PROBLEM ON PARTICLES & SYMMETRIES

Non-relativistic limit of the Klein-Gordon equation

1. The *Klein-Gordon* (*KG*) equation reads as [1],

$$[\, \partial_\mu \partial^\mu + \frac{c^2}{\hbar^2}\, m^2 \,]\, \phi(x^\nu) \;=\; 0 \,, \tag{1}$$

where $\partial_\mu \hat{=} \frac{\partial}{\partial x^\mu}$ is the 4-vector derivative with respect to space-time, μ is a *Lorentz* index, m denotes a mass, c is the light velocity and $\hbar$ the Planck constant.

   (a) What does $\phi(x^\nu)$ represent? What does the *KG* equation describe physically?

   (b) Express the *KG* equation in terms [2] of the 4-momentum p^μ, by using the form of the free solution $\phi(x^\nu) = Ne^{-\frac{i}{\hbar}x^\mu p_\mu}$. For this purpose, first calculate the *Lorentz* product $x^\mu p_\mu$ as well as the *d'Alembertian* differential operator.

   (c) Based on the previous result, compare the dimensions of the two operators entering the *KG* equation.

2. From now on, let us consider the non-relativistic limit, $\epsilon = \frac{\vec{p}^{\,2} c^2}{m^2 c^4} \ll 1$.

   (a) Calculate the square root of the relativistic expression of E^2 to determine the total energy E at first order in the expansion parameter ϵ. Then demonstrate that the non-relativistic condition can take the form, $E - mc^2 \ll mc^2$.

   (b) Deduce the factorisation, $\phi(x^\nu) = Ne^{-\frac{i}{\hbar}x^\mu p_\mu} = \psi(x^\nu) e^{-\frac{i}{\hbar}mc^2 t}$, and provide $\psi(t, \vec{x})$.

   (c) Justify the following approximate condition, relying on a general quantum principle:

$$\left| i\hbar \frac{\partial \psi(x^\nu)}{\partial t} \right| \approx \left| (E - mc^2)\psi(x^\nu) \right| \ll \left| mc^2 \psi(x^\nu) \right| \,. \tag{2}$$

   (d) Calculate the time derivative $\frac{\partial \phi(x^\nu)}{\partial t}$ in terms of $\psi(x^\nu)$ (without replacing $\psi(x^\nu)$ by its expression).

   (e) Now calculate $\frac{\partial^2 \phi}{\partial t^2}$ by neglecting the second order derivative, $\frac{\partial^2 \psi}{\partial t^2}$, with respect to the first one, $\frac{\partial \psi}{\partial t}$, as inspired from the time derivative of Equation (2).

   (f) Insert the result of previous question into Equation (1) and comment about the obtained equation.

[1] Throughout all the Particle Physics part of the exam, we consider the usual metric $g^{\alpha\beta} = diagonal(+1, -1, -1, -1)$.

[2] Recall that $p^\mu = (\frac{E}{c}, \vec{p})$.

Towards the reaction $e^+e^- \to \mu^+\mu^-$

Let us consider the reaction $S_a^- S_a^+ \to S_b^- S_b^+$ between massive spinless particles (scalar fields) with charges ± 1 in the relativistic quantum framework of a generic theory where these elementary particles interact with a massless neutral spin-one boson V. Such a process, generated by the V exchange, is depicted on the following *Feynman* diagram.

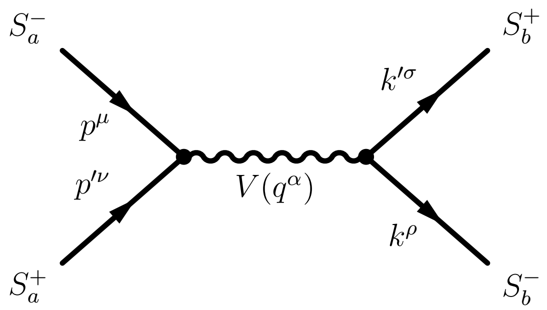

Figure 1: *Feynman* diagram for the two to two-body reaction. The 4-momentum (q^α, p^μ, p'^ν, k^ρ or k'^σ) associated to each particle is indicated [the greek indices are *Lorentz* indices such that for example $\mu = 0, 1, 2, 3$]. The arrows show the propagation flow directions. Extending this reaction to fermions coupled to a photon could lead to the QED process $e^+e^- \to \mu^+\mu^-$ involving electrons and muons.

1. Make a comment about the charge flow in the studied reaction of Figure 1.

2. Using the anti-particle prescription, draw the *Feynman* diagram for the process of Figure 1, but considering instead the S_a^- and S_b^+ propagations exclusively ($S_a^- S_a^- \to S_b^+ S_b^+$). Indicate the corresponding 4-momentum along the four external legs and specify the two axes of the plane of your *Feynman* diagram.

3. Write [3] the probability amplitude, $-i\mathcal{M}$, for the reaction $S_a^- S_a^- \to S_b^+ S_b^+$ in terms of the 4-momenta q^α, p^μ, p'^ν, k^ρ, k'^σ and the coupling constant g of the theory.

4. Make a complete comment [4] about the possibility for the propagator denominator, entering $-i\mathcal{M}$, to vanish.

5. Express the 4-momentum q^μ in terms of k^α and k'^β. Then express the 4-momentum p'^ν as a function of the other 4-momenta, thanks to the 4-momentum conservation relation.

6. Based on previous question, give the *Lorentz* product $(p^\mu - p'^\mu)(k'_\mu - k_\mu)$ only in terms of p^μ, k^ρ and k'^σ. Simplify the result by using the fact that the 4-momentum P_i^μ of an initial or final state particle satisfies $P_i^\mu P_{i\mu} = m_i^2$.

[3] Apply directly the table of *Feynman* rules provided during the lectures in the case where the V interaction to $S_i^\pm S_j^\pm$ does not change the species: $i = j$.

[4] Mathematically and physically (using the correct terminology).

7. Calculate the *Lorentz* product $q^\alpha q_\alpha$ within the center-of-mass frame (defined by $\vec{p}+\vec{p'} = \vec{0}$) and present the result as a function only of the total energies E_k, $E_{k'}$ of the two final particles.

8. Give a simplified form of the reaction amplitude $\mathcal{M}$ thanks to the above answers.

***