PROBLEM OF PARTICLE PHYSICS

Rotated spinors

1. **Momentum operator.-** The generic expression for the *Dirac* spinor [with index $j = 1, 2, 3, 4$] of a fermionic particle (with mass m) reads as,

$$\psi_j(x^\alpha) \;=\; u_j \; e^{-\frac{i}{\hbar} p_\alpha x^\alpha}$$

where the *Lorentz* [1] product $p_\alpha x^\alpha$ involves the 4-momentum components $p^\alpha = (E, \vec{p})$ and the 4-vector coordinates $x^\alpha = (t, \vec{x})$, α being a *Lorentz* index: $\alpha = 0, 1, 2, 3$. Calculate the eigenvalues of this spinor with respect to the momentum operator, $\hat{P}^k = -i\hbar\partial_k$ ($k = 1, 2, 3$).

2. **Helicity operator.**

   (a) Let us consider the momentum, $\vec{p} = (p \sin\theta, 0, p\cos\theta)$, with norm $p \equiv ||\vec{p}||$. Interpret geometrically this angle θ. Then show that

$$\frac{1}{p}\,\vec{p}.\vec{\sigma} \;=\; \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

   where σ^k denote the three 2×2 *Pauli* matrices.

   (b) Show that the eigenvalues of $\vec{p}.\vec{\sigma}/p$ are ± 1. We denote, in an obvious correspondence, $V_\pm$ as the two associated eigenvectors.

   (c) Give the conditions that must satisfy the two V_- components.

   (d) Find out V_- by rewriting [2] the conditions of previous question in terms of $\theta/2$ rather than θ. Assume V_- to be normalised to unity and real, for simplicity.

   (e) Similarly, obtain the two V_+ components.

   (f) The 4×4 Helicity operator is defined by,

$$\hat{h} \;=\; \frac{\hbar}{2p}\begin{pmatrix} \vec{p}.\vec{\sigma} & 0 \\ 0 & \vec{p}.\vec{\sigma} \end{pmatrix} .$$

   Verify that the following spinors (up to normalisation factors) are eigenvectors of $\hat{h}$,

$$u_j^{(1)} = \begin{pmatrix} V_+ \\ \frac{\vec{p}.\vec{\sigma}}{E+m} V_+ \end{pmatrix},\; u_j^{(2)} = \begin{pmatrix} V_- \\ \frac{\vec{p}.\vec{\sigma}}{E+m} V_- \end{pmatrix},\; u_j^{(3)} = \begin{pmatrix} \frac{\vec{p}.\vec{\sigma}}{E-m} V_+ \\ V_+ \end{pmatrix},\; u_j^{(4)} = \begin{pmatrix} \frac{\vec{p}.\vec{\sigma}}{E-m} V_- \\ V_- \end{pmatrix},$$

   and provide their respective eigenvalues.

[1] We consider the metric, $g^{\alpha\beta} = diagonal(+1, -1, -1, -1)$.

[2] Recall the trigonometrical formula, $\cos\theta = 1 - 2\sin^2(\theta/2) = 2\cos^2(\theta/2) - 1$, and, $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$.

3. **Hamiltonian operator.**

   (a) Write the *Dirac* equation satisfied by the $u_j^{(n)}$ spinors (where $n = 1, 2, 3, 4$).

   (b) Express the 4×4 Hamiltonian ($\hat{H}$) eigenvectors [3] in terms of $u_j^{(n)}$ and give their energy eigenvalues as a function of E, specifying its sign in each case.

4. **The rotation.**

   (a) Give a set of four *Dirac* spinors being common eigenvectors of $\hat{h}$, $\hat{P}^k$ and $\hat{H}$ (that can be embedded into a 4-momentum operator $\hat{P}^\alpha$). Can they constitute an orthonormal-basis of the whole *Hilbert* space? Show that this common basis is unique, by considering the eigenvalues/eigenvectors correspondences; how is called such a set of operators $\{\hat{h}, \hat{P}^\alpha\}$?

   (b) Based on Question n'2e, compare the obtained eigenvalues and eigenvectors $\left(u_j^{(n)} e^{-\frac{i}{\hbar} p_\alpha x^\alpha}\right)$ of the observables $\hat{h}, \hat{P}^\alpha$ with the case [4] of a momentum along the third spatial axis, $\vec{p} = (0, 0, p)$. Then make a physical comment.

   (c) In the case $\vec{p} = (0, 0, p)$, the first spinor is given by,

$$u_j^{(1)}|_{p_z} \; = \; \begin{pmatrix} 1 \\ 0 \\ \frac{p}{E+m} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{pmatrix} .$$

   Calculate the effect on this spinor of the matrix,

$$U(\theta) \; = \; \begin{pmatrix} U_{2\times2}(\theta) & 0 \\ 0 & U_{2\times2}(\theta) \end{pmatrix} \; \text{with} \; U_{2\times2}(\theta) \; = \; \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} .$$

   Compare the result with the $u_j^{(1)}$ spinor found in Questions n'2b, 2e and 2f. Deduce the precise physical interpretation of the matrix $U(\theta)$. Show that

$$U_{2\times2}(\theta) \; = \; \cos(\theta/2) - i\sigma^2 \sin(\theta/2) \; .$$

   One can then demonstrate that $U_{2\times2}(\theta) = e^{-i\theta\frac{\sigma^2}{2}}$. What is the name in group theory of the operator $\frac{\sigma^2}{2}$?

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[3] In the *Dirac-Pauli* representation.

[4] Studied during the N.P. lectures.