# PROBLEM ON PARTICLES & SYMMETRIES

**A spinless particle coupled to the photon**

The framework is the relativistic quantum mechanics.

1. Let us consider the covariant Lagrangian density $\mathcal{L} = \mathcal{L}_F + \mathcal{L}_\phi$ with $\mathcal{L}_F = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}$ and,

$$\mathcal{L}_\phi = \frac{1}{2}(\partial_\mu \phi + i\frac{q}{\hbar c} A_\mu \phi)\,(\partial^\mu \bar{\phi} - i\frac{q}{\hbar c} A^\mu \bar{\phi}) - \frac{m^2}{2}\frac{c^2}{\hbar^2}\,\bar{\phi}\phi\,,$$

$\mu, \nu$ being *Lorentz* indices running from 0 to 3, $F^{\mu\nu}$ a **rank-2 tensor** (depending only on $A^\mu$), $\partial_\mu$ the **4-vector** derivation, $\phi$ the **scalar** wave function ($\bar{\phi}$ the complex conjugate) of a spinless particle [with electric charge $q$ and mass $m$] and $A_\mu$ the electromagnetic field (photon) which behaves as a **4-vector** in the covariant formalism [1].

   (a) By using the *Lorentz* matrix $\Lambda^{\mu}_{\cdot\,\nu}$ (and its inverse), show that $\mathcal{L}_F$ is a *Lorentz* invariant.

   (b) Demonstrate that $(\partial_\mu \phi) A^\mu = (\partial^\mu \phi) A_\mu$ using the *Minkowski* metric [2].

2. Apply the following *Euler-Lagrange* equation on $\mathcal{L}$, and, comment about the obtained result.

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial [\partial_\mu \phi]}\,.$$

3. Based on previous question, comment about this equation:

$$\left[\, \bar{D}_\mu \bar{D}^\mu + \frac{m^2 c^2}{\hbar^2} \,\right] \bar{\phi} \;=\; 0\,, \text{with, } \bar{D}_\mu \hat{=} \partial_\mu - i\frac{q}{\hbar c} A_\mu\,. \qquad (1)$$

4. Calculate the linear combination $\phi \times (1) - \bar{\phi} \times (\bar{1})$ of Equation (1) [3].

   (a) Express the resulting equation exclusively in terms of quantities of the type $\phi(\partial_\mu \partial^\mu \phi)$, $2\phi\phi \times (\dots)$ or $2\phi(\partial_\mu \phi) \times (\dots)$ [possibly with $\bar{\phi}$ instead of some $\phi$].

   (b) Write the obtained equation under the covariant form $\partial_\mu j^\mu = 0$ (provide explicitly the 4-current $j^\mu$). Is $j^\mu$ a 4-vector?

   (c) Setting $j^\mu \hat{=} (\rho c, \vec{j})$, express the condition $\partial_\mu j^\mu = 0$ in terms [4] of the $j^\mu$ components and the gradient vector $\vec{\nabla}$. What does represent physically the obtained equation? Calculate the covariant $j_\mu$ components.

[1] Using standard notations, $\hbar$ denotes the *Planck* constant, $c$ the light velocity and $\mu_0$ the vacuum magnetic permeability.
[2] Throughout all the Particle Physics part of the exam, we consider the convention $g^{\mu\nu} = diagonal(+1, -1, -1, -1)$.
[3] $(\bar{1})$ denotes the complex conjugate of Equation (1) with $A_\mu$ taken to be real.
[4] Recall that $\partial_\mu \hat{=} \frac{\partial}{\partial x^\mu}$ where the contravariant space-time coordinates read as $x^\mu = (ct, x, y, z)$.

5. For an isolated charge $q = -e$ in a central *Coulomb* potential $V(r)$ created by an atomic nucleus containing $Z$ protons [5], one has $A^0(r) = ZeV(r)$, $\vec{A} = \vec{0}$ and a stationary solution of Equation (1) has the form [$E$ is the charged particle energy]:

$$\phi(\vec{r}, t) = f(\vec{r})\, e^{-i\frac{Et}{\hbar}}.$$

   (a) For such a solution, calculate the quantity: $\mathcal{Q} = q(\phi\partial^0\bar{\phi} - \bar{\phi}\partial^0\phi - 2i\frac{q}{\hbar c}A^0\phi\bar{\phi})$.

   (b) Make physical remarks about the previous result regarding in particular the obtained sign.

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[5] $V(r) \propto 1/r$ for large $r$ values (where $r^2 = x^2 + y^2 + z^2$) with respect to the nuclear radius.