PROBLEM ON PARTICLES & SYMMETRIES

About fundamental equations

1. **Lorentz invariance of Maxwell equations.-** Once the field strength $F^{\mu\nu}$ is defined (Lorentz indices $\mu, \nu \equiv 0, 1, 2, 3$), classical Maxwell equations can be written under a covariant form :

$$\partial_\mu \, F^{\mu\nu} \;=\; \mu_0 \, j^\nu \tag{1}$$

where ∂_μ is the 4-vector derivative and j^ν is the 4-current of charge. The goal of the exercise is to demonstrate the Lorentz invariance of this equation.

   (a) As a preliminary, write the Lorentz transformation for a generic 4-vector V^μ (expression of V'^μ in the frame $\mathcal{F}'$ in terms of the Lorentz matrix $\Lambda^{\alpha}_{\;\cdot\,\beta}$ and V^μ in the frame $\mathcal{F}$) [1].

   (b) Multiply the expression obtained in the previous question by the inverse Lorentz matrix in order to rather express V^μ in terms of $(\Lambda^{-1})^{\alpha}_{\;\cdot\,\beta}$ and V'^μ.

   (c) Using the Question 1b, rewrite Equation (1) by expressing ∂_μ as a function of ∂'_μ and the rank-two tensor $F^{\mu\nu}$ in terms of $F'^{\mu\nu}$. Then show that the obtained equation reads as,

$$\delta^\beta_\alpha \, \partial'_\beta \, \left(\Lambda^{-1}\right)^{\nu}_{\;\cdot\,\gamma} \, F'^{\alpha\gamma} \;=\; \mu_0 \, j^\nu \tag{2}$$

   where δ^β_α denotes the Kronecker symbol.

   (d) Replace j^ν by its expression in terms of j'^ν in Equation (2) and multiply the obtained equality by a Λ matrix to get rid of all the Lorentz matrices. Conclude.

2. **Interpretations of the Schrödinger equation.-** Within the non-relativistic framework of quantum mechanics, we consider the following Lagrangian density, involving the wave function (complex scalar field) $\phi(t, \vec{x})$ for a particle of mass m,

$$\mathcal{L} \;=\; \frac{i\hbar}{2} \, (\phi^* \partial_t \phi - \phi \, \partial_t \phi^*) \;-\; \frac{\hbar^2}{2m} \sum_{k=1}^{3} \partial_k \phi \, \partial_k \phi^* \;-\; V(t, \vec{r}) \, \phi \, \phi^* \, . \tag{3}$$

$\partial_t = \partial/\partial t$, $\partial_k = \partial/\partial x_k$ [no covariant formalism] are respectively the time and space partial derivatives, the exponent * stands for the complex conjugate and V is some energy potential.

   (a) To find out the equation of motion, apply the Euler-Lagrange equation,

$$\frac{\partial \mathcal{L}}{\partial \phi} \;=\; \partial_t \, \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \;+\; \sum_{j=1}^{3} \partial_j \, \frac{\partial \mathcal{L}}{\partial(\partial_j \phi)} \tag{4}$$

   to the Lagrangian (3). Comment on the obtained equation.

[1] The names and positions of the greek indices must be chosen properly in all the presented expressions.

(b) Calculate the following quantity, by using Equation (3),

$$\mathcal{Q} \;=\; \phi\,\frac{\partial\mathcal{L}}{\partial\phi} \;+\; \partial_t\phi\,\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} \;+\; \sum_{k=1}^{3}\,\partial_k\phi\,\frac{\partial\mathcal{L}}{\partial(\partial_k\phi)} \tag{5}$$

and compare the resulting $\mathcal{Q}$ with the Lagrangian $\mathcal{L}$ itself. Same question for the Quantity (5) with the replacement $\phi \to \phi^*$ [but same $\mathcal{L}$].

(c) Let us now define the two new objects,

$$R \;=\; -\frac{i}{\hbar}\left\{\phi\,\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} \;-\; \phi^*\,\frac{\partial\mathcal{L}}{\partial(\partial_t\phi^*)}\right\}\;,\; C_j \;=\; -\frac{i}{\hbar}\left\{\phi\,\frac{\partial\mathcal{L}}{\partial(\partial_j\phi)} \;-\; \phi^*\,\frac{\partial\mathcal{L}}{\partial(\partial_j\phi^*)}\right\}\;. \tag{6}$$

Calculate the combination $i\hbar(\partial_t R + \partial_j C_j)$ only by using Equations (4), (6) and the previous question (without calculating explicitly R and C_j through the $\mathcal{L}$ definition) [2]. Interpret physically the result as well as R and C_j.

(d) Calculate both R and C_j by injecting the Lagrangian (3) into Equalities (6). Give C_j as an imaginary part.

(e) Deduce from previous question the expression of C_j as a function of R and v_j, for the solution $\phi = Ne^{-\frac{i}{\hbar}(E\,t-\vec{p}.\vec{x})}$ of the free Schrödinger equation ($V = 0$) where N is a normalisation factor, E the energy of the particle and $p_j = m\,v_j$ [$j = 1,2,3$] its spatial momentum.

(f) Justify mathematically this equality (for $j = 1,2,3$),

$$\partial_j\mathcal{L} \;=\; \left[\frac{\partial\mathcal{L}}{\partial\phi}\,\partial_j\phi \;+\; \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\,\partial_j\partial_t\phi \;+\; \sum_{k=1}^{3}\,\frac{\partial\mathcal{L}}{\partial(\partial_k\phi)}\,\partial_j\partial_k\phi \;+\; \{\phi\to\phi^*\}\right] \;-\; \phi\,\phi^*\,\partial_j V\;. \tag{7}$$

(g) In Equation (7), replace $\partial\mathcal{L}/\partial\phi$ by its expression (4). Then rewrite Equation (7) with global derivatives as follows (specify the quantities M_j and σ_{jk} in terms of $\mathcal{L}$) [3],

$$\partial_t\,M_j \;=\; \sum_{k=1}^{3}\,\partial_k\,\sigma_{jk} \;-\; |\phi|^2\,\partial_j V\;. \tag{8}$$

(h) Give the quantity $-\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\,\partial_j\phi - \{\phi\to\phi^*\}$ in terms of m, ϕ and v_j only. Make use of Lagrangian (3) and Questions (2d), (2e).

(i) Apply an integration over a volume $\mathcal{V}$ to Equation (8) (*i.e.* apply $\int_{\mathcal{V}} d^3x$), without integrating. Then rewrite the term involving σ_{jk} as a function of σ_{jk} and dS_k [= infinitesimal surface area vector] by invoking the Gauss's theorem. This term represents a constraint on a surface area. Interpret physically the two remaining terms with respect to the fundamental (second) Newton's law of motion [4].

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[2]Noticing in Equation (6), that some terms arise by replacing ϕ with ϕ^*, might help to have more compact expressions.
[3]The $\partial_j\mathcal{L}$ term can be taken into account via a Kronecker symbol.
[4]One may remind the standard quantum interpretation of $|\phi(t,\vec{x})|^2$.