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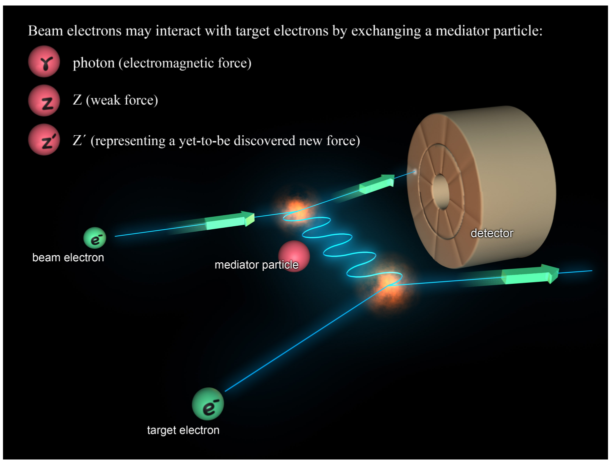
Particles and Symmetries

Grégory Moreau*

Laboratoire de Physique Théorique, Bât. 210, Université Paris-sud 11
F-91405 Orsay Cedex, France

Abstract

The present notes provide exclusively a synthesis of the formalism presented in the lecture series "Particles and Symmetries" (part of the Major Course "Particles, Nuclei and Universe") at the 1st year Master *General Physics* of the Paris-Saclay University. After recalling the basics about the covariant formalism in special relativity, I introduce the Klein-Gordon equation. Then the time-dependent perturbation theory is presented and the Fermi's golden rule demonstrated. The special case of electromagnetic interactions is described within the covariant formalism (adopting a Lagrangian approach). All these physical tools are then used to explain the Quantum ElectroDynamics [Q.E.D.] theory, in the simplified context of spinless charged particles. The Feynman rules are derived, and, applied to the calculation of amplitudes – and cross sections – for simple scattering reactions.



*moreau@th.u-psud.fr

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1 Special relativity: the covariant formalism

From the principle of relativity the fundamental laws have the same form in any Galilean frame.

$$\underline{\text{Galilean transformation}}\begin{cases} x'^i &= R^i_{.j}\, x^j - V'^i t + a^i \\ \\ t' &= (\pm)t + t^0 \end{cases}$$

$$\underline{\text{Lorentz transformation :}}\ \ x'^\mu = \Lambda^{\mu}_{.\nu}\ x^\nu + a^\mu$$

$$\text{where}\quad x^\nu = (ct, x, y, z)$$

$$\Lambda^{\mu}_{.\nu}\ (\beta) = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \text{ Det } \equiv \gamma^2 - \gamma^2\beta^2 = \gamma^2(1-\beta^2) = 1$$

"Algebric measure" (in its own direction): $\quad \beta = \overline{V}/C$

$$\gamma = 1/\sqrt{1-\beta^2}$$

$$\text{Limiting case } \gamma \xrightarrow[\beta \ll 1]{} 1 : \begin{cases} ct' &= \gamma ct - \gamma\beta x \\ x' &= -\gamma\beta ct + \gamma x \\ y' &= y \qquad z' = z \end{cases}$$

$\beta c = \overline{V} = \parallel \vec{V}_{F'/F} \parallel = V'^1 > 0$

Remark : $[\Lambda^{-1}]^{\mu}_{.\nu}(\beta) = \Lambda^{\mu}_{.\nu}(-\beta)$

The principle of relativity dictates that :
$\exists$ some equation structures invariant as changing Lorentz frame.

ex : $x'^\mu x'_\mu = x^\nu x_\nu \longmapsto$ "covariant form" i.e. invariance explicit for $x^\nu x_\nu$

$$\Lambda^\mu_{\ \nu}\ x^\nu\ \Lambda_\mu^{\ \beta}\ x_\beta = x^\nu x_\nu \qquad \boxed{\Lambda^\mu_{\ \nu}\ \Lambda_\mu^{\ \beta} = \delta^\beta_\nu} \qquad \text{with} \qquad [\Lambda^{-1}]^{\mu}_{.\nu}(\beta) = \Lambda_{\nu.}^{.\mu}(\beta)\ .$$

Direct use $\Rightarrow dx'^{\mu}\ dx'_{\mu} = dx^{\nu}\ dx_{\nu}$.

Mathematical framework: Riemann space R_4.

$$g^{\mu\nu}\ x_{\nu} = x^{\mu}, \quad \text{with the metric tensor for flat space} \quad g^{\mu\nu} \equiv \begin{pmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$$\begin{aligned} c^2dt^2\left(1-\left(\frac{V}{c}\right)^2\right) &= c^2dt^2 - d\vec{x}^2 = ds^2 \\ &= dx_{\mu}\ g^{\mu\nu}\ dx_{\nu} \\ &= dx^{\nu}\ dx_{\nu} \end{aligned}$$

- Property:

$$g^{\mu\nu}g_{\nu\rho} = \delta^{\mu}_{\rho} \Rightarrow g^{\mu\nu}g_{\nu\mu} = 4$$

- 4-Vectors:

$$\begin{aligned} &\text{ex :} \quad dx'^{\nu} = \Lambda^{\nu\cdot}_{\cdot\mu}\ dx^{\mu} \\ &\text{ex :} \quad p^{\mu} = (E/c, \vec{p}) = (\gamma mc, \gamma m\vec{V}) = mV^{\mu} \end{aligned}$$

$$\begin{aligned} \text{Indeed,}\ p^{\mu}p_{\mu} &= \gamma^2m^2c^2(1-\beta^2) = m^2c^2 \qquad \left(\Rightarrow E = \frac{mc^2}{(1-\beta^2)^{1/2}}\right) \\ &= \frac{E^2}{c^2} - \vec{p}^2\ \widehat{=}\ m^2c^2 \\ (\beta \ll 1) \qquad &\simeq mc^2 + \frac{1}{2}mv^2 + \ldots \end{aligned}$$

- Tensors:

$$T'^{\mu\nu}\rho = \Lambda^{\mu\cdot}_{\cdot\alpha}\ \Lambda^{\nu\cdot}_{\cdot\beta}\ \Lambda^{\cdot\gamma}_{\rho}\ T^{\alpha\beta}\gamma$$

- Fields:

$$\begin{aligned} \delta\phi(x^{\alpha}) &= \frac{\partial\phi(x^{\alpha})}{\partial x^{\nu}}\ \delta\ x^{\nu} \\ \text{For a ``scalar field''}\ \phi'(x'^{\mu}) &= \phi(x^{\mu}) \\ \text{not like}\ A'^{\mu}(x'^{\nu}) &= \Lambda^{\mu\cdot}_{\cdot\alpha}\ A^{\alpha}(\Lambda^{\nu\cdot}_{\cdot\beta}x^{\beta}) \\ \Rightarrow \frac{\partial}{\partial x^{\nu}}\ \widehat{=}\ \partial_{\nu}\ \text{must be covariant}\ \Rightarrow \partial'_{\mu} &= \Lambda^{\cdot\nu}_{\mu\cdot}\ \partial_{\nu} \end{aligned}$$

$$\boxed{\partial_{\nu}\ \widehat{=}\ \left(\frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)}$$

- Operators:

$$\text{ex : d'Alembertian} \quad \Box^{(2)} \equiv \partial^\mu \partial_\mu = \frac{\partial^2}{c^2 \partial t^2} - \partial_x^2 - \partial_y^2 - \partial_z^2 = \frac{\partial^2}{c^2 \partial t^2} - \underbrace{\vec{\nabla}^2}_{\text{``}Laplacian\text{''}\,\Delta}$$

- Integration:

$$d^4x' = \left| \frac{\partial x'^\mu (x^\nu)}{\partial x^\nu} \right| d^4x = |\text{Det } \Lambda^{\mu\cdot}_{\cdot\nu}|\ d^4x = d^4x \quad \text{(Jacobian)}.$$

$\boxed{\text{Natural unit system}}$

To simplify calculations, it is customary to set, at a first level, $\hbar = c = 1$. Then,

$$\begin{array}{||lcl}
E = mc^2 & \longrightarrow & M = [E], \text{ as } E = m \\
E = p^2/2m & \longrightarrow & P^2 = [E]^2 \\
\Delta x \Delta p \gtrsim h & \longrightarrow & L = [E]^{-1} \\
\Delta t \Delta E \gtrsim \hbar & \longrightarrow & T = [E]^{-1} \\
F = m\frac{d^2x}{dt^2} & \longrightarrow & [F] = [E]^{1-1-(-2)} \\
[S] = [\hbar] & \longrightarrow & [S] = [E]^0 = 1 \\
\vec{F} = \frac{q_1 q_2}{4\pi r^2}\ \hat{r} & \longrightarrow & [q]^2 = [E]^2 L^2 = 1 \qquad \text{(Coulomb's law)}
\end{array}$$

with absorbed inverse vacuum permittivity $1/\epsilon_0$ ($[\epsilon_0] = N^{-1} m^{-2} C^2$) and $\epsilon_0\ \mu_0 = \frac{1}{c^2} = 1$ so that $\mu_0 = 1 \Rightarrow \epsilon_0 = 1$.

2 The Klein-Gordon equation

2.1 The Schrödinger equation

$$\rho = \left|\psi(x^i,t)\right|^2 = \underbrace{\frac{d^3P(x)}{dx^3}}_{\substack{|\psi|^2 d^3x \\ \rho\ d^3x}}$$

1.− classical energy for a free particle system

$$E : \vec{p}^2/2m$$

2.− substituting the differential operators

$$\begin{cases} E & \longrightarrow & i\hbar\partial_t \\ \vec{p} & \longrightarrow & -i\hbar\vec{\nabla} \end{cases}$$

3.− one obtains the Schrödinger equation

$$\boxed{\frac{-\hbar^2}{2m}\ \Delta\psi(\vec{x},t) = i\hbar\ \partial_t\psi(\vec{x},t)}$$

<u>Probability current:</u>

Condition of conservation of the probability of presence :

$$-\int_V d^3x\ \frac{\partial\ n\rho}{\partial t} = \int_{S(V)} n\ \vec{j}.d\vec{S} = \int_V \vec{\nabla}.n\vec{j}\ d^3x \qquad \text{(from the Gauss theorem)}$$

$$\forall\ V\ \Rightarrow \qquad \boxed{\vec{\nabla}.\vec{j} + \tfrac{\partial\rho}{\partial t} = 0} \quad \text{with}\ \ \dot{\rho} \mathrel{\hat{=}} \frac{\partial\rho}{\partial t}\ \ \text{(continuity equation)}$$

$$\left.\begin{array}{rcl} \dot{\rho} & = & \partial_t(\psi\psi^*) = \dot{\psi}\psi^* + \psi\dot{\psi}^* = \left(\frac{i\hbar}{2m}\Delta\psi\right)\psi^* + \psi\left(-\frac{i\hbar}{2m}\Delta\psi^*\right) = \frac{i\hbar}{2m}(\Delta\psi\psi^* - \psi\Delta\psi^*) \\ \vec{j} & = & \frac{i\hbar}{2m}(\vec{\nabla}\psi\psi^* - \psi\vec{\nabla}\psi^*) \quad \Rightarrow \quad \vec{\nabla}.\vec{j} = -\frac{i\hbar}{2m}(\Delta\psi\psi^* + \cancel{\vec{\nabla}\psi\vec{\nabla}\psi^*} - \cancel{\vec{\nabla}\psi\vec{\nabla}\psi^*} - \psi\Delta\psi^*) \end{array}\right\}$$

The sum of these two lines vanishes and one recovers the above continuity equation.

Check of the (free) solution of the Schrödinger equation:

$$\begin{aligned}
\psi(\vec{x},t) &= N e^{\frac{i}{\hbar}(\vec{p}.\vec{x}-Et)} \\
&\Rightarrow -\frac{\hbar}{2m}\left[\left(\frac{i}{\hbar}\,p_x\right)^2+\left(\frac{i}{\hbar}\,p_y\right)^2+\left(\frac{i}{\hbar}\,p_z\right)^2\right]\cancel{\psi}=i\left(\frac{-iE}{\hbar}\right)\cancel{\psi} \\
\underbrace{\vec{p}^2/2m}_{H_{\text{free}}\ \text{eigenvalues}} &= E \ \checkmark
\end{aligned}$$

$$\begin{cases}
\vec{p} = \hbar\vec{k} \ ; \ k = 2\pi/\lambda \quad \text{(wave length)} \\
E = \hbar\omega \ ; \ \omega = 2\pi/T \quad \text{(period)}
\end{cases}$$

Stationary solution:

$$\vec{j} = -\frac{i\hbar}{2m}\left(N^* e^{-\frac{i}{\hbar}p^\mu x_\mu} N\left(\frac{i\vec{p}}{\hbar}\right)e^{-\frac{i}{\hbar}(p^\mu x_\mu)} - N e^{-\frac{i}{\hbar}p^\mu x_\mu} N^*\left(\frac{i\vec{p}}{\hbar}\right)e^{-\frac{i}{\hbar}p^\mu x_\mu}\right)$$

$$\boxed{\vec{j} = \tfrac{|N|^2}{m}\,\vec{p}} \quad\longrightarrow\quad \vec{j} = \frac{\rho}{m}\,\vec{p} = \rho\vec{v} \quad \text{with} \quad \boxed{\rho = |\psi|^2 = |N|^2}$$

2.2 Extension to the relativistic framework

The Schrödinger equation has not a covariant form [∂_t instead of ∂_μ] and is not suitable for the relativistic energy expression ($E = \vec{p}^2/2m$).

- First approach:

$$\begin{aligned}
\widehat{H}_{free}\psi &= i\hbar\partial_t\psi \\
\widehat{H}^2_{free}\psi &= i\hbar\partial_t\,(H_{free}\,\psi) = -\hbar^2\partial_t^2\psi
\end{aligned}$$

$$\text{prescription}\begin{cases}
\widehat{\hbar} \neq \widehat{\hbar}(t)(\neq f(\partial_t)\ldots) \longrightarrow \left[\hbar^2\partial_t^2 - \hbar^2c^2\Delta + m^2c^4\right]\psi = 0 \\
\text{with time-dependent states}
\end{cases}$$

$$\begin{aligned}
\text{(classical relativistic energy)} \qquad E^2 &= \vec{p}^2c^2 + m^2c^4 \\
\Longrightarrow \widehat{H}^2_{free} &= -(\hbar c)^2\Delta + m^2c^4\mathbb{1}
\end{aligned}$$

$$\boxed{\left[\underbrace{\frac{\partial^2}{\partial(ct)^2} - \Delta}_{\Box=\partial^\mu\partial_\mu} + m^2\frac{c^2}{\hbar^2}\right]\psi(x^\nu) = 0}$$

• Second approach:

$$E^2 = \vec{p}^2 c^2 + m^2 c^4 \longrightarrow \quad -\hbar^2 \partial_t^2 \psi = -(\hbar c)^2 \Delta\psi + m^2 c^4 \psi$$

Note in passing that the fundamental relativistic quantum equation for a free massive particle with spin-1/2 reads as (Dirac equation),

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \qquad \text{(with } \hbar = c = 1)$$

2.2.1 Density flux

Let us recover a continuity equation,

$$KG \times (-i\psi)^* - KG^* \times (-i\psi) = \frac{1}{c^2}\partial_t^2 \psi(-i\psi^*) - \Delta\psi(-i\psi^*) + m^2 \frac{c^2}{\hbar}(-i\psi^*)\psi = 0$$

$$\left(\frac{i}{c^2}\partial_t\psi^*\partial_t\psi - \frac{i}{c^2}\partial_t\psi\partial_t\psi^*\right) - \frac{1}{c^2}\partial_t^2\psi^*(-i\psi) + \Delta\psi^*(-i\psi)$$
$$-m^2\frac{c^2}{\hbar^2}(-i\psi)\psi^* + (-i\vec{\nabla}\psi^*.\vec{\nabla}\psi + i\vec{\nabla}\psi.\vec{\nabla}\psi^*) = 0$$

$$\partial_t \underbrace{\left(\frac{i}{c^2}\left\{\partial_t\psi^*\psi - \partial_t\psi\psi^*\right\}\right)}_{\rho} + \vec{\nabla}.\underbrace{\left(i\left\{-\vec{\nabla}\psi^*\psi + \vec{\nabla}\psi\psi^*\right\}\right)}_{\vec{j}} = 0$$

which has the expected form, $\boxed{\partial_t\rho + \vec{\nabla}.\vec{j} = 0}$

ρ cannot be a probability density as $\left[\begin{array}{l} \neq \psi\psi^* = |\psi(x,t)|^2 \\ \\ \rho \text{ possibly} \begin{cases} < 0 \\ \text{imaginary} \end{cases} \end{array}\right.$

Check of the (free) solution of the Klein-Gordon equation:

$$\begin{aligned} \psi(\vec{x},t) &= N e^{\frac{i}{\hbar}(\vec{p}.\vec{x} - Ect)} = N e^{\frac{i}{\hbar}p^\mu x_\mu} \\ &\Rightarrow \frac{1}{c^2}\left(-\frac{iE}{\hbar}\right)^2\psi - \left(\frac{i}{\hbar}\right)^2 \vec{p}^2\psi + m^2\frac{c^2}{\hbar^2}\psi = 0 \end{aligned}$$

$$-\frac{E^2}{c^2}+\vec{p}^2+m^2c^2=0 \quad \checkmark$$

$$\text{with,}\quad \widehat{H}^2_{free}\psi=-\hbar^2c^2\left(\frac{-\vec{p}^2}{\hbar^2}\right)\psi+m^2c^4\psi=\left(\vec{p}^2c^2+m^2c^4\right)\psi$$

The current components are,

$$\rho=-\frac{i}{c^2}\left\{|N|^2\left(\frac{iE}{\hbar}-\left(-\frac{iE}{\hbar}\right)\right)\right\}=\boxed{\tfrac{|N|^2}{\hbar c^2}2E}$$

$$\vec{j}=-i|N|^2\left\{-(-\frac{i\vec{p}}{\hbar})+(\frac{i\vec{p}}{\hbar})\right\}=\boxed{\frac{|N|^2}{\hbar}2\vec{p}}$$

leading to a covariant form (4-current),

$$\boxed{j^\mu=i\left(\psi^*\partial^\mu\psi-\psi\partial^\mu\psi^*\right)}=\left(\rho c,\vec{j}\right)$$

$$\overset{\text{check}}{\Longrightarrow}\begin{cases} j^0=i\left(\psi^*\frac{1}{c}\partial_t\psi-\psi\frac{1}{c}\partial_t\psi^*\right)=\rho c \\ \\ j^i=i\left(\psi^*(-\partial_{x^i})\psi-\psi(-\partial_{x^i})\psi^*\right)=\vec{j} \quad \checkmark \end{cases}$$

$$\text{since}\quad \partial^\mu=g^{\mu\nu}\partial_\nu=\left(\frac{1}{c}\partial_t,-\partial_{x^i}\right)$$

Covariant form of the continuity equation $\longmapsto \boxed{\partial_\mu j^\mu=0}$

$$\overset{\text{check}}{\Longrightarrow}\quad 0 \;=\; \partial_0 j^0+\partial_i j^i=\frac{1}{\cancel{c}}\partial_t\rho\cancel{c}+\partial_{x^i}j^i=\partial_t\rho+\vec{\nabla}.\vec{j} \quad \checkmark$$

The free solution $\psi=N\exp\left(-\frac{i}{\hbar}p^\mu x_\mu\right)$ gives rise to:

$$j^\mu \;=\; (\rho c,\vec{j})=\left(\frac{|N|^2}{\hbar c}2E,\frac{|N|^2}{\hbar}2\vec{p}\right)=\frac{|N|^2}{\hbar}2\left(\frac{E}{c},\vec{p}\right)$$

$$j^\mu \;=\; \boxed{2\frac{|N|^2}{\hbar}p^\mu}$$

Let us now make a comment on the $\rho\, d^3x$ Lorentz transformation...

• ⓐ

$$\text{In F'}\quad \left|\begin{array}{l} cdt' = \gamma cdt - \gamma\beta dx = 0 \\ dx' = -\gamma\beta cdt + \gamma dx, \qquad dx' = -\beta(+\gamma\beta dx) + \gamma dx = dx\gamma(-\beta^2+1) = \sqrt{1-\beta^2}dx \\ dy' = dy \qquad\qquad dz' = dz \end{array}\right.$$

so that $\quad d^3x \longrightarrow d^3x' = \sqrt{1-\beta^2}\,\underbrace{dxdydz}_{d^3x}$

• ⓑ

$$\begin{aligned} p'^0 &= \gamma \times p^0 - \gamma\beta p_x \\ \frac{E'}{c} &= \gamma\frac{E}{c} \\ \rho &= constant \times E \implies \boxed{\rho' = \gamma\rho} \end{aligned}$$

• ⓐ + ⓑ

$$\boxed{\rho' d^3x' = \gamma\rho \times \sqrt{1-\beta^2}d^3x = \rho d^3x}$$

Lorentz invariance as in quantum mechanics:

$$\rho d^3x = d^3P(x) = \frac{\#\text{ particles in } d^3x}{\#\text{ total of particles}}$$

2.2.2 Application: the energy sign

The relativistic expression, $E = \pm\sqrt{p^2c^2 + m^2c^4}$, raises the question of the unacceptable negative energy. Historically,

- in 1927, Dirac devised the relativistic wave function equation linear in $\partial_t, \partial_{x^i}$ but the negative energy problem remained still unsolved. Although he did not succeed in overcoming this problem, he got the unexpected bonus of a spin-1/2 description (!).

- In 1934, Pauli and Weisskopf revived the Klein-Gordon equation by inserting "-e" in j^μ to interpret it as a charge 4-current (probability density $\rightarrow$ charge density).

- Stückelberg (1941) and Feynman (1948) proposed then the following prescription:

$$
j^\mu(e^-, p^\mu) = (-e)\ 2\frac{|N|^2}{\hbar}p^\mu \quad \& \quad \begin{aligned} j^\mu(e^+, p^\mu) &= (+e)\ 2\frac{|N|^2}{\hbar}p^\mu \\ &= (-e)\ 2\frac{|N|^2}{\hbar}\underbrace{\left(-\frac{E}{c}, -\vec{p}\right)}_{-p^\mu} \\ &= j^\mu(e^-, -p^\mu) \end{aligned}
$$

3 Time-dependent perturbation theory

3.1 Quantum state decomposition

So far, we have discussed the wave functions, 4-current j^μ, 4-momentum p^μ for the free particles.
In order to compute transition amplitudes of scattering reactions, one needs obviously to introduce interactions.
This requires to consider the perturbation theory. Given the complexity of the relativistic quantum theory, we start with a non-relativistic framework.

The starting point is to assume that one knows the solutions of the (free) Schrödinger equation,

$$\underbrace{H_0}_{free}\,\phi_n = E_n\phi_n(\vec{x}), \quad \phi_n(\vec{x}) = \langle \vec{x}, n\rangle \implies \phi_n(\vec{x},t) = \phi_n(\vec{x})e^{-i\frac{E_n t}{\hbar}}$$

$$\text{Solution of} \quad \underbrace{-\frac{\hbar^2}{2m}}_{H\neq H_0(t)}\,\Delta\psi = i\hbar\,\partial_t\psi$$

$$e^{-i\frac{E_n t}{\hbar}}H_0\phi_n = i\hbar\left(\frac{-E_n}{\hbar}\right)\phi_n e^{-i\frac{E_n t}{\hbar}}$$

$$\text{with } \langle n,m\rangle = \delta_{nm} = \int_V d^3x\,\overbrace{\langle n|\vec{x}\rangle}^{\phi_n*(\vec{x})}\,\overbrace{\langle\vec{x}|m\rangle}^{\phi_m(\vec{x})}$$

The objective is to solve the Schrödinger equation with an interaction term,

$$\big[\overbrace{H_0 + V(\vec{x},t)}^{H}\big]\psi = i\hbar\,\partial_t\psi(\vec{x},t) \tag{1}$$

The solution is necessary of the form,

$$\underbrace{\langle\vec{x}|\psi(t)\rangle}_{\psi(\vec{x},t)} = \sum_n C_n(t)\,\underbrace{\langle\vec{x}|n\rangle}_{\phi_n(\vec{x})} \quad \text{as at t=0} \quad \langle\vec{x}|\psi(t=-T/2)\rangle = \sum_n C_n(t=-T/2)\langle\vec{x}|n\rangle$$

It is useful to redefine $C_n(t)$ accordingly to,

$$\psi(\vec{x},t) = \sum_n \underbrace{a_n(t)e^{-i\frac{E_n t}{\hbar}}}_{C_n(t)}\,\phi_n(\vec{x}) \tag{2}$$

Substituting into Eq. (1), one gets,

$$\sum_n a_n(t) e^{-i\frac{E_n t}{\hbar}} \underbrace{H_0\phi_n} E_n\phi_n + \sum_n V a_n(t) e^{-i\frac{E_n t}{\hbar}}\phi_n$$

$$= i\hbar \sum_n \left[\partial_t a_n(t) e^{-i\frac{E_n t}{\hbar}} + a_n(t)\left(\frac{-iE_n}{\hbar}\right) e^{-i\frac{E_n t}{\hbar}}\right]\phi_n$$

$$\sum_n \left(\int_V d^3x\ \phi_f^* \phi_n V\right) a_n(t) e^{-i\frac{E_n t}{\hbar}} = i\hbar \sum_n \left(\int_V d^3x\ \phi_f^*\phi_n\right)\partial_t a_n(t) e^{-i\frac{E_n t}{\hbar}}$$

$$\boxed{\frac{\partial a_f(t)}{\partial t} = -\frac{i}{\hbar}\sum_n \left\{a_n(t) e^{-i(E_n E_f)\frac{t}{\hbar}}\left(\int_V d^3x\ \phi_f^* V(\vec{x},t)\phi_n\right)\right\}} \tag{3}$$

At the initial state:

$$a_i\left(t = -\frac{T}{2}\right) = 1, \tag{4}$$

$$a_{n(\neq i)}\left(t = -\frac{T}{2}\right) = 0 \Rightarrow \begin{cases} \psi(\vec{x}, -T/2) = \phi_i(\vec{x})(e^{+iE_iT/2\hbar}) \\ \frac{\partial a_f(t)}{\partial t} = -\frac{i}{\hbar}\ e^{i(E_f - E_i)\frac{t}{\hbar}} \int_V d^3x\ \phi_f^* V(\vec{x},t)\phi_i,\ t = -T/2 \end{cases}$$

<u>*Remark*</u> : interaction during $\left[-\frac{T_0}{2}, +\frac{T_0}{2}\right]$ with $T > T_0 > 0$

1st approximation: use Eq. (4) for $\partial a_f(t)/\partial t$ at the initial time $t = -T/2$ for any time. Integrating Eq. (4) gives rise to,

$$a_f(t) - a_f\left(-\frac{T}{2}\right) = \frac{-i}{\hbar}\int_{-\frac{T}{2}}^{t} dt'\ e^{\frac{it'}{\hbar}(E_f - E_i)} \int_V d^3x\ \phi_f^* V(\vec{x},t')\phi_i$$

$\longrightarrow$ solution $\psi(\vec{x},t)$ fixed in Eq. (2)

After interaction has ceased, at $t = T/2$, one can reorder as,

$$T_{fi}^{(T)} \widehat{=} a_f\left(\frac{T}{2}\right) = \frac{-i}{\hbar}\int_{-\frac{T}{2}}^{\frac{T}{2}} dt \int_V d^3x \left[\phi_f(\vec{x}) e^{-i\frac{E_f t}{\hbar}}\right]^* V(\vec{x},t) \overbrace{\left[\phi_i(\vec{x}) e^{-i\frac{E_i t}{\hbar}}\right]}^{\psi_i}$$

$$\boxed{T_{fi} \simeq \frac{-i}{\hbar}\int d^4x\ \phi_f^*(x^\mu) V(x^\nu)\psi_i(x^\sigma)}$$

3.2 The Fermi's golden rule

Probability interpretation:

$$P_t^{i\to f} = |\langle f|i(t)\rangle|^2 = \left|\left\langle f\middle|\left(\sum_n a_n(t)e^{-i\frac{E_n t}{\hbar}}\right)n\right\rangle\right|^2 = \left|a_f(t)e^{-i\frac{E_n t}{\hbar}}\right|^2$$

$$\boxed{P_{t=T/2}^{i\to f} = \left|T_{fi}^{(T)}\right|^2}$$

$$\begin{aligned}
\underline{\text{Case:}}\ V(\vec{x},t) &= V(\vec{x}) \\
T_{fi}^{t\to\infty} &= \frac{-i}{\hbar}V_{fi}\int_{-\infty}^{+\infty} dt\ e^{i(E_f-E_i)\frac{t}{\hbar}} = \frac{-i}{\cancel{\hbar}}V_{fi}\int_{-\infty}^{+\infty} \hbar\ dt'\ e^{i(E_f-E_i)t'}1_{(t')} \\
\text{with}\quad V_{fi} &\mathrel{\widehat{=}} \int_V d^3x\ \phi_f^*(\vec{x})V(\vec{x})\phi_i(\vec{x}) \qquad (5)\\
T_{fi} &= -2\pi_i\delta(E_f-E_i)V_{fi}
\end{aligned}$$

Let us define a transition probability per unit time,

$$\begin{aligned}
W \mathrel{\widehat{=}} \lim_{T\to\infty}\frac{|T_{fi}^{(T)}|^2}{T} &= \lim_{T\to\infty}\frac{1}{T}\left(-iV_{fi}\int_{-\frac{T}{2}}^{\frac{T}{2}} dt'\ e^{i(E_f-E_i)t'}\right)\left(2\pi_i\delta(E_f-E_i)V_{fi}^*\right) \\
&= \lim_{T\to\infty}\frac{2\pi}{T}|V_{fi}|^2\int_{-\frac{T}{2}}^{\frac{T}{2}} dt'\ \delta(E_f-E_i) \\
&= 2\pi\ |V_{fi}|^2\delta(E_f-E_i)
\end{aligned}$$

$$\begin{aligned}
W_{fi} &= \int^{(E_i\in)\Delta E_f} dE_f\ \rho(E_f)2\pi|V_{fi}|^2\delta(E_f-E_i) \\
W_{fi} &= \boxed{2\pi\ |V_{fi}|^2\rho(E_i)} \quad \textbf{(Fermi's golden rule)}
\end{aligned}$$

since $\displaystyle\int \frac{\text{Proba}^{\to[E+dE]f}}{T} \equiv \int \frac{\text{Proba}^{\to E_f}}{T}\times dN_f$ and transition probability $\forall E_f$.

3.3 Higher order solution

To improve the $a_f(t)$ solution, let us insert the first order approximation for $a_f(t)$ in the original Eq.(3),

$$\frac{\partial af(t)}{\partial t} = \left(\frac{-i}{\hbar}\right) e^{i(E_f-E_i)\frac{t}{\hbar}}\ V_{fi} + \left(\frac{-i}{\hbar}\right)^2 \sum_n \left[\int_{-\frac{T}{2}}^{\frac{T}{2}} dt' e^{\frac{it'}{\hbar}(E_n-E_i)}\ V_{ni}\right] e^{i(E_n-E_f)\frac{t}{\hbar}}\ V_{f_n} + \cdots$$

Let us integrate to find out $a_f(t)$ and in turn $a_f\left(\frac{T}{2}\right) = T_{fi}^{(T)}$,

$$T_{fi}^{(T)} = \left(\frac{-i}{\hbar}\right) V_{fi} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt\ e^{i(E_f-E_i)\frac{t}{\hbar}}$$

$$+ \left(\frac{-i}{\hbar}\right)^2 \sum_n \left\{ \int_{-\frac{T}{2}}^{\frac{T}{2}} dt\ e^{i(E_f-E_n)\frac{t}{\hbar}} V_{f_n} \int_{-\frac{T}{2}}^{t} dt'\ e^{i(E_n-E_i)\frac{t'}{\hbar}} V_{ni} \right\} + \ldots$$

$$T_{fi}^{(\infty)} = -2\pi\ i\delta(E_f-E_i)V_{fi} + \left(\frac{-i}{\hbar}\right)^2 \sum_n \underbrace{\int_{-\infty}^{+\infty} dt\ e^{i(E_f-E_n)\frac{t}{\hbar}}}_{\delta(E_f-E_i-i\epsilon)}\ V_{f_n} \frac{e^{i(E_n-E_i-i\epsilon)\frac{t}{\hbar}}}{i(E_n-E_i-i\epsilon)\frac{t}{\hbar}}\ V_{ni} + \ldots$$

$$\text{since} \quad \int_{-\infty}^{t} dt' e^{i(E_n-E_i-i\epsilon)\frac{t'}{\hbar}} = \frac{e^{i(E_n-E_i-i\epsilon)\frac{t}{\hbar}}}{i(E_n-E_i-i\epsilon)} - 0\ .$$

$$\begin{aligned} T_{fi}^{(\infty)} &= -2\pi i V_{fi}\delta(E_f-E_i) + i\sum_{n\neq i} \frac{V_{f_n}V_{ni}}{-(E_i-E_n-i\epsilon)} 2\pi\delta(E_f-E_i-i\epsilon) + \ldots \\ &= \boxed{-2\pi i\delta(E_f-E_i)\left[V_{fi} + \sum_{n\neq i} V_{fn}\frac{1}{E_i-E_n(+i\epsilon)}\ V_{ni} + \ldots\right]} \end{aligned}$$

Second order term: $\displaystyle\sum_{\substack{n\neq m\\ m\neq i}} V_{fn}\frac{1}{E_m-E_n}V_{nm}\frac{1}{E_i-E_m}V_{ni}$

Remark: $\displaystyle\frac{d}{dt}\langle\psi(t)|H|\psi(t)\rangle = \frac{1}{i\hbar}\ \langle[H,H]\rangle_{|\psi(t)>} + \left\langle \frac{\partial H}{\partial t}\right\rangle_{|\psi(t)>} = 0$

3.4 Electromagnetic interaction

$||\vec{E},\vec{B}|| \propto e^{i(\vec{k}.\vec{x}-\omega t)} \overset{rk.}{\Leftrightarrow} \psi_\alpha \propto e^{\frac{i}{\hbar}(\vec{p}.\vec{x}-Et)}$

ⓐ New time dependence:

$$\begin{aligned} V \propto e^{-i\omega t}(\cdots) \Rightarrow T_{fi}^{(\infty)} &= \frac{-i}{\hbar}\int dt\ e^{-i\omega t+i\frac{E_f}{\hbar}t-i\frac{E_i}{\hbar}t}\ V_{fi} + \text{2nd order}\ldots \\ &= -2\pi i\delta(E_f-E_i-\hbar\omega)V_{fi} \\ &\quad -2\pi i\delta(E_f-E_i-2\hbar\omega)\sum_{n\neq i} V_{fn}[E_i-E_n+\hbar\omega]V_{ni} + \text{2nd order}\ldots \end{aligned}$$

since Eq.(3) gives rise to $\dfrac{\partial af(t)}{\partial t} \propto \sum_n a_n(t)e^{i(E_n-E_f)\frac{t}{\hbar}}\int d^3x\ \psi_f^* V(\vec{x},t)\psi_n$

<u>*Remark :*</u> $E_i+\hbar\omega = E_n$ possible $\longmapsto +i\epsilon$

ⓑ Fermi's rule:

$$\begin{aligned} W^{1st} &\propto \int dt' e^{it'(E_f-E_i-\hbar\omega)}V_{fi}^* V_{fi} \\ &\propto \delta(E_f-E_i-\hbar\omega)|V_{fi}|^2 \Longrightarrow W_{fi} = 2\pi|V_{fi}|^2\rho(E_i+\hbar\omega) \\ W^{2nd} &\propto \left|\delta(E_f-E_i-\hbar\omega)V_{fi} + \delta(E_f-E_i-2\hbar\omega)\sum_n V_{fn}[E_i'-E_n]^{-1}V_{ni}\right|^2 \Longrightarrow W_{fi} = \ldots \end{aligned}$$

ⓒ Anti-particle discussion:

$$\begin{aligned} T_{fi} &\propto \int dt\int d^3x \Big[\phi_f e^{-iE_f\frac{t}{\hbar}}\Big]^* V(\vec{x})e^{-i\omega t}\Big[\phi_i e^{-iE_i\frac{t}{\hbar}}\Big] \\ &\propto \int dt\int d^3x \Big[\phi_i e^{-i(-E_i)\frac{t}{\hbar}}\Big]^* V(\vec{x})\Big[e^{-i(-\frac{E_\gamma}{\hbar})t}\Big]^*\Big[\phi_f e^{-i(-E_f)\frac{t}{\hbar}}\Big] \end{aligned}$$

and in T_{fi} calculated , $\delta(E_f-E_i-\hbar\omega) \;\Leftrightarrow\; \delta(-E_f+E_i+\hbar\omega)$
$\Leftrightarrow\; -(-E_f)+(-E_i)+(-\hbar\omega)$

4 The covariant equations of electromagnetism

We need to describe the interaction considered within a covariant formalism, before applying perturbation theory (more concretely) within the special relativity framework.

4.1 The classical theory: Maxwell's equations and Lorentz force

4-potential : $A^\mu = \begin{pmatrix} \phi/c \\ \vec{A} \end{pmatrix}$ Electromagnetic tensor : $\boxed{F^{\mu\nu} \mathrel{\widehat{=}} \partial^\mu A^\nu - \partial^\nu A^\mu}$

$$\begin{aligned} F^{0\prime} &= \partial^0 A' - \partial' A^0 = \frac{1}{c}\partial_t A_x - (-\partial_x)\frac{\phi}{c} = -\frac{1}{c}E_x \\ \text{since, } \vec{E} &= -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} &= \vec{\nabla} \wedge \vec{A} \end{aligned}$$

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_{x/c} & -E_{y/c} & -E_{z/c} \\ & 0 & -B_z & +B_y \\ & & 0 & -B_x \\ & (-) & & 0 \end{bmatrix}$$

$$\boxed{\partial_\mu F^{\mu\nu} = \mu_0 j^\nu \quad , \qquad \partial_\mu F^{\mu\nu} = 0}$$

$$\begin{cases} \star F^{\mu\nu} & \widehat{=} \quad \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \qquad \text{(dual of } F^{\mu\nu}\text{, Hodge operator)} \\ j^\nu & \equiv \quad \binom{q\rho c}{q\vec{j}} \end{cases}$$

$$\boxed{\partial_\mu F_{\nu\rho} + \partial_\rho F_{\rho\nu} + \partial_\nu F_{\rho\nu} = 0} \quad \overset{\mu\neq\nu\neq\rho}{\longmapsto} \quad 2(\partial_\mu \star F^{\mu\sigma} + \partial_\rho \star F^{\rho\sigma} + \partial_\nu \star F^{\mu\nu}) = 0$$

$$\overset{\text{ex: } \mu=\nu}{\longmapsto} \quad \partial_\mu F_{\mu\rho} + 0 + \partial_\mu F_{\rho\mu} = 0 \Longleftrightarrow \partial_\mu (F_{\mu\rho} + F_{\rho\mu}) = 0$$

$$\begin{aligned} \partial_\mu F^{\mu_0} = \mu_0 j^0 \quad &\Leftrightarrow \quad \frac{1}{c}\partial_t F^{00} + \partial_i F^{i0} = \mu_0 q\rho c \\ &\Leftrightarrow \quad 0 + \frac{1}{c}\vec{\nabla}.\vec{E} = \end{aligned} \qquad \boxed{\vec{\nabla}.\vec{E} = \frac{q\rho}{\epsilon_0}}$$

$$\nu = i \Longrightarrow \boxed{\vec{\nabla} \wedge \vec{B} - \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j}} \qquad \boxed{\vec{\nabla}.\vec{B} = 0}$$

$$
\begin{aligned}
\partial_\mu \frac{1}{2}\epsilon^{\mu 0\rho\sigma}F_{\rho\sigma} &= 0\\
0+\partial_\mu\epsilon^{1023}F_{23}+\partial\epsilon^{1032}F_{32}+\partial_2 E^{2013}F_{13}+\partial_2\epsilon^{2031}F_{31}+\partial_3\epsilon^{3012}F_{12}+\partial_3\epsilon^{3021}F_{21} &= 0\\
-\partial_1(-B_x)+\partial_1(+B_x)+\partial_2(+B_y)-\partial_2(-B_y)-\partial_3(-B_z)+\partial_3(+B_z) &= 0\\
+2[\partial_1(B_x)+\partial_2(B_y)+\partial_3(B_z)] &= 0\\
\vec{\nabla}.\vec{B} &= 0
\end{aligned}
$$

Similarly $\nu = i \Rightarrow \boxed{\vec{\nabla}\wedge\vec{E}+\frac{\partial\vec{B}}{\partial t}=\vec{0}}$

$\boxed{\frac{dp^\mu}{d\tau}\widehat{=}f^\mu = qF^{\mu\nu}V_\nu}$ with $V^\nu = \begin{pmatrix}\gamma c\\ \gamma\vec{V}\end{pmatrix}$ (Lorentz force)

$\underline{\mu = 0}$:

$$
\frac{d}{d\tau}\frac{E}{c} = qF^{0i}\Big(-\gamma V_i\Big) = q\left(\frac{-E_i}{c}\right)\gamma\Big(-V_i\Big) = q\frac{\gamma}{c}\vec{E}.\vec{V}
$$

$$
\frac{dE}{d\tau} = \overbrace{\vec{F}_{el}}^{q\vec{E}}.(\gamma\vec{V}) \qquad \int\frac{dE}{\cancel{dt}} = \int\vec{F}_{el}.\frac{d\vec{x}}{\cancel{dt}} \qquad (\vec{V}\perp\vec{F}_B) \qquad \Delta E = -W_{\vec{F}_{el}}
$$

$\underline{\mu = i}$:

$$
\gamma m\frac{dV^1}{d\tau} = f^1_{rel.} = qF^{10}\gamma c + 0 + qF^{12}(-\gamma V_y)+qF^{13}(-\gamma V_2) = q\gamma E_x + q\gamma V_y B_z - q\gamma V_z B_y
$$

$$
\boxed{m\frac{d\gamma\vec{v}}{d\tau} = q(\gamma\vec{E}+\gamma\vec{V}\times\vec{B})}
$$

The gauge symmetry:

The Maxwell equations and Lorentz force are invariant under $\begin{cases} j^\nu \longmapsto j^\nu\\ A^\alpha \longmapsto A^\alpha - \partial^\alpha\lambda(\overbrace{\vec{x},t}^{x^\alpha})\end{cases}$

$$
F^{\mu\nu}\longmapsto \partial^\mu[A^\nu - \cancel{\partial^\nu\lambda}] - \partial^\nu[A^\mu - \cancel{\partial^\mu\lambda}] = F^{\mu\nu}
$$

Possible choice not affecting the EOM's: $\partial_\alpha A^\alpha(x^\mu)\longmapsto \underbrace{\partial_\alpha A^\alpha(x^\mu) - \Box\lambda(x^\mu)}_{=0 \text{ by } \lambda \text{ choice}}$

$$
\mu_0 j^\nu = \partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu\cancel{A^\mu}) = \Box A^\nu
$$

Local conservation: $\partial_\mu j^\mu(x^\nu) = 0$

Global conservation: $\displaystyle\int_{\mathbb{R}^3} d^3x\ \partial_0 q\rho c = \int_{\mathbb{R}^3} -\vec{\nabla}.\vec{j}qd^3x = -\int_{S_{\mathbb{R}^3}} q\vec{j}.d\vec{S} = 0 \Leftrightarrow \partial_t \underbrace{\int_{\mathbb{R}^3} q\rho d^3x}_{Q} = 0$

4.2 The quantum theory: a Lagrangian description

Free particle:

$$S_{\text{part.}} = \frac{1}{c}\int d^4x\ \mathcal{L}_{\text{part.}} \text{ with } \mathcal{L}_{\text{part.}} = \frac{1}{2}(\partial_\mu\psi)(\partial^\mu\psi^*) - V_m \text{ and } V_m = \frac{m^2c^2}{2\hbar^2}|\psi|^2$$

Dimensions (natural unit system): $[\psi]^2 \equiv [E]^2$, $\left[\dfrac{dE}{d^3x}\right] = [E]^4 = [V_m]$

$$\begin{aligned}
\frac{\mathcal{L}}{\partial\psi} \boxed{=} \partial_\mu \frac{\mathcal{L}}{\partial(\partial_\mu\psi)} \quad \longmapsto \quad -\frac{k^2}{2}\psi^* \quad &\boxed{=} \quad \partial_\mu\left[\frac{1}{2}\frac{\partial}{\partial(\partial_\mu\psi)}\partial_\nu\psi\partial^\nu\psi^*\right] \\
&= \quad \partial_\mu[\partial^\nu\psi^*\delta^\mu_\nu] \\
&= \quad \partial_\mu\partial^\mu\psi^* \qquad \text{(Klein-Gordon equation)}
\end{aligned}$$

Anti-particle $(-e \to +e)$: $\psi_\alpha\ e^{-\frac{i}{\hbar}p^\mu x_\mu} \quad \longrightarrow \quad \psi^*_\alpha\ e^{-\frac{i}{\hbar}p^\mu x_\mu}$

Free radiation:

$$\mathcal{L}(A^\mu) \Rightarrow \partial_\mu F^{\mu\nu} = 0 \quad \text{if} \quad \mathcal{L}(A^\mu) = \frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}, \qquad [A^\mu] \underbrace{=}_{\mu_0=1} [E]$$

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial A_\alpha} \quad &\boxed{=} \quad \partial_\beta\left[\frac{\partial\mathcal{L}}{\partial(\partial_\beta A_\alpha)}\right] \\
0 \quad &\boxed{=} \quad \left[\frac{\partial\mathcal{L}}{\partial(\partial_\beta A_\alpha)}\left\{(\partial_\mu A_\nu - \partial_\nu A_\mu)(\underbrace{\partial^\mu A^\nu}_{\partial_\rho g^{\rho\mu}g^{\nu\sigma}A_\sigma} - \underbrace{\partial^\nu A^\mu}_{\partial_\rho g^{\rho\nu}g^{\mu\sigma}A_\sigma})\right\}\right]\left(-\frac{1}{4\mu_0}\right)
\end{aligned}$$

$$
\begin{aligned}
0 &= \partial_\beta\Big[(\delta^\beta_\mu\delta^\alpha_\nu - \delta^\beta_\nu\delta^\alpha_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) + (\partial_\mu A_\nu - \partial_\nu A_\mu)(g^{\rho\mu}g^{\nu\sigma}\delta^\beta_\rho\delta^\alpha_\sigma - g^{\rho\nu}g^{\mu\sigma}\delta^\beta_\rho\delta^\alpha_\sigma)\Big] \\
&= \partial_\beta\Big[(\partial^\beta A^\alpha - \partial^\alpha A^\beta - \partial^\alpha A^\beta + \partial^\beta A^\alpha) + (\partial^\beta A^\alpha - \partial^\alpha A^\beta - \partial^\alpha A^\beta + \partial^\beta A^\alpha)\Big] \\
&= \partial_\beta\Big[\overbrace{\cancel{4}\partial^\beta A^\alpha - \cancel{4}\partial^\alpha A^\beta}^{F^{\beta\alpha}}\Big]\Big(-\frac{1}{4\mu_0}\Big)
\end{aligned}
$$

<u>Interacting matter / radiation:</u>

$$
\text{The gauge symmetry is a theoretical guide:}\quad
\begin{cases}
j^\nu \longrightarrow j^\nu(x^\alpha) \\
\psi \longrightarrow e^{iq\alpha}\ \psi(x^\alpha) \\
A^\mu \longrightarrow A^\mu - \partial^\mu\lambda(x^\alpha)
\end{cases}
$$

$$
\boxed{\mathcal{L}^{(s=0)}_{QED} = \frac{1}{2}(D_\mu\psi)(D^\mu\mu)^* - \frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - V_m} \qquad (D^\mu = \partial^\mu + iqA^\mu)
$$

$$
\begin{aligned}
D^{\mu *}\psi^* = (\partial^\mu - iqA^\mu)\psi^* \quad &\longmapsto \quad (\partial^\mu - iqA^\mu + iq\partial^\mu\lambda)e^{-iq\lambda}\psi^* \\
&= \quad -(iq\partial^\mu\lambda)e^{-iq\lambda}\psi^* + e^{-iq\lambda}\partial^\mu\psi^* - iqA^\mu e^{-iq\lambda}\psi^* - iq(\partial^\mu\lambda)e^{-iq\lambda}\psi^* \\
&= \quad e^{-iq\lambda}\ \underbrace{(\partial^\mu - iqA^\mu)}_{D^{\mu *}}\ \psi^*
\end{aligned}
$$

so that,

$$
(D^\mu\psi)^* \longmapsto e^{-iq\lambda}(D^\mu\psi)^*
$$

$$
(D_\nu\psi) \longmapsto (D_\nu\psi)e^{iq\lambda}
$$

To calculate the Euler-Lagrange equations it is useful to write,

$$
\begin{aligned}
\frac{1}{2}(D_\mu\psi)(D^\mu\psi)^* &= (\partial_\mu\psi + iqA_\mu\psi)(\partial^\mu\psi^* - iqA^*_\mu\psi^*)\frac{1}{2} \\
&= \frac{1}{2}\Big(\partial_\mu\psi\partial^\mu\psi^* + iq\Big[A_\mu\psi\partial^\mu\psi\psi^* - \partial_\mu\psi A^\mu\psi^*\Big] + q^2A^2|\psi|^2\Big)
\end{aligned}
$$

Euler-Lagrange equations for the scalar fields:

$$\begin{cases} \frac{\partial \mathcal{L}^{\text{new}}}{\partial \psi^*} = \frac{1}{2}\Big(iq[-A^*_\mu \partial^\mu \psi] + q^2 A^2 \psi \Big) \\ \partial_\alpha \frac{\partial \mathcal{L}^{\text{new}}}{\partial(\partial_\alpha \psi)^*} = \partial_\alpha \frac{1}{2}\Big(iq\Big[A_\mu \psi g^{\mu\rho} \delta^\alpha_\rho \Big]\Big) = i\frac{q}{2}\partial_\alpha [A^\alpha \psi] \end{cases}$$

so that,

$$-\frac{m^2}{2}\psi - \frac{1}{2} iqA^*_\mu(\partial^\mu\psi) + \frac{1}{2}q^2|A|^2\psi \boxed{=} \partial_\mu\partial^\mu\psi\frac{1}{2} + i\frac{q}{2}\partial_\mu[A^\mu\psi]$$

$$0 = [\partial^\mu + iqA^{\mu *}][\partial_\mu\psi + iqA_\mu\psi] + m^2\psi \quad \text{with the compact form:} \quad \boxed{[D^\mu D_\mu + m^2]\psi = 0}$$

Euler-Lagrange equations for the vector field:

$$\begin{aligned} \frac{\partial \mathcal{L}^{\text{new}}}{\partial A_\alpha} &= \frac{1}{2}\Big(iq\Big[\delta^\alpha_\mu \psi \partial^\mu \psi^*\Big] + q^2|\psi|^2 \delta^\alpha_\nu A^{\nu *}\Big) \\ &= \frac{1}{2}\Big(iq\psi\partial^\alpha\psi^* + q^2|\psi|^2 A^{\alpha *}\Big) \end{aligned}$$

$$-\frac{iq}{2}\left[-\psi\underbrace{(\partial^\alpha - iqA^{*\alpha})}_{D^{\alpha *}}\psi^* \right] = \partial_\beta F^{\beta\alpha}\left(-\frac{1}{\mu_0}\right)$$

$$\boxed{\partial_\beta F^{\beta\alpha} = \mu_0 j^\alpha \quad \text{with} \quad j^\alpha = \frac{iq}{2}\Big[\psi^*(D^\alpha\psi) - (D^\alpha\psi)^*\psi\Big]}$$

$$\Rightarrow \quad 0 = \underbrace{\partial_\alpha\partial_\beta}_{S.}\underbrace{F^{\beta\alpha}}_{A.S.} = \mu_0\partial_\alpha j^\alpha$$

To be compared with the previous free current, $j^\mu = i(\psi^*\partial^\mu\psi - \psi(\partial^\mu\psi)^*) \equiv (\rho c, \vec{j})$

5 Spinless Quantum ElectroDynamics (QED)

$$\begin{cases} \text{perturbation theory in a covariant formalism} \\ \text{based on the electromagnetic interactions} \end{cases}$$

5.1 Scattering off an electromagnetic field

5.1.1 Matter equation of motion

At first order in $q = -e$ (electron charge), where α is the fine structure constant with $\frac{1}{137} = \alpha = \frac{e^2}{4\pi(\hbar c)}$ ($\hbar c = 1 \Rightarrow [e] = 1$),

$$\begin{aligned} 0 &= [(\partial^\mu + iqA^{\mu *})(\partial_\mu + iqA_\mu) + m^2]\psi \\ &\simeq \Box\psi - ie(A^{\mu *}\partial_\mu\psi + \partial^\mu(A_\mu\psi)) + m^2\psi \\ &= [\, \underbrace{\Box + m^2}_{K.-G.} \; \underbrace{-ie(A^{\mu *}\partial_\mu + \partial^\mu A_\mu)}_{V} \,]\psi \end{aligned}$$

5.1.2 Perturbation theory

Covariant formalism / first order.

$T_{fi} \simeq \frac{-i}{\hbar}\int d^4x\ \psi_f^*(x^\mu)V(x^\nu)\psi_i(x^\sigma)$ (no more standard probability interpretation)

$\rightarrow$ from $\mathcal{L}_{QED} : [\psi] = [E]$, $[T] = 1$ so $[V] = [E]^2$

$$\begin{aligned} T_{fi} \simeq \frac{-e}{\hbar}\int d^4x\ \psi_f^* A^{\mu *}\partial_\mu\psi_i + \int \underbrace{d^4x\ \psi_f^*\partial^\mu(A_\mu\psi_i)}_{} \\ - \int d^4x\ (\partial^\mu\psi_f)A_\mu\psi_i + \int d^4x\ \partial^\mu(\psi_f^* A_\mu\psi_i) \end{aligned}$$

$$T_{fi} \simeq \frac{-i}{\hbar}\int d^4x\ (-ie)[\psi_f^*(\partial_\mu\psi_i) - (\partial_\mu\psi_f^*)\psi_i]A^\mu$$

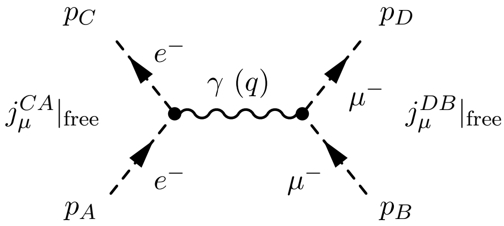
$$\boxed{T_{fi} \simeq \frac{-i}{\hbar}\int d^4x\ j_\mu^{fi}|_{\text{free}}\ A^\mu}$$

with,

$$\begin{aligned} j_\mu^{fi}|_{\text{free}} &= (-ie)\Big[N_f N_i \begin{pmatrix} -\frac{i}{\hbar}\frac{E_i}{c} \\ -\frac{i}{\hbar}(-p_i^\alpha) \end{pmatrix} - N_f N_i \begin{pmatrix} \frac{i}{\hbar}.\frac{E_f}{c} \\ \frac{i}{\hbar}.(-p_f^\alpha) \end{pmatrix}\Big] e^{-\frac{i}{\hbar}[E_i t - E_f t + \vec{p}_f.\vec{x} - \vec{p}_i.\vec{x}]} \\ &= (-ie)N_f N_i\Big(-\frac{i}{\hbar}[p_{i\mu} + p_{f\mu}]e^{-\frac{i}{\hbar}(p_i - p_f)^\nu x_\nu}\Big) \end{aligned}$$

5.2 Application : scattering off a muon

The Feynman diagram for the scattering of an electron off a muon is:



One needs first to write, $\Box A^\mu = \mu_o j^\mu_{DB}|_{\text{free}}$ so that $A^\mu = -\frac{\hbar^2\mu_o}{\underline{q}^2} j^\mu_{DB}|_{\text{free}}$ with $q^\nu = p^\nu_D - p^\nu_B$ and $\Box e^{\frac{i}{\hbar}q^\nu x_\nu} = \frac{-1}{\hbar^2}[\frac{E^2}{c^2} - \vec{q}^2] = \frac{-\underline{q}^2}{\hbar^2}$ as $\Box A^\mu = -\frac{\hbar^2\mu_o}{\underline{q}^2}\Box j^\mu_{DB}|_{\text{free}} = -\frac{1}{\underline{q}^2}(\frac{-\underline{q}^2}{\hbar^2} j^\mu_{DB}|_{\text{free}})\hbar^2\mu_o$.

$$
\begin{aligned}
T_{CD,AB} \simeq\ & \left(\frac{-i}{\hbar}\right)\int d^4x\ (-ie)N_C N_A\left(\frac{-i}{\hbar}\right)(p_A + p_C)_\mu \\
& \times\ e^{\frac{i}{\hbar}(p_c - p_A + p_D - p_R)\underline{.x}} \left[-\frac{\hbar^2\mu_o}{\underline{q}^2}\right](-ie)N_D N_B\left(\frac{-i}{\hbar}\right)(p_D + p_B)^\mu
\end{aligned}
$$

$$
\begin{aligned}
\simeq\ & \left(\frac{-i}{\hbar}\right)N_C N_A N_B N_D\ (-1)\underbrace{\int d^4x\ e^{\frac{i}{\hbar}(p_C + p_D - p_A - p_B)\underline{.x}}}_{\hbar^4(2\pi)^4\delta^{(4)}(p_C + p_D - p_A - p_B)} \\
& \times\ \underbrace{(ie)(p_A + p_C)^\nu\left[\frac{-ig_{\mu\nu}}{\underline{q}^2}\right](ie)(p_D + p_B)^\mu}_{-i\mathcal{M}}\ (-i\mu_o)
\end{aligned}
$$

5.3 The cross section

5.3.1 Normalisation

Inspired by the non-relativistic framework where $\int_V |\psi|^2 d^3x = 1,\ N = \frac{1}{\sqrt{V}}$.
Free solution: ρ can be related to the probability with a careful normalisation, $\rho = \mathcal{N}_{\text{tot}}\frac{dP}{d^3x}$, $\int_V \rho\ d^3x = \int_V |N|^2\ 2E\ d^3x = 2E$.

5.3.2 Transition rate per unit volume

As W definition but $\begin{cases} 2\text{ bodies} \\ 1/V \end{cases}$ $\quad W_{AB\to CD} \hat{=} \lim_{T\to\infty}\frac{P^{A\to C}_{t=T/2}}{TV} = \lim_{T\to\infty}\frac{|T^{(T)}_{CD,AB}|^2}{TV}$

$$W_{AB\to CD} = \lim_{T\to\infty} \tfrac{1}{TV} |N_A N_B N_C N_D|^2 \; |\mathcal{M}|^2 \; (2\pi)^4 \; \delta^{(4)}(p_C + p_D - p_A - p_B)$$

$$\times \int d^4x \; e^{i(p_C+p_D-p_A-p_B).x} \; 1_x$$

$$\boxed{W_{AB\to CD} = \frac{1}{V^4} \; |\mathcal{M}|^2 \; (2\pi)^4 \; \delta^{(4)}(p_C + p_D - p_A - p_B)}$$

5.3.3 Cross section notion

The experimental results for an $AB \to CD$ scattering reaction are usually quoted in the form of a "cross section":

Probability of interaction per VT $\left[\begin{array}{l} \equiv \text{Initial Flux} \times \textbf{Cross Section} \times \text{Density of Target} \\ = W_{AB\to CD} \times \text{Number of Final States} \end{array}\right.$

The cross section represents the surface area around a charged particle, in which another charged particle, passing through, would interact with the former particle (effective area 'seen' by the incoming particle). This notion thus includes the interaction strength.

Typically, $\phi_n \propto e^{\pm\frac{i}{\hbar}(p_x^n x)} \to$ Boundary Condition $\sin(p_x^n x)$ so that $\sin(o) = 0$ and $\sin(p_x^n \; L) = 0$, $p_x^n \; L = 2n_x\pi$ ($n_x \in \mathcal{N}^*$), $dn_x = L \; dp_x^n/2\pi$. Finally, $\Delta n = V d^3p/(2\pi)^3$.

$$\frac{\text{Number of states in } d^3p \text{ (in V)}}{\text{Number of particles (in V)}} = \frac{\Delta n}{2E} = \frac{V d^3p/(2\pi)^3}{2E} \equiv \text{Number of states per particle}$$

So finally, Number of final states $= 1_{e^-} \times \frac{V d^3p_C}{(2\pi)^3 2E_C} \times 1_{\mu-} \times \frac{V d^3p_D}{(2\pi)^3 2E_D}$.

5.3.4 Cross section expression

$$d\sigma = \tfrac{1}{V^4} \; |\mathcal{M}|^2 \; \underbrace{(2\pi)^4 \; \delta^{(4)}(p_C + p_D - p_A - p_B) \; \frac{V d^3p_C}{(2\pi)^3 2E_C} \; \frac{V d^3p_D}{(2\pi)^3 2E_D}}_{dP} \; \underbrace{\frac{V}{2E_A} \; \frac{V}{2E_B} \; \frac{1}{||\vec{v}_A - \vec{v}_B||}}_{1/F}$$

$$d\sigma = \tfrac{|\mathcal{M}|^2}{F} dP \quad \left\{\begin{array}{l} |\mathcal{M}|^2\text{: Lorentz invariant (squared amplitude) - theoretical prediction} \\ dP\text{: Lorentz invariant (phase space factor) “dLips” - available energy} \\ F\text{: not Lor. invariant (flux term) - experimental conditions} \end{array}\right.$$

For dP, the dimension counting goes like, $[E^{0-\cancel{4}+\cancel{2}+\cancel{2}-2}] \Rightarrow \times(\hbar c)^2 : [\triangle p \; c \; \; \triangle \; x \; c]^2 = [E]^2 L^2$.

5.3.5 Lorentz invariant cross section

Adding the following assumption, the cross section can become Lorentz invariant : parallel beams (impact along an accelerator axis: collinear collisions).

$$\begin{aligned}
F'' &= 4E_AE_B\left[\left|\left|\frac{\vec{P}_A}{E_A}-\frac{\vec{P}_B}{E_B}\right|\right|^2\right]^{1/2} \quad \text{since} \quad \vec{v}=\frac{\gamma m\vec{v}}{\gamma m(c)}=\frac{\vec{p}}{E/(c)}\\
&= 4E_AE_B\left[\frac{\vec{p}_A^{\,2}}{E_A^2}+\frac{\vec{p}_B^{\,2}}{E_B^2}-2\frac{\vec{p}_A.\vec{p}_B}{E_AE_B}\right]^{1/2}\\
&= 4\left[\underbrace{E_B^2\vec{p}_A^{\,2}+E_A^2\vec{p}_B^{\,2}-2E_AE_B\ \vec{p}_A.\vec{p}_B}_{\mathcal{C}}\right]^{1/2}
\end{aligned}$$

where,

$$\begin{aligned}
\boxed{\mathcal{C}=(p_A^\mu p_{B_\mu})^2-m_A^2m_B^2} &= (E_AE_B-\vec{p}_A.\vec{p}_B)^2-(E_A^2-\vec{p}_A^{\,2})(E_B^2-\vec{p}_B^{\,2})\\
&= \cancel{E_A^2E_B^2}+\cancel{[|p_A||p_B|(-1)]^2}+2E_AE_B|p_A||p_B|\\
&\quad -\left[\cancel{E_A^2E_B^2}+\cancel{|p_A|^2|p_B|^2}-E_A^2\vec{p}_B^{\,2}-E_B^2\vec{p}_A^{\,2}\right]
\end{aligned}$$

5.3.6 Center-of-mass frame

This frame is possibly equivalent to the laboratory frame but is different from fixed target frames.

For a process $AB\to CD$, such a frame is defined by $\boxed{\vec{p}_A+\vec{p}_B=\vec{o}}$ where $p_A^\mu=\begin{pmatrix}E_A\\ \vec{p}_A\end{pmatrix}$.

$$\begin{aligned}
dP_{c.m.} &= \frac{1}{(2\pi)^2}\frac{d^3p_C}{2E_C}\frac{d^3p_D}{2E_D}\delta^{(4)}(p_C+p_D-p_A-p_B)\\
&= \frac{1}{(2\pi)^2}\frac{d^3p_C}{2E_C}\frac{1}{2E_D}\ d^3p_D\ \delta^{(3)}(\vec{p}_C+\vec{p}_D-\vec{p}_A-\vec{p}_B)\ \delta(E_C+E_D-E_A-E_B)\\
&= \frac{d\Omega_C}{(2\pi)^2}\frac{dp_C}{4E_CE_D}\ |p_C|^2\ \delta(\sqrt{s}-E_C-E_D)
\end{aligned}$$

$$\begin{aligned}
\text{since } s &= (p_A+p_B)^\mu(p_A+p_B)_\mu\\
&= (E_A+E_B)^2-\vec{o}^{\,2} \quad \text{with} \quad d^3p=|p|^2dp\sin\theta\ d\theta\ d\phi
\end{aligned}$$

$$\begin{aligned}
&= \frac{d\Omega_f}{(2\pi)^2}\frac{1}{4E_CE_D}\ |p_f|^2\ \delta(\sqrt{s}-E_C+E_D)\ \frac{d\sqrt{s}}{|p_f|(1/E_C+1/E_D)}\\
&= \frac{d\Omega_f|p_f|}{4(2\pi)^2(E_D+E_C)} \quad = \quad \boxed{\frac{|p_f|d\Omega_f}{16\pi^2\sqrt{s}}}
\end{aligned}$$

We have used the relation, $\sqrt{s} = E_C + E_D = (\vec{p}_f^2 + m_C^2)^{1/2} + (\vec{p}_f^2 + m_D^2)^{1/2}$, so that $\frac{d\sqrt{s}}{dp_f} = \frac{1}{2}\frac{2|p_f|}{(|p_f|^2+m_C^2)^{1/2}} + \frac{|p_f|}{E_D}$.

$$F''_{c.m.} = 4E_A E_B \left(\left|\frac{p_A}{E_A}\right| + \left|\frac{p_B}{E_B}\right|\right) = 4E_A E_B\left(\frac{|p_i|}{E_A} + \frac{|p_i|}{E_B}\right)$$
$$= 4|p_i|\ [E_B + E_A] = 4|p_i|\sqrt{s}$$

$$\boxed{\left.\frac{d\sigma}{d\Omega_f}\right|_{c.m.} = |\mathcal{M}|^2_{c.m.}\ \frac{1}{64\pi^2 s}\ \frac{|p_f|}{|p_i|}}$$

5.3.7 High-energy limit

The relativistic regime is realistic for high-energy colliders (exploring new physics domains),

$$\boxed{E^2 \hat{=} |p|^2 + m^2 \simeq |p|^2}$$

In this limiting case, the squared amplitude becomes,

$$|\mathcal{M}|^2_{c.m.} = \left| e^2 \begin{pmatrix} 2E \\ \vec{p}_i + \vec{p}_f \end{pmatrix} \cdot \begin{pmatrix} 2E \\ -\vec{p}_i - \vec{p}_f \end{pmatrix} \frac{1}{\begin{pmatrix} 0 \\ -\vec{p}_f + \vec{p}_i \end{pmatrix}^2} \right|^2$$
$$= e^4\ \left[\frac{4E^2 + (\vec{p}_i + \vec{p}_f)^2}{-(\vec{p}_i - \vec{p}_f)^2}\right]^2$$
$$= e^4\ \left[\frac{4E^2 + 2|p_i|^2 + 2|p_i|^2\cos\theta}{-(2|p_i|^2 - 2|p_i|^2\cos\theta)}\right]^2 = e^4\ \left[\frac{6 + 2\cos\theta}{2 - 2\cos\theta}\right]^2$$
$$= e^4\ \left[\frac{3 + \cos\theta}{1 - \cos\theta}\right]^2$$

which leads to the following differential cross section,

$$\boxed{\left.\frac{d\sigma}{d\Omega f}\right|_{c.m.} \simeq \underbrace{\left(\frac{e^2}{4\pi}\right)^2}_{\alpha^2} \frac{1}{4s}\left[\frac{3+\cos\theta}{1-\cos\theta}\right]^2}$$

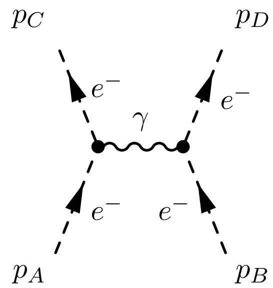
<u>Cross section units:</u> σ in barn(s) $\equiv 10^{-28}$ m^2.
Smaller units: 1 picobarn $\equiv 10^{-12}$ barn, 1 femtobarn $\equiv 10^{-15}$ barn.

5.4 Other examples of processes

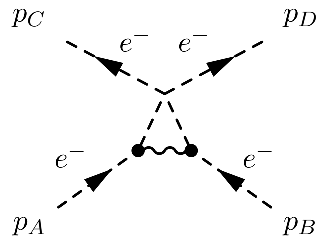
5.4.1 Electron-electron scattering

The Feynman diagrams for the Møller scattering are:

"t-channel"



$+$



"u-channel"

Those two Feynman diagrams sum up in the total amplitude. The Feynman rules give,

$$-i\mathcal{M} = (ie)(p_A{+}p_C)^\nu \underbrace{\frac{-i}{(p_D - p_B)^2}}_{t}(p_B{+}p_D)_\nu(ie) \;+\; (ie)(p_A{+}p_D)^\nu \underbrace{\frac{-i}{(p_C - p_B)^2}}_{u}(p_B{+}p_C)_\nu(ie)$$

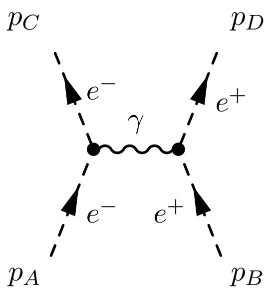
with the *Mandelstam* variable definitions (making use of the 4-momentum conservation),

$$t \hat{=} (p_D - p_B)^2 \;=\; (p_A - p_C)^2$$

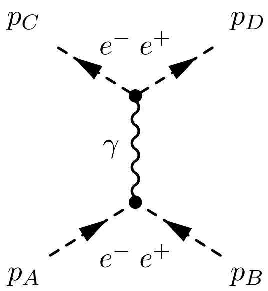
$$u \hat{=} (p_C - p_B)^2 \;=\; (p_A - p_D)^2$$

5.4.2 Electron-positron scattering

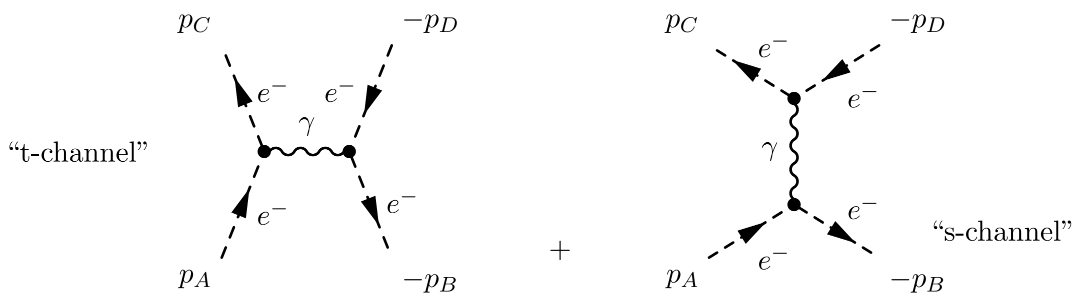
The Feynman diagrams for the Bhabha scattering are:



$+$



These two Feynman diagrams are respectively equivalent to the following ones (in terms of the corresponding anti-particles).



The Feynman rules give rise to the following sum of amplitudes,

$$-i\mathcal{M} = (ie)(p_A{+}p_C)^\mu \, \frac{-i}{\underbrace{(p_D - p_B)^2}_{t}} (ie)(-p_D{-}p_B)_\mu \;+\; (ie)(p_C{-}p_D)^\mu \, \frac{-i}{\underbrace{(p_D + p_C)^2}_{s}} (ie)(p_A{-}p_B)_\mu$$

which can be re-expressed through,

$$\begin{aligned} (p_A + p_C)^\mu (p_D + p_B)_\mu &= (p_A + [p_A + p_B - p_D])^\mu (p_D + p_B)_\mu \\ &= 2p_A^\mu (p_D + p_B)_\mu + \underbrace{(-p_D + p_B)^\mu (p_D + p_B)_\mu}_{-\cancel{m_D^2}+\cancel{m_B^2}+\cancel{p_D^\mu p_{B\mu}}-\cancel{p_D^\mu p_{B\mu}}} = 2p_A.p_D + 2p_A.p_B \\ &= s - u \end{aligned}$$

with the third *Mandelstam* variable definition (using of 4-momentum conservation),

$$s \,\hat{=}\, (p_D + p_C)^2 = (p_A + p_B)^2$$

Similarly, one has $(p_C - p_D)^\mu (p_A - p_B)_\mu = u - t$.

<u>Property</u> : $s + t + u \;=\; \sum_i m_i^2 + 2\, \cancel{p_A^2} + 2\, \cancel{p_A}.\underbrace{(p_B - p_D - p_C)}_{-\cancel{p_A}} \;=\; \sum_{i=A,B,C,D}\; m_i^2$.

Finally, the amplitude takes the simple form,

$$\boxed{\mathcal{M} = (-e^2)\left(\frac{u-s}{t} + \frac{u-t}{s}\right)}$$