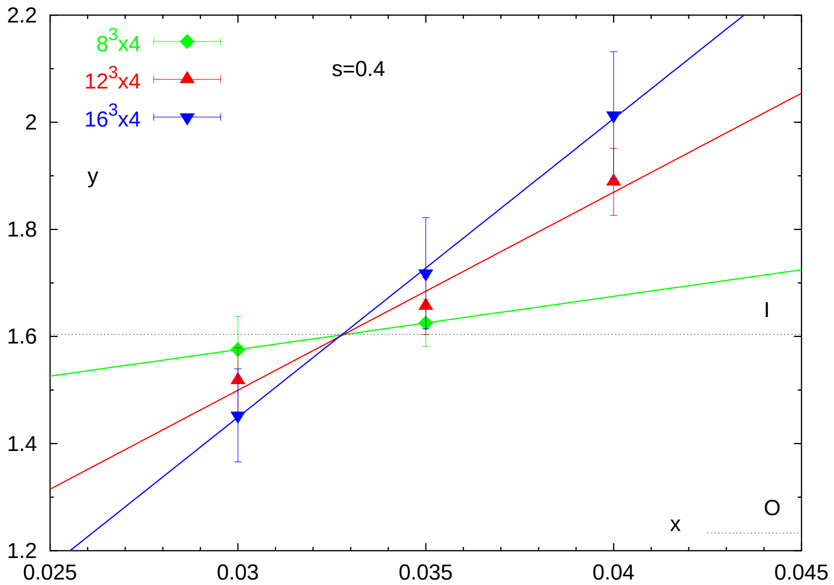
- intersection for various V yields critical value of m
- value of B_4 is universal
- corrections from V finite and 'order parameter not matched correctly'

standard action



$m_c \simeq 4\, m_u^{phys} \Leftrightarrow m_{PS} \simeq 290$ MeV

[Bielefeld; deForcrand,Philipsen]



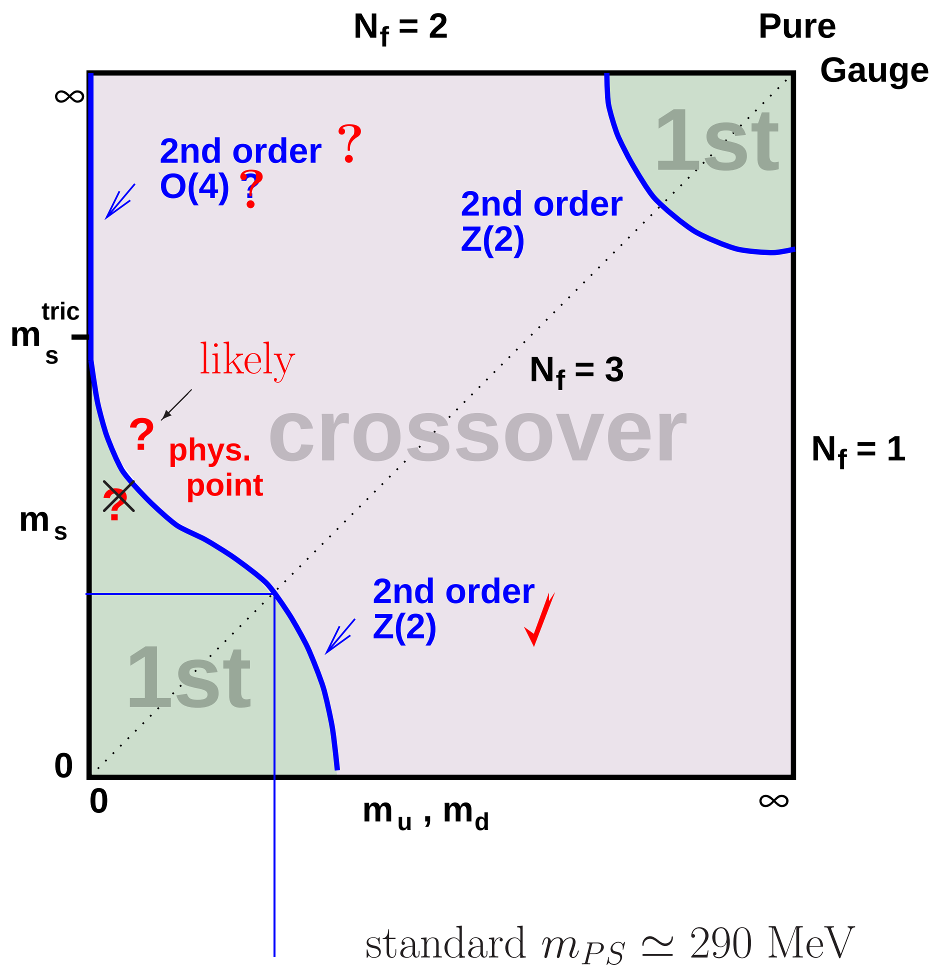
magnetization-like order parameter $\mathcal{M}$

not identical with chiral condensate $\langle \overline{q} q \rangle$

(chiral symmetry broken by $m_q \neq 0$ anyway)

$$\mathcal{M} = \langle \overline{q} q \rangle + s\, S$$

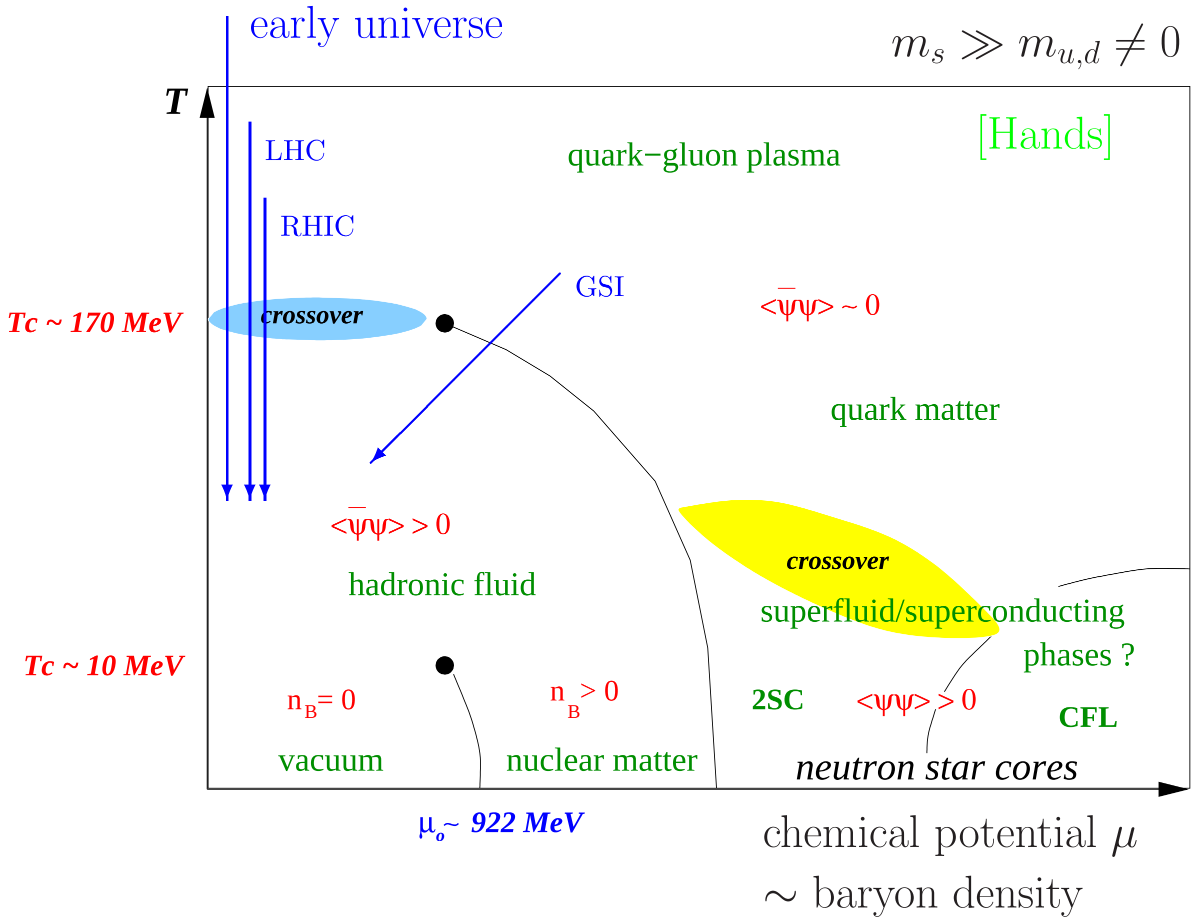
phase diagram in the $m_{u,d} - m_s$ plane at $\mu = 0$



standard $m_{PS} \simeq 290$ MeV
$\rightarrow m_{PS} \simeq 70$ MeV improved

most probably,
in the real world the transition
is not a (strongly) first order one

expected properties :



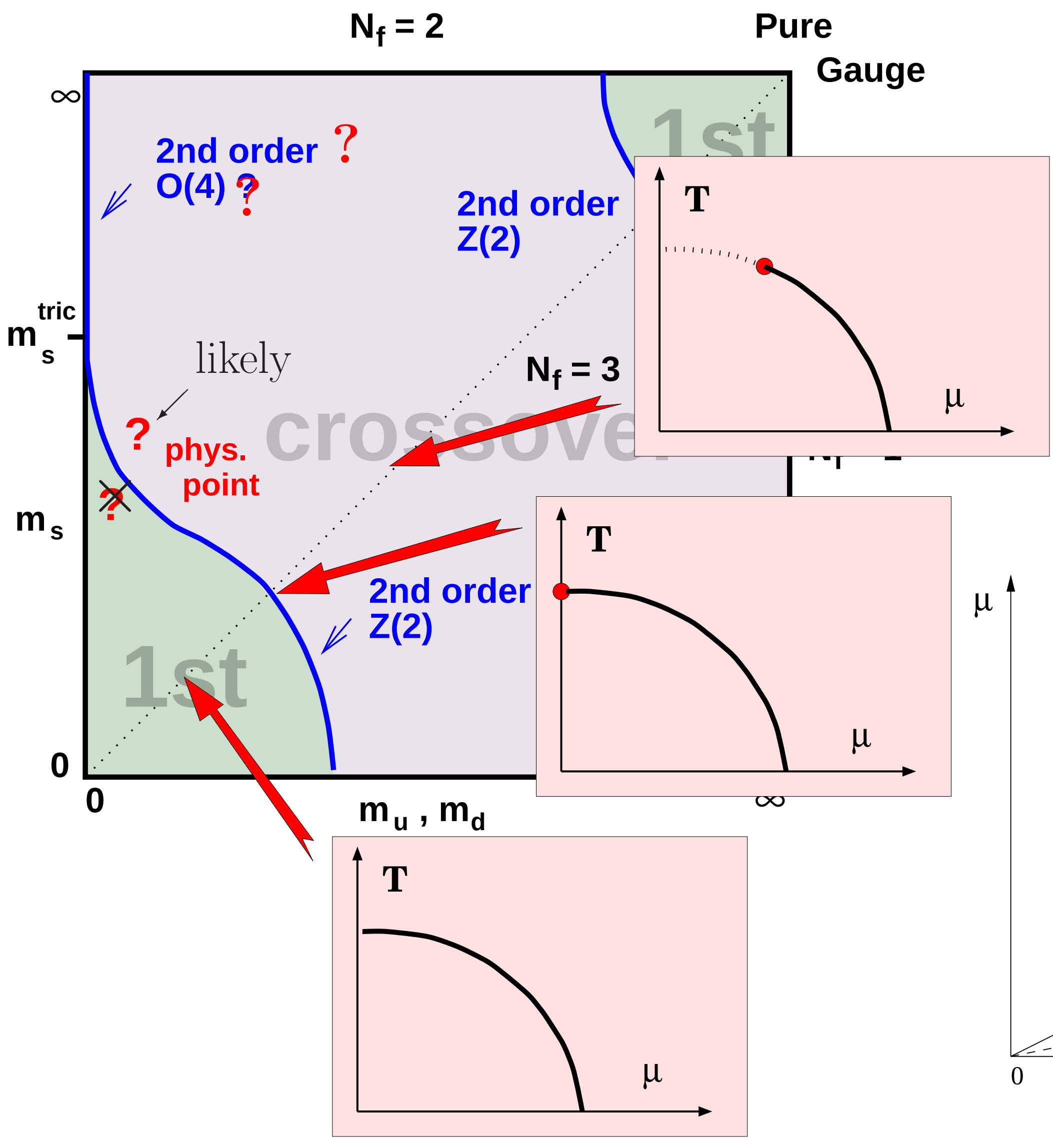
in detail dependent on masses of light flavors

$m_{u,d} \ll m_s \quad N_F = 2$

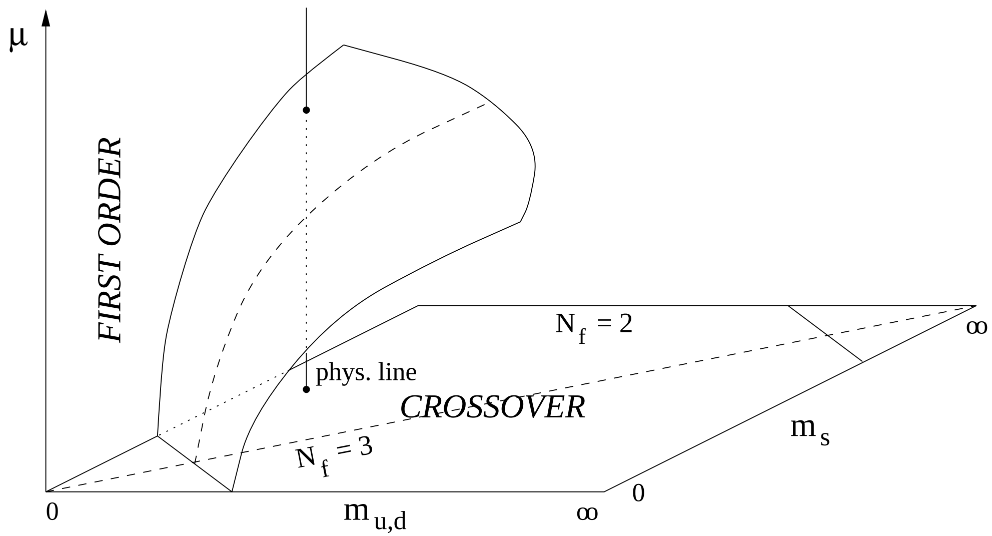
$m_{u,d} < m_s \quad N_F = 2 + 1$

$m_{u,d} \simeq m_s \quad N_F = 3$

see e.g. Rajagopal, Wilczek, hep-ph/0011333


Nf = 2
Pure
Gauge
∞
2nd order ?
O(4) ?
2nd order
Z(2)
1st
tric
ms
likely
crossover
Nf = 3
? phys.
point
ms
2nd order
Z(2)
1st
0
0
mu , md
∞
T
μ
T
μ
T
μ

summarized:


μ
FIRST ORDER
Nf = 2
phys. line
CROSSOVER
Nf = 3
ms
0
mu,d
∞
0
∞

the problem :

$$Z_{GC}(T,V,\mu) = \int \mathcal{D}U_\mu \, \mathcal{D}q \, \mathcal{D}\overline{q} \; \exp\{-S_G(U) + \overline{q}M(\mu)q\}$$

integrate over quark fields

$$Z_{GC}(T,V,\mu) = \int \mathcal{D}U_\mu \; \det M(\mu) \; \exp\{-S_G(U)\}$$

- for $\mu \neq 0$: $\det M(\mu)$ complex $\Rightarrow$ can not be used as statistical weight in Monte Carlo
- reformulate: $\det M(\mu) = |\det M(\mu)| \, e^{i\Theta}$ and use phase Θ as (part of the) observable:

$$\langle \, \mathcal{O} \, \rangle_{\det M} = \langle \, \mathcal{O} \; e^{i\Theta} \, \rangle_{|\det M|} \; / \; \langle \, e^{i\Theta} \, \rangle_{|\det M|}$$

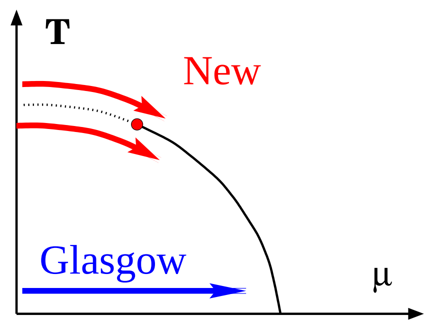
- but : $\langle \, e^{i\Theta} \, \rangle_{|\det M|} \sim e^{-V}$ 'sign problem'

∗ 'Reweighting'

[Glasgow; Fodor,Katz]

simulate at parameters $p_0 = (g, m, \mu)_0$ and reweight to $p = (g, m, \mu)$

$$\mathcal{D}U e^{-S_G(p)} \det M(p) = \underbrace{\mathcal{D}U e^{-S_G(p_0)} \det M(p_0)}_{\text{simulation}} \star \underbrace{e^{-[S_G(p)-S_G(p_0)]} \frac{\det M(p)}{\det M(p_0)}}_{\text{correction-factor}}$$



- limited by overlap

∗ 'Taylor-expansion'

[Bielefeld-Swansea; Gavai,Gupta]

$$\langle \mathcal{O} \rangle \left(\frac{\mu}{T}\right) = \langle \mathcal{O} \rangle_{\mu=0} + \langle \tilde{\mathcal{O}}_2 \rangle_{\mu=0} \star \left(\frac{\mu}{T}\right)^2 + \langle \tilde{\mathcal{O}}_4 \rangle_{\mu=0} \star \left(\frac{\mu}{T}\right)^4 + \ldots \qquad \text{with} \quad \tilde{\mathcal{O}}_k = \frac{1}{k!} \frac{\partial^k \mathcal{O} \det M}{\partial \mu^k}$$

- limited by convergence radius

∗ 'imaginary μ'

[Forcrand,Philipsen; D'Elia,Lombardo]

- $\mu = i\mu_I \Rightarrow \det M$ real and positive
- analytic continuation to real μ

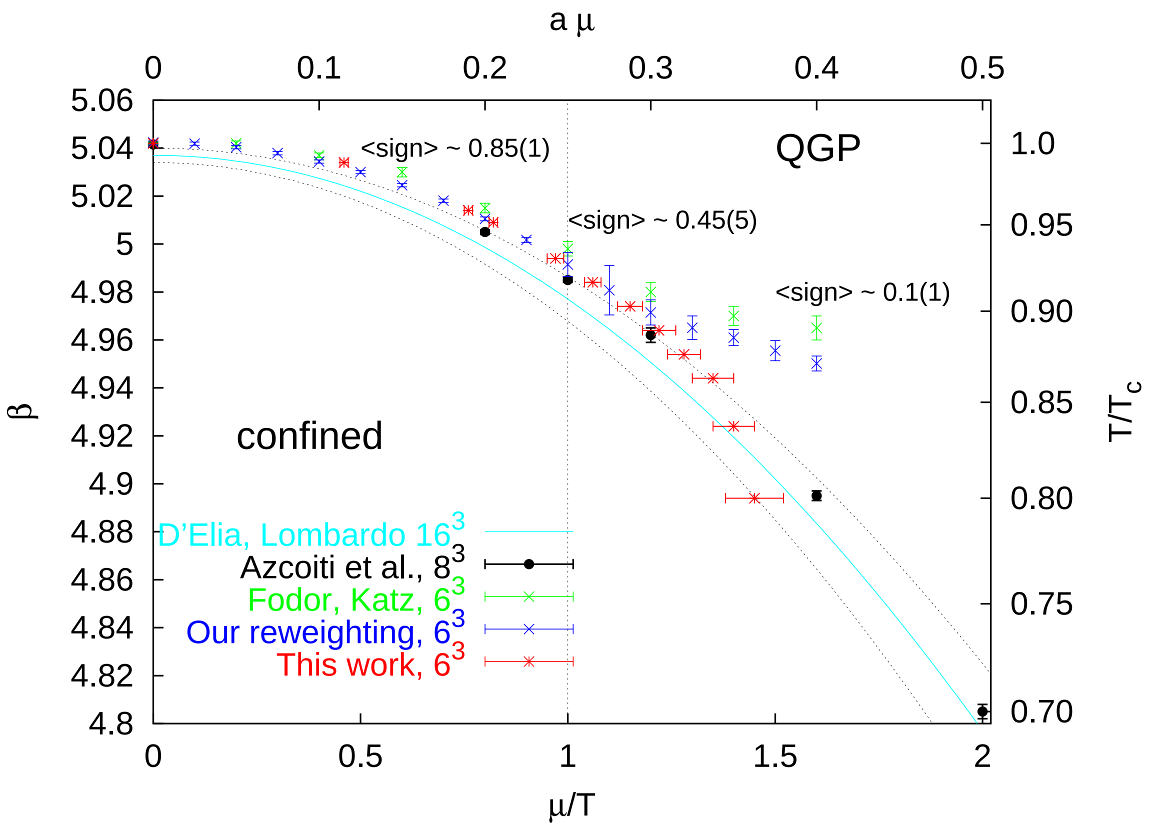
- limited by $Z(\mu_I/T) = Z(\mu_I/T + 2\pi/3)$

$*$ **'canonical'** [Forcrand,Kratochvila]

$$Z_C(B) = \frac{1}{2\pi} \int d\left(\frac{\mu_I}{T}\right) \exp\left\{-i3B\frac{\mu_I}{T}\right\} Z_{GC}(\mu = \mu_I)$$

- sample at fixed μ_I
- Fourier transform each determinant $\longrightarrow$ work $\sim N_\sigma^9 \times N_\tau$
- combine with reweighting in μ_I
- back to $Z_{GC}(\mu)$ by $\Sigma_B \exp\left\{+3B\frac{\mu}{T}\right\} Z_C(B)$

Phase Diagram $T - \mu$: comparing apples with apples



taken from Forcrand
see hep-lat/0602024

canonical partition fct.

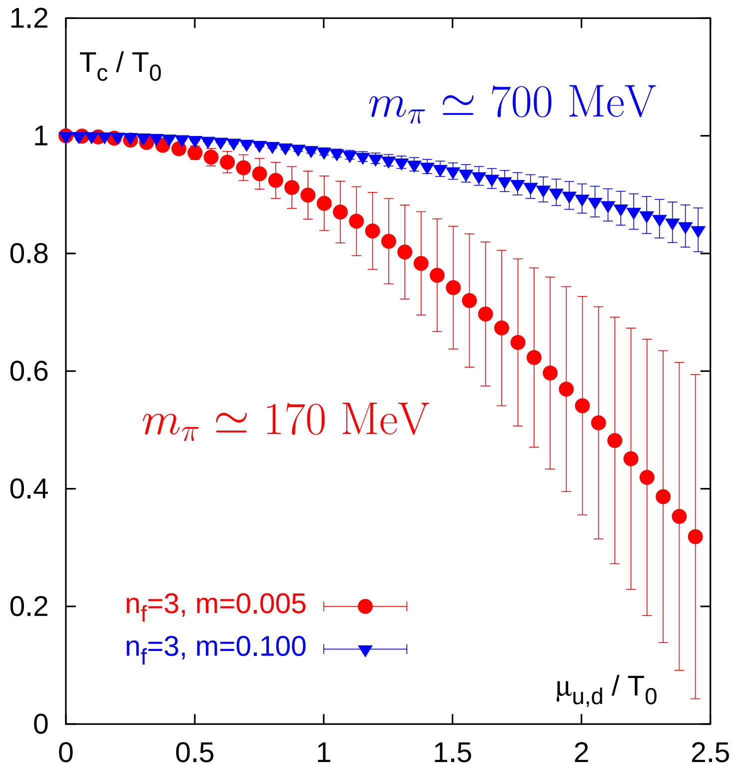
$N_F = 4$

small lattices

agreement at small μ/T
seems to hold also at
- $N_F < 4$
- bigger lattices

i) reweighting becomes unreliable

- applicable at small values for μ in the phenomenologically relevant range for RHIC, LHC
- first, exploratory results in qualitative agreement, further systematic investigations required
- in particular at smaller quark masses :



$N_F = 3$, improved action

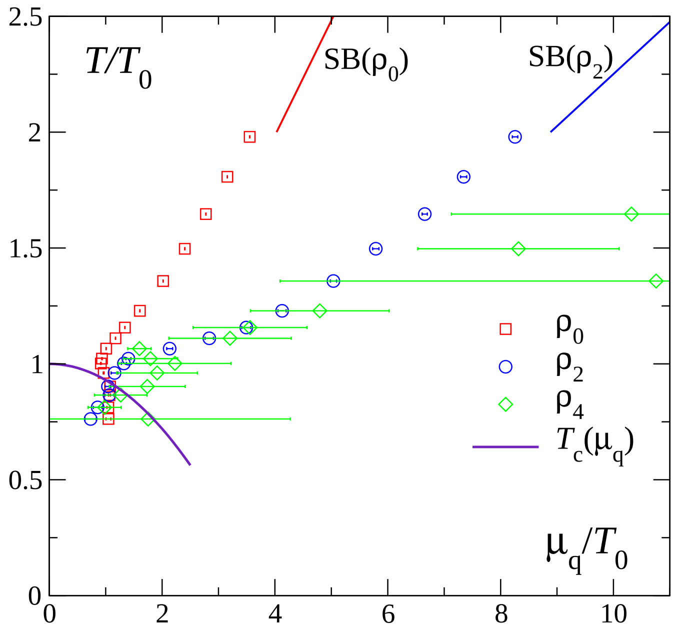
$$\frac{T_c(\mu)}{T_c(0)} = 1 - 0.025(6) \left(\frac{\mu}{T_c(0)} \right)^2$$

$$\frac{T_c(\mu)}{T_c(0)} = 1 - 0.114(46) \left(\frac{\mu}{T_c(0)} \right)^2$$

(perturbative β-function $d\beta_c / d \ln a$)

- considerable quark mass dependence

convergence radius



nearest (complex) singularity determines convergence radius

$$\rho = \lim_{k\to\infty} \rho_k = \lim_{k\to\infty} \sqrt{\left|\frac{c_k}{c_{k+2}}\right|}$$

- SB limit: $\rho_k = \infty$ for $k \geq 4$
- for T big: approaching SB limit

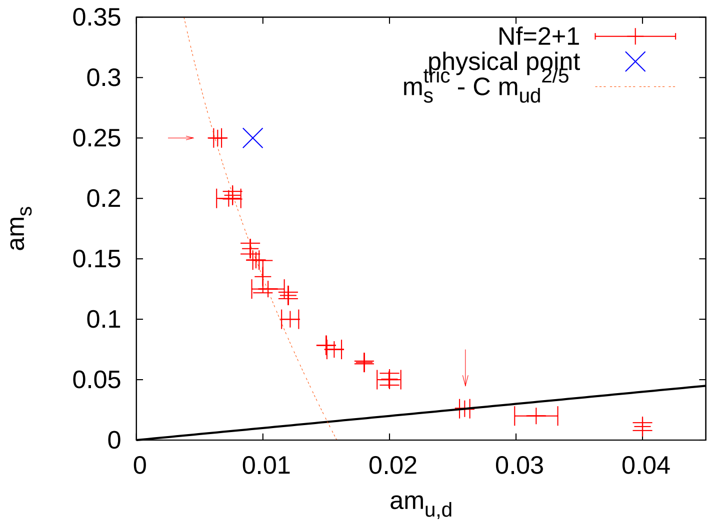
- at $T_c(\mu)$: $\rho_k \simeq 1$

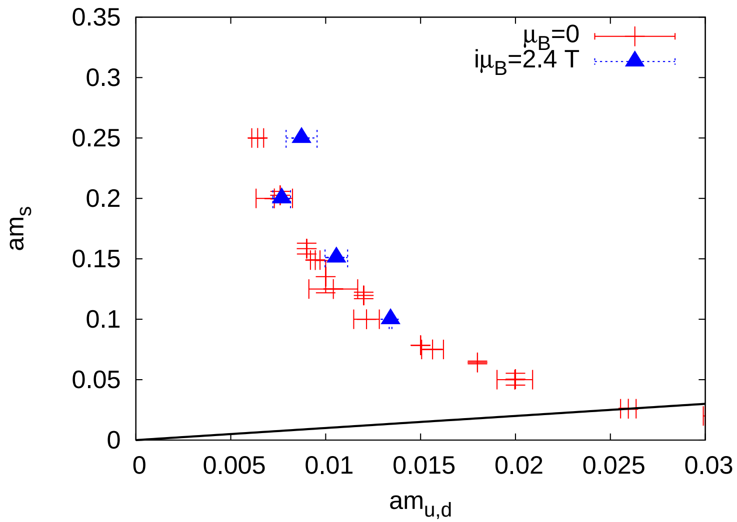
- $c_k > 0 \Rightarrow$ convergence radius indicates critical point

Results:

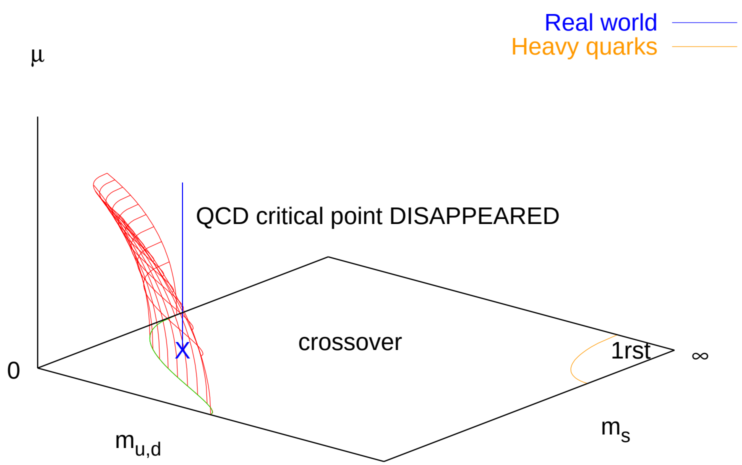
- Gavai, Gupta: $\mu_{B,E} \simeq 180\text{MeV}$ (Taylor expansion)
- Fodor, Katz: $\mu_{B,E} \simeq 360\text{MeV}$ (Lee-Yang zeroes)
- Bi-Swansea: LGT consistent with HRG, HRG analytic (Taylor expansion)

existence of a critical endpoint ? [deForcrand, Philipsen]





critical region has the tendency to grow with $\mu_I \Rightarrow$ shrink with real μ



but: finer lattices/improved actions needed

$N_\tau = 4$ here

Choice of fermions

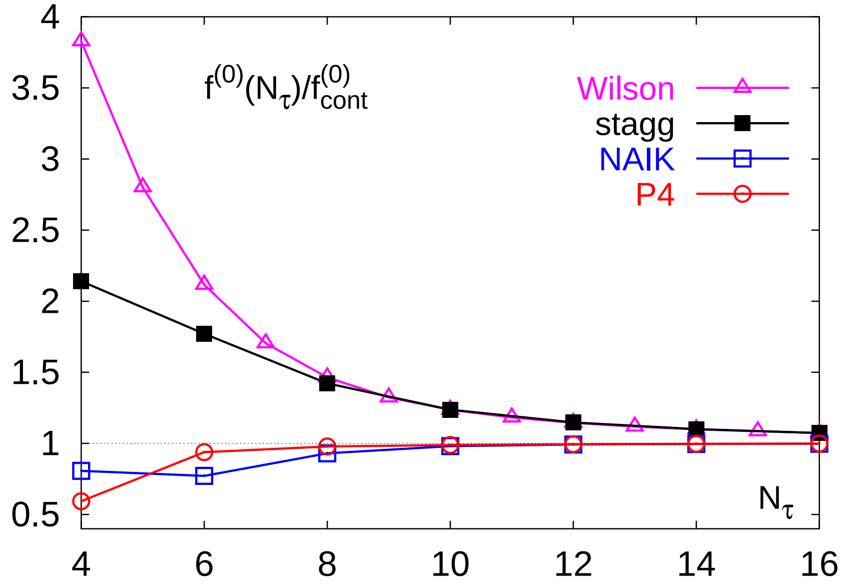
- free energy density, for instance (see later): $f/T^4 \sim N_\tau^4 \times \text{signal}$

$\Rightarrow$ signal $\sim 1/N_\tau^4$

$\Rightarrow$ keep N_τ small

$\Rightarrow$ coarse lattices $a = 1/N_\tau T$

$\Rightarrow$ improved actions



- to improve thermodynamics: Naik or `p4`

- to improve flavor/taste symmetry: fat links start from energy-momentum tensor $\frac{\Theta^\mu_\mu(T)}{T^4} = \frac{\epsilon - 3p}{T^4} = T\frac{d}{dT}(p/T^4)$

where $p = \frac{T}{V}\ln Z(T,V) - \lim_{T\to 0}\frac{T}{V}\ln Z(T,V)$ subtracting T = 0 normalization

thus $$\frac{p(T)}{T^4} - \frac{p(T_0)}{T_0^4} = \int_{T_0}^{T} dT' \frac{1}{T'^5}\Theta^\mu_\mu(T')$$

now $Z(T,V;g,m_l,m_s) = Z(N_\tau,N_\sigma,a;\beta,\hat{m}_l,\hat{m}_s) \rightarrow Z(N_\tau,N_\sigma,a;\beta,m_\pi,m_K)$

and tune bare lattice parameters $\hat{m}_l, \hat{m}_s$ with β such that $m_{\pi,K} = \text{const} \Rightarrow \hat{m}_{l,s}(\beta), a(\beta)$ **LoCP**

$$\Rightarrow \quad \frac{\Theta^\mu_\mu(T)}{T^4} = -R_\beta(\beta)N_\tau^4\left(\left\langle\frac{d\bar{S}}{d\beta}\right\rangle_T - \left\langle\frac{d\bar{S}}{d\beta}\right\rangle_{T=0}\right)$$

with $$R_\beta(\beta). = T\frac{d\beta}{dT} = -a\frac{d\beta}{da}$$

furthermore, will need $(\hat{m}_s(\beta) = \hat{m}_l(\beta) \times h(\beta))$

$$R_m(\beta). = \frac{1}{\hat{m}_l(\beta)}\frac{d\hat{m}_l(\beta)}{d\beta} \qquad R_h(\beta). = \frac{1}{h(\beta)}\frac{dh(\beta)}{dh}$$

action $S = \beta\, S_G \,+\, 2\,\hat{m}_l(\beta)\;\bar{\psi}_l\psi_l \,+\, \hat{m}_s(\beta)\;\bar{\psi}_s\psi_s \,+\, \beta$ independent

such that Θ_μ^μ consists of three pieces

$$\frac{\Theta_G^{\mu\mu}(T)}{T^4} = R_\beta\; N_\tau^4\; \Delta\langle \bar{S}_G\rangle$$

where $\Delta\langle\mathcal{O}\rangle = \langle\mathcal{O}\rangle_{T=0} - \langle\mathcal{O}\rangle_T$

$$\frac{\Theta_F^{\mu\mu}(T)}{T^4} = -R_\beta\, R_m\; N_\tau^4\; \{\, 2\,\hat{m}_l\,\Delta\langle\bar{\psi}\psi\rangle_l \,+\, \hat{m}_s\,\Delta\langle\bar{\psi}\psi\rangle_s \,\}$$

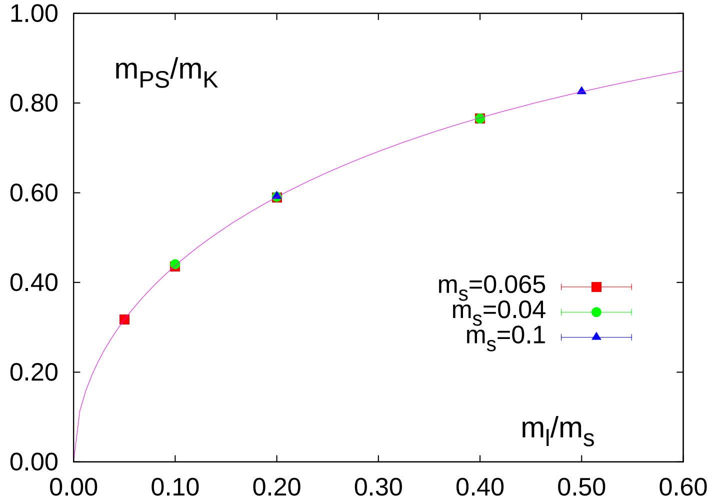
$$\frac{\Theta_h^{\mu\mu}(T)}{T^4} = -R_\beta\, R_h\; N_\tau^4\; \hat{m}_s\,\Delta\langle\bar{\psi}\psi\rangle_s$$

need: β functions $R_\beta(\beta), R_m(\beta), R_h(\beta)$

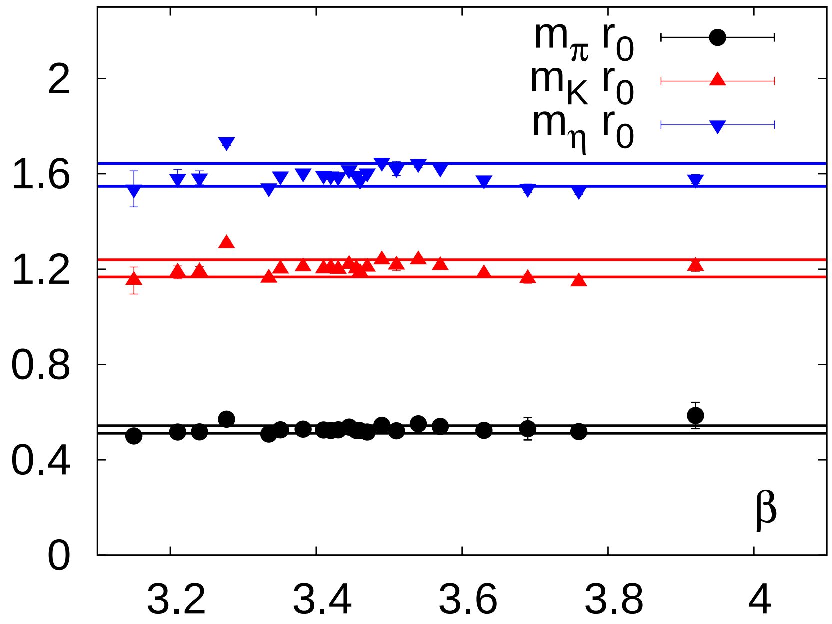
"action differences" $\Delta\bar{S}_G, \Delta\langle\bar{\psi}\psi\rangle_{l,s}$

$m_{\pi,K} =$ const: **Line of Constant Physics (LoCP)**

- to sufficient precision,
  m_π/m_K depends on $h = \hat{m}_s/\hat{m}_l$ only
  $\Rightarrow$ fix $h = 10$
  $\Rightarrow$ $R_h(\beta) = 0$

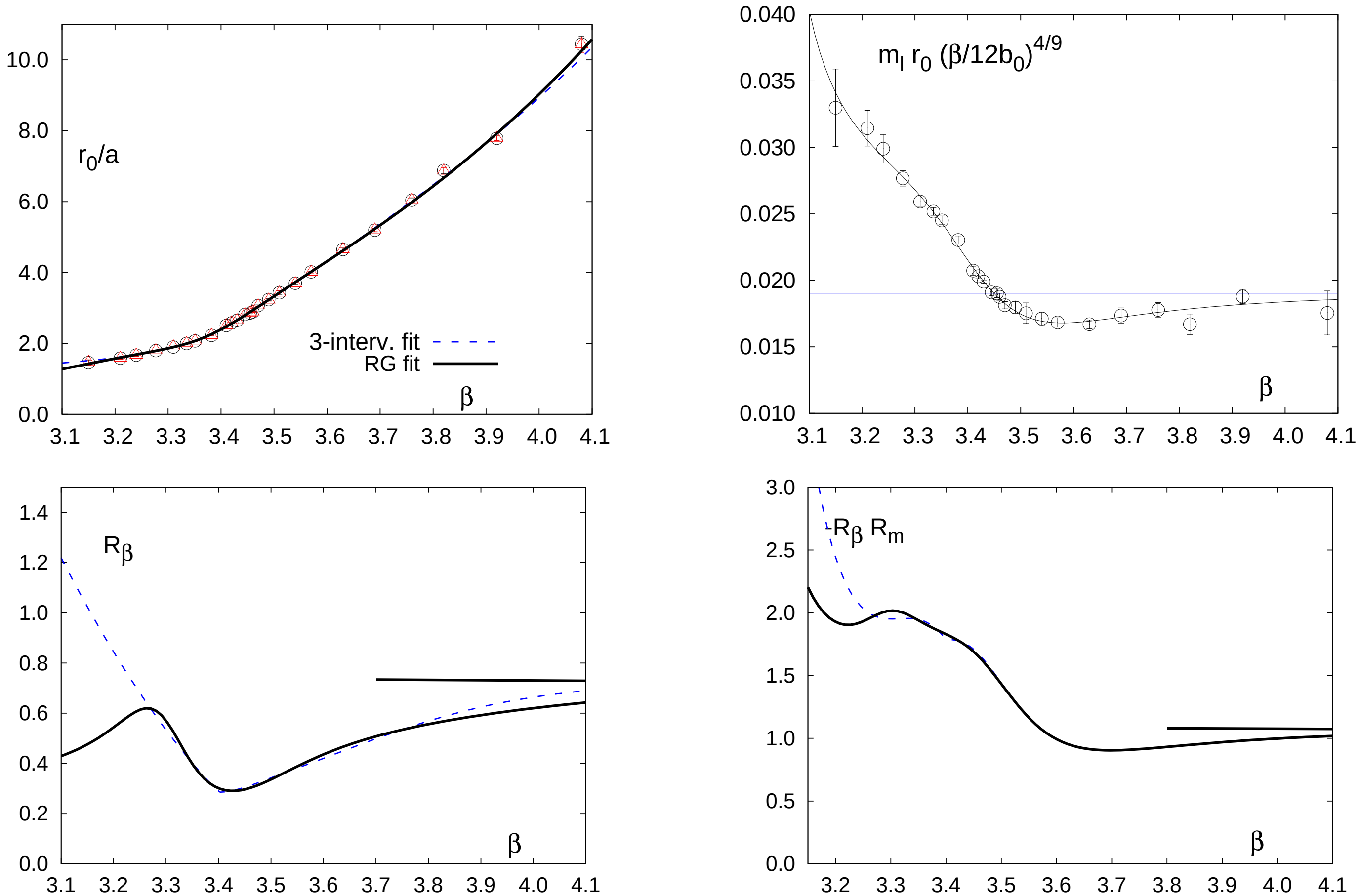


- fine tune $\hat{m}_l(\beta)$



$m_K \simeq m_K^{\rm phys}$

$m_\pi \simeq 220\,\mathrm{MeV}$



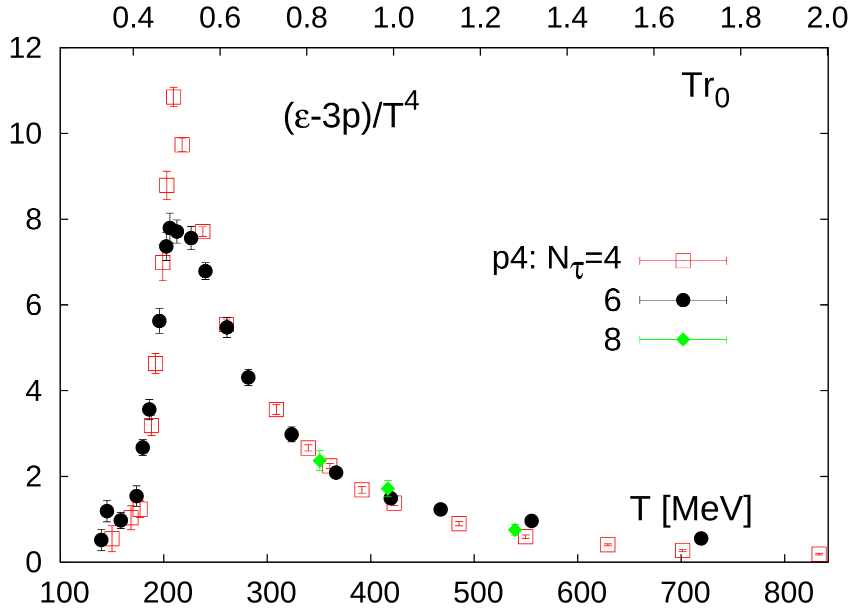
Allton inspired parametrization with rational fct. in $\hat{a}(\beta) = R_\beta^{(2-loop)}(\beta)/R_\beta^{(2-loop)}(\beta = 3.4)$

$$\frac{a}{r_0} = a_r R_\beta^{(2-loop)} \frac{1 + b_r \hat{a}^2 + c_r \hat{a}^4 + d_r \hat{a}^6}{1 + e_r \hat{a}^2 + f_r \hat{a}^4} \qquad \Rightarrow \quad R_\beta = \frac{r_0}{a}\left(\frac{dr_0/a}{d\beta}\right)^{-1}$$

$$\hat{m}_l = a_m R_\beta^{(2-loop)} \left(\frac{12 b_0}{\beta}\right)^{4/9} \frac{1 + b_m \hat{a}^2 + c_m \hat{a}^4 + d_m \hat{a}^6}{1 + e_m \hat{a}^2 + f_m \hat{a}^4 + g_m \hat{a}^6} \qquad \Rightarrow \quad R_m$$

the results:

$\Theta^{\mu}_{\mu}(T)/T^4$ (the “raw” data)

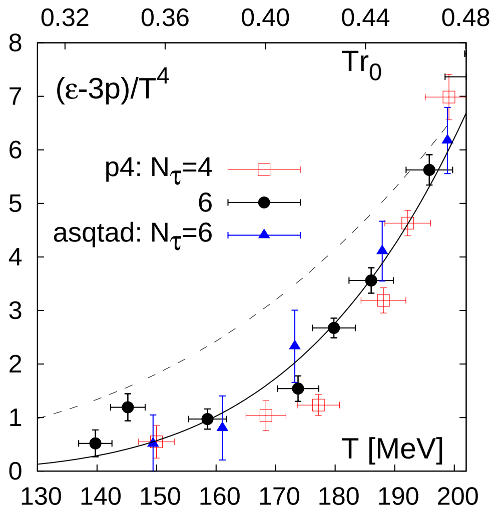


⋆ small discretization effects

⋆ $N_\tau = 8$ data corroborating

⋆ agreement with [MILC]
i.e. different, asqtad action

low T



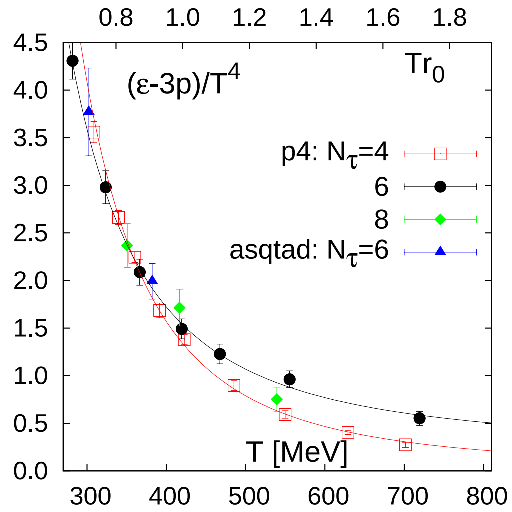
⋆ shift in T_c

⋆ dashed line:
hadron gas

high T

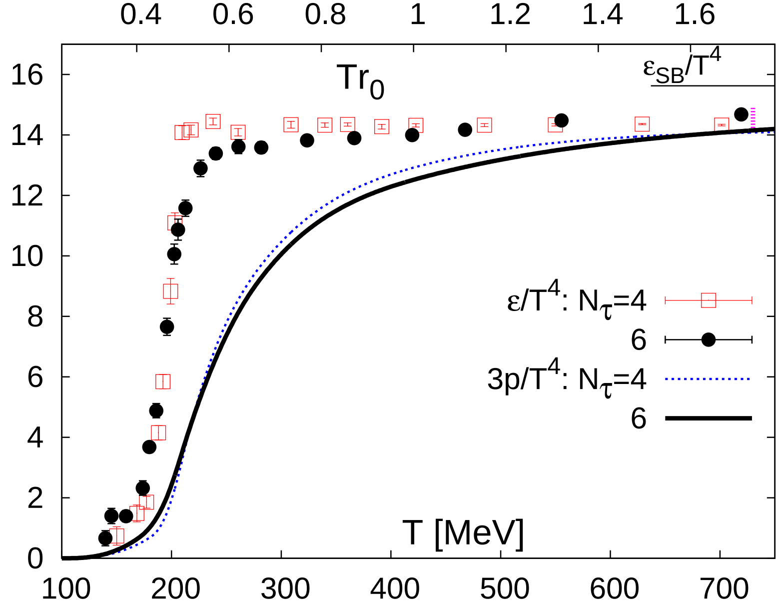
⋆ deviations from
conformal symm.

⋆ get discretization
effects controlled

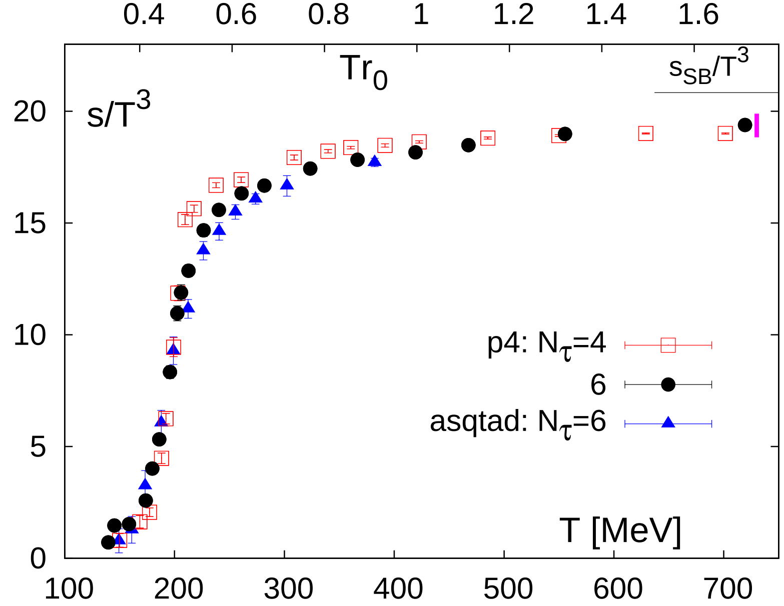


the results:

pressure and energy density



entropy density $s/T^3 = (\epsilon + p)/T^4$



⋆ discretization errors small, also good agreement with `asqtad` action

⋆ integration error: see little bar to the right

⋆ in comparison with **Stefan-Boltzmann**: 10 % below at 2 - 3 T_c

→ compare with dim.red. [Laine et al.]

Equation of State at $\mu \neq 0$: **Taylor expansion**

$$\frac{p}{T^4} = \frac{1}{VT^3} \ln Z(T, \mu_u, \mu_d, \mu_s) = \sum_{i,j,k} c_{ijk}(T) \left(\frac{\mu_u}{T}\right)^i \left(\frac{\mu_d}{T}\right)^j \left(\frac{\mu_s}{T}\right)^k$$

with $(\hat{\mu} = \mu/T)$

$$c_{ijk} = \frac{1}{i!j!k!} \frac{\partial^i}{\partial \hat{\mu}_u^i} \frac{\partial^j}{\partial \hat{\mu}_d^j} \frac{\partial^k}{\partial \hat{\mu}_s^k} \left(\frac{p}{T^4}\right)\Bigg|_{\vec{\mu}=0}$$

for instance

$$c_{200} = \frac{N_\tau}{2N_\sigma^3} \left(\frac{1}{4} \left\langle \frac{\partial^2 \ln \det M}{\partial \hat{\mu}_u^2} \right\rangle + \frac{1}{16} \left\langle \left(\frac{\partial \ln \det M}{\partial \hat{\mu}_u}\right)^2 \right\rangle\right)$$

with

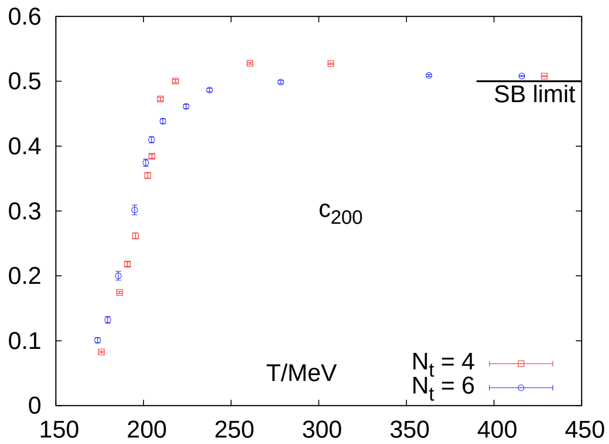
$$\frac{\partial^2 \ln \det M}{\partial \hat{\mu}_u^2} = \mathrm{tr}\left(M^{-1} \frac{\partial^2 M}{\partial \hat{\mu}_u^2}\right) - \mathrm{tr}\left(M^{-1} \frac{\partial M}{\partial \hat{\mu}_u} M^{-1} \frac{\partial M}{\partial \hat{\mu}_u}\right)$$

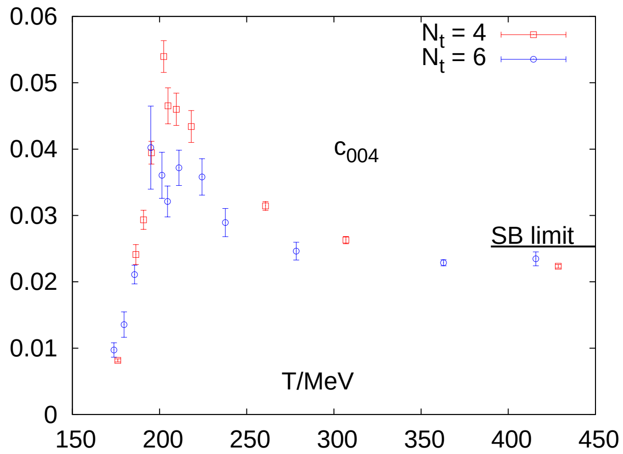
For degenerate u and d quarks: $\mu_u = \mu_d \equiv \mu_q \;\Rightarrow$ e.g.

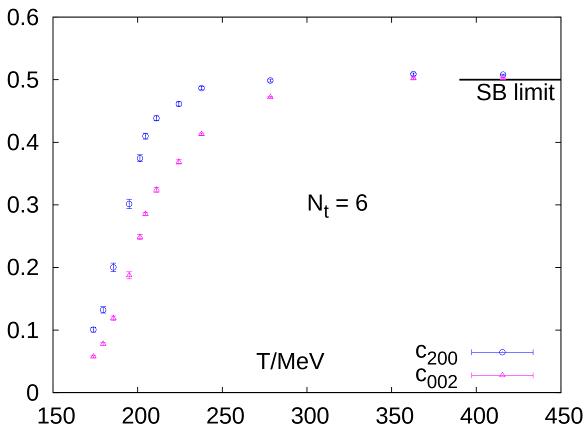
$$c_{20}^{qs} = c_{200} + c_{020} + c_{110}$$

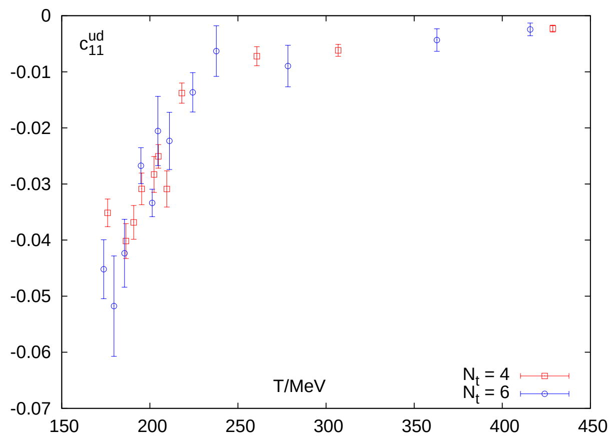
note: $c_{ijk} = 0$ for $i + j + k$ odd because of charge symmetry

some examples of the coefficients









⋆ some but not large discretization effects

⋆ approaching SB limit quickly

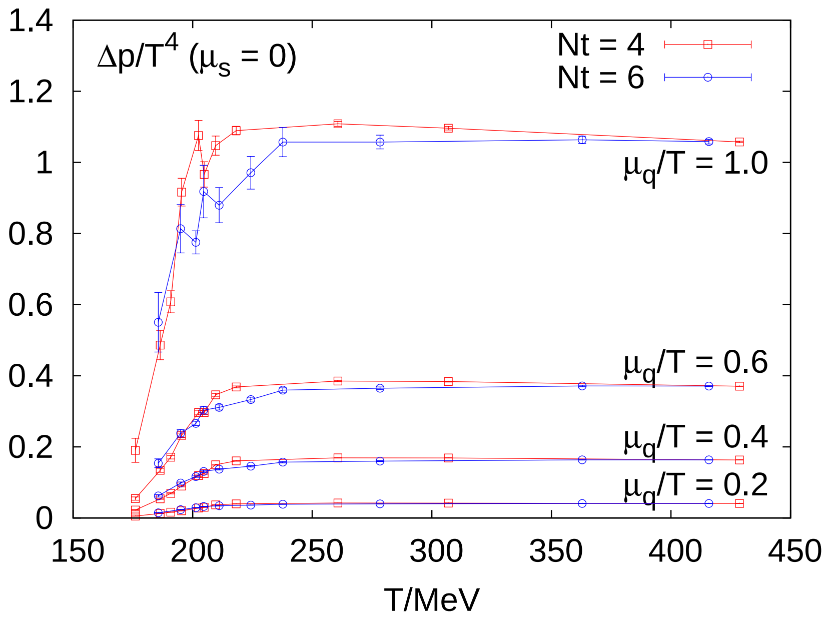
⋆ rapid rise in quadratic coeff.

⋆ peaks in quartic coeff.

⋆ small but non-vanishing off diagonal coeff.

⇒ **pressure** difference

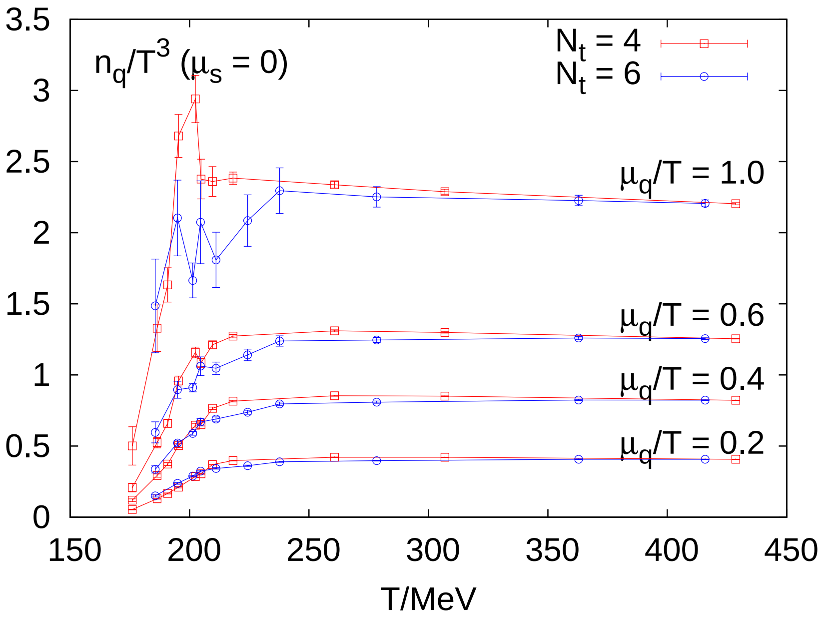
$$\Delta p = p(\vec{\mu}) - p(\vec{\mu} = 0)$$



⋆ small discretization effects

⋆ rapid rise at $T \simeq 200$ MeV

⋆ small contribution compared to $\mu = 0$



⇒ **number density**

$$\frac{n_q}{T^3} = 2\, c_{20}^{qs}\, \hat{\mu}_q + \ldots$$

⋆ vanishing at $\mu_q = 0$

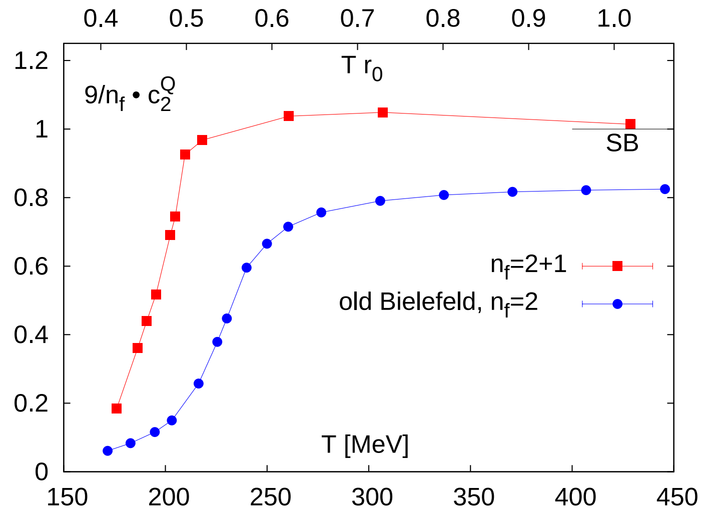
⋆ with rising μ_q , developing a peak more physical in terms of B, S, Q quantum numbers

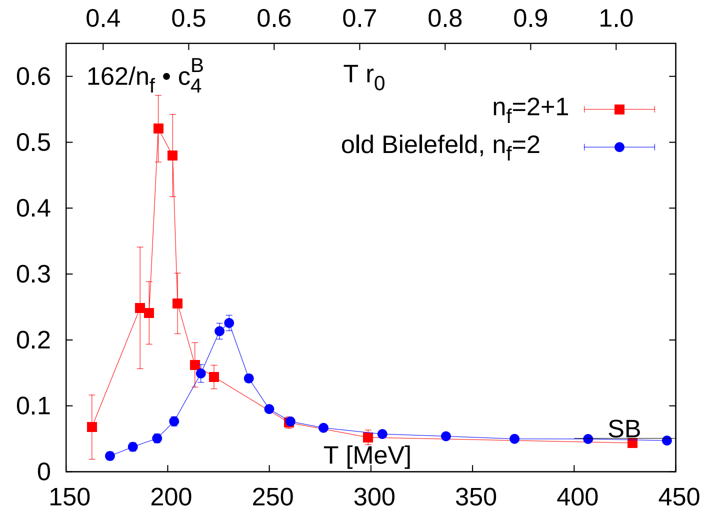
$$\frac{p}{T^4} = \sum_{i,j,k} c_{ijk}^{BSQ}(T) \left(\frac{\mu_B}{T}\right)^i \left(\frac{\mu_S}{T}\right)^j \left(\frac{\mu_Q}{T}\right)^k$$

where $\mu_u = \frac{1}{3}\mu_B + \frac{2}{3}\mu_Q \qquad \mu_d = \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q \qquad \mu_s = \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q - \mu_S$

for instance $c_{400}^{BSQ} = \frac{1}{81}\left(c_{40}^{qs} + c_{31}^{qs} + c_{22}^{qs} + c_{13}^{qs} + c_{04}^{qs}\right)$

leading to





$c_4^B = c_{400}^{BSQ}$

$c_2^Q = c_{002}^{BSQ}$

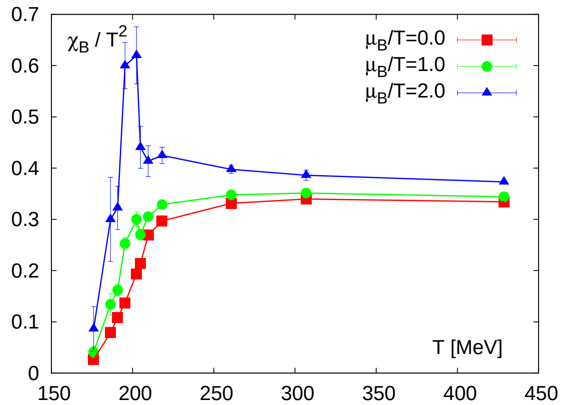
old Bielefeld: $m_{PS} \simeq 700$ MeV

⇒ investigate **quantum number fluctuations** → event-to-event fluctuations in HIC

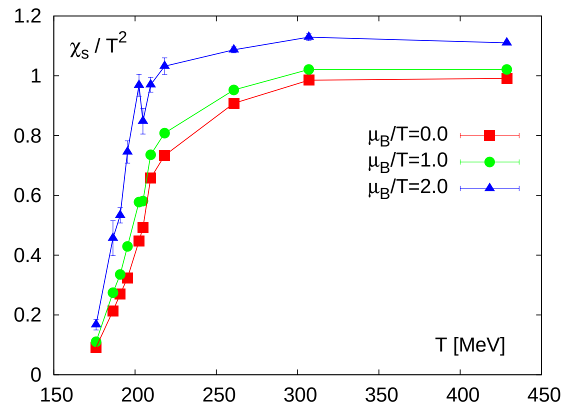
at $\mu_S = 0$: (note: at $\mu_u = \mu_d$ follows $\mu_Q = 0$)

$$\chi_X \sim \langle n_X^2 \rangle - \langle n_X \rangle^2$$

$$\frac{\chi_B}{T^2} = 2\, c_{200}^{BSQ} + 12\, c_{400}^{BSQ} \left(\frac{\mu_B}{T}\right)^2 + \dots$$



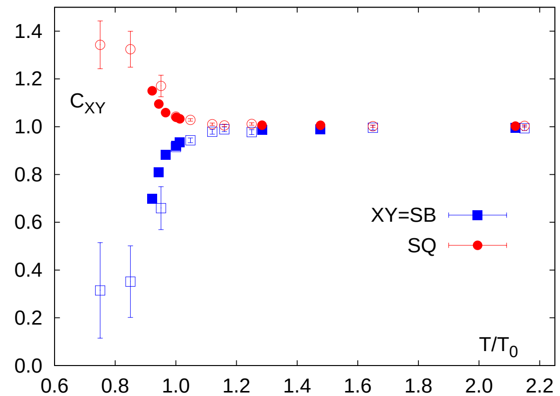
$$\frac{\chi_S}{T^2} = 2\, c_{020}^{BSQ} + 2\, c_{220}^{BSQ} \left(\frac{\mu_B}{T}\right)^2 + \dots$$



⋆ peaks developing

suggest approaching the critical endpoint

$3\langle n_S n_Y \rangle / \langle n_S^2 \rangle$ [Gavai, Gupta]



⋆ $C_{SY} \simeq 1$ signals

$S = 1$ carried by $B, Q = \pm 1/3$

Outlook

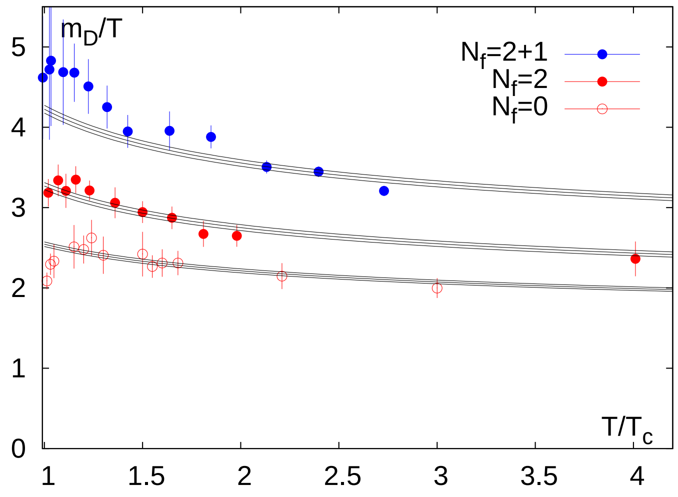
further subjects:

- forces (between static quarks)

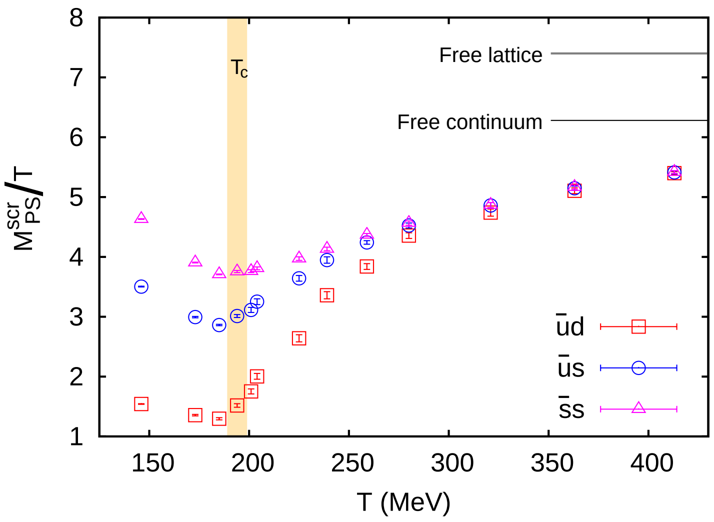
$$< L(0)L(R) > \sim \; e^{-F(R)/T}$$

screening

$$F(R) = -\frac{4}{3}\frac{\alpha(T)}{R}e^{-m_D(T)R}$$



- screening masses/lengths (glueballs, mesons)



- real time properties

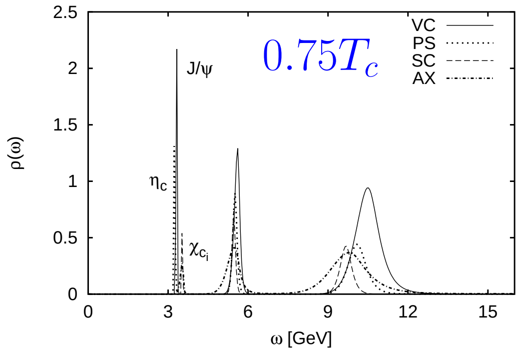
$$\sigma(\omega, \vec{p}) \sim \int d^4x \; e^{ipx} \;\; < [\Phi(x) \, , \, \Phi(0)] >$$

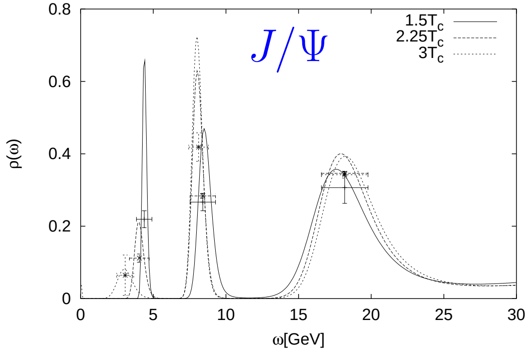
$\rightsquigarrow$ dilepton rates $\quad dW \sim \sigma_V$

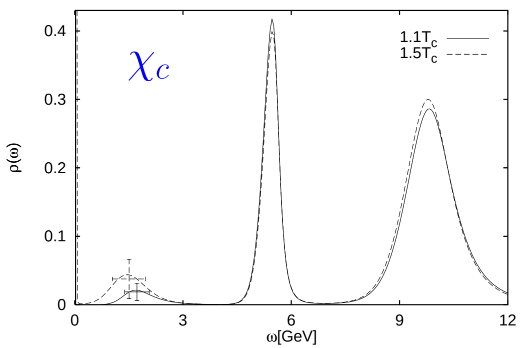
$\rightsquigarrow$ transport coefficients, e.g. $\rho \sim d\sigma_V/d\omega$

heavy quarks

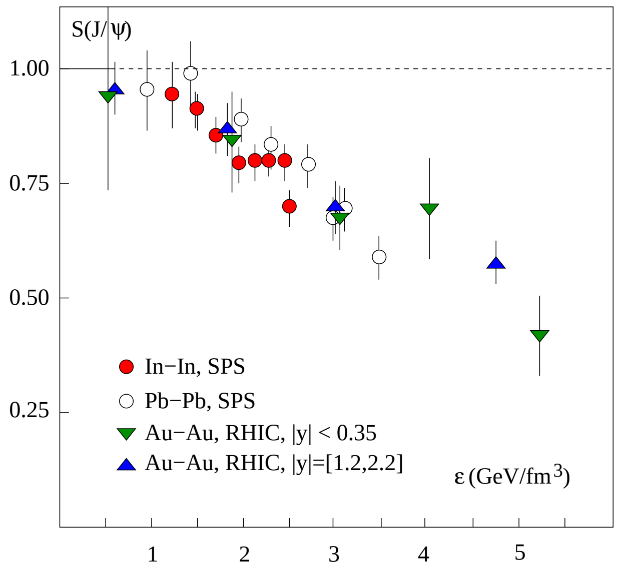
e.g. J/Ψ suppression







- individual "melting" temperatures $> T_c$
- suppression patterns



LHC:
- large abundance of c-quarks
- feed down from b-quarks