PROBLEM ON PARTICLES & SYMMETRIES

The stress tensor

(lecture notes not allowed; write on a separate sheet; show the number of the treated question; distinguish clearly the demonstration and the result; justify your answers)

1. The theoretical framework in this part is quantum mechanics. Let us consider the Lagrangian density [1],

$$\mathcal{L} = -\frac{1}{2m}\vec{\nabla}\phi^*.\vec{\nabla}\phi - \frac{1}{2i}\left(\phi^*\frac{\partial\phi}{\partial t} - \frac{\partial\phi^*}{\partial t}\phi\right) - \phi^*\, V(\vec{r},t)\,\phi\;,$$

   for a spinless particle of mass m described by a wave function $\phi(\vec{r},t)$. The exponent * stands for the complex conjugation, $\vec{\nabla}$ for the gradient vector and V is an energy potential.

   (a) Find the dimension of the ϕ field entering this Lagrangian density within the natural unit system. Is the result consistent with the standard wave function definition?

   (b) In analytical mechanics, with $\phi, \dot{\phi} = \frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial r_i}$ $[i = 1, 2, 3]$ as the fundamental variables, the Hamiltonian density is defined as a component of **the stress tensor**: $\mathcal{H} = \dot{\phi}^*\frac{\partial\mathcal{L}}{\partial\dot{\phi}^*} + \dot{\phi}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \mathcal{L}$. Calculate this $\mathcal{H}$.

   (c) Compute the Hamiltonian by integrating $\mathcal{H}$ over the whole physical space. Apply an integration by part for one of the two terms. Comment on the obtained result (field value at infinity and Hamiltonian operator).

2. The framework of this part [2] is the relativistic quantum mechanics. We consider the covariant Lagrangian density (natural unit system),

$$\mathcal{L}_c = \frac{1}{2}\,\partial_\mu\phi^*\,\partial^\mu\phi - \frac{m^2}{2}\,\phi^*\phi\;,$$

   of a free spinless particle with mass m. $\phi(\vec{r},t)$ is the *Klein-Gordon* equation solution, the exponent * stands for the complex conjugation, μ is a *Lorentz* index running from 0 (time component) to 3 and $\partial_\mu = \frac{\partial}{\partial x^\mu}$ is the 4-vector derivation, x^μ being the 4-coordinates.

   (a) By using the *Lorentz* matrix $\Lambda^\mu_{\cdot\nu}$ (and its inverse), show in details that $\mathcal{L}_c$ is a *Lorentz* invariant. Give the physical interpretation of this invariance. Is the corresponding action invariant as well?

   (b) Demonstrate that $\partial_\mu\phi^*\partial^\mu\phi = \partial^\mu\phi^*\partial_\mu\phi$ using the *Minkowski* metric $g^{\mu\nu}$.

[1] Using the natural unit system where $\hbar = c = 1$.

[2] Independent from part 1.

(c) The *Lorentz* invariance leads [3] to the local conservation relation $\partial_\mu T^{\mu\nu} = 0$ where

$$T^{\mu\nu} = \partial^\mu \phi^* \frac{\partial \mathcal{L}_c}{\partial[\partial_\nu \phi^*]} + \partial^\mu \phi \frac{\partial \mathcal{L}_c}{\partial[\partial_\nu \phi]} - \mathcal{L}_c \, g^{\mu\nu}$$

is **the stress tensor**. Calculate $T^{\mu\nu}$ (let the third term as it is). What is its rank?

(d) Calculate [4] the Hamiltonian density $\mathcal{H} = T^{00} = \partial^0 \phi^* \partial^0 \phi - \mathcal{L}_c \, g^{00}$, defined in analytical mechanics with $\phi, \partial^\mu \phi$ as the fundamental variables. Express $\mathcal{H}$ in terms of $\phi^{(*)}$, $\vec{\nabla}\phi^{(*)}$ and $\frac{\partial \phi^{(*)}}{\partial t}$.

(e) Assuming that $\phi(\vec{r}, t) = \sqrt{\frac{1}{E_n L^3}} \, e^{i(\vec{p}_n . \vec{r} \pm E_n t)}$ within a volume L^3, calculate $\phi^* \phi$ and $\frac{\partial \phi^*}{\partial t} \frac{\partial \phi}{\partial t}$. $\vec{p}_n$ and E_n represent respectively the momentum and energy eigenvalues for the quantum state level n.

(f) For the $\phi(\vec{r}, t)$ solution form of previous question, calculate $\mathcal{H}$ as a function of E_n and L^3 exclusively, by using the classical relativistic energy expression $E_n^2 = \vec{p}_n^{\,2} + m^2$.

(g) Integrate the Hamiltonian density over the volume L^3 to get the Hamiltonian. What can be concluded?

(h) In the present analytical mechanics context, $P^\nu = \int \int \int_{-\infty}^{+\infty} d^3x \, T^{0\nu}$ represents the total 4-momentum. Indeed, show that P^0 represents the Hamiltonian and demonstrate the global conservation relation $\frac{\partial P^\nu}{\partial t} = 0$ (for the latter purpose, integrate the relation $\partial_\mu T^{\mu\nu} = 0$ over the whole space and apply the *Gauss* theorem to the part $\sum_{i=1}^{3} \partial_i T^{i\nu}$).

(i) The stress tensor is also called **the Energy-Momentum tensor**. Express the relation $\partial_\mu T^{\mu 0} = 0$ as a continuity equation (knowing that $T^{i0} = T^{0i}$). Interpret physically this equation.

(j) Comment on the complementary relations $\partial_\mu T^{\mu i} = 0$ [$i = 1, 2, 3$].

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[3] Through the *Noether*'s theorem.

[4] Throughout all the Particle Physics part of the exam, consider the metric convention $g^{\mu\nu} = diagonal(+1, -1, -1, -1)$.