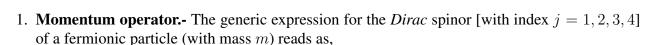
PROBLEM OF PARTICLE PHYSICS

Rotated spinors



$$\psi_j(x^{\alpha}) = u_j e^{-\frac{i}{\hbar}p_{\alpha}x^{\alpha}}$$

where the Lorentz¹ product $p_{\alpha}x^{\alpha}$ involves the 4-momentum components $p^{\alpha} = (E, \vec{p})$ and the 4-vector coordinates $x^{\alpha} = (t, \vec{x})$, α being a Lorentz index: $\alpha = 0, 1, 2, 3$. Calculate the eigenvalues of this spinor with respect to the momentum operator, $\hat{P}^k = -i\hbar\partial_k$ (k = 1, 2, 3).

2. Helicity operator.

(a) Let us consider the momentum, $\vec{p} = (p \sin \theta, 0, p \cos \theta)$, with norm $p \equiv ||\vec{p}||$. Interpret geometrically this angle θ . Then show that

$$\frac{1}{p}\vec{p}.\vec{\sigma} = \begin{pmatrix} \cos\theta & \sin\theta\\ \sin\theta & -\cos\theta \end{pmatrix}$$

where σ^k denote the three 2 × 2 *Pauli* matrices.

- (b) Show that the eigenvalues of $\vec{p}.\vec{\sigma}/p$ are ± 1 . We denote, in an obvious correspondence, V_{\pm} as the two associated eigenvectors.
- (c) Give the conditions that must satisfy the two V_{-} components.
- (d) Find out V_{-} by rewriting ² the conditions of previous question in terms of $\theta/2$ rather than θ . Assume V_{-} to be normalised to unity and real, for simplicity.
- (e) Similarly, obtain the two V_+ components.
- (f) The 4×4 Helicity operator is defined by,

$$\hat{h} = \frac{\hbar}{2p} \begin{pmatrix} \vec{p}.\vec{\sigma} & 0\\ 0 & \vec{p}.\vec{\sigma} \end{pmatrix} .$$

Verify that the following spinors (up to normalisation factors) are eigenvectors of h,

$$u_j^{(1)} = \begin{pmatrix} V_+ \\ \frac{\vec{p}.\vec{\sigma}}{E+m}V_+ \end{pmatrix}, \ u_j^{(2)} = \begin{pmatrix} V_- \\ \frac{\vec{p}.\vec{\sigma}}{E+m}V_- \end{pmatrix}, \ u_j^{(3)} = \begin{pmatrix} \frac{\vec{p}.\vec{\sigma}}{E-m}V_+ \\ V_+ \end{pmatrix}, \ u_j^{(4)} = \begin{pmatrix} \frac{\vec{p}.\vec{\sigma}}{E-m}V_- \\ V_- \end{pmatrix},$$

and provide their respective eigenvalues.

¹ We consider the metric, $g^{\alpha\beta} = diagonal(+1, -1, -1, -1)$.

² Recall the trigonometrical formula, $\cos \theta = 1 - 2\sin^2(\theta/2) = 2\cos^2(\theta/2) - 1$, and, $\sin \theta = 2\sin(\theta/2)\cos(\theta/2)$.

- (a) Write the *Dirac* equation satisfied by the $u_i^{(n)}$ spinors (where n = 1, 2, 3, 4).
- (b) Express the 4×4 Hamiltonian (\hat{H}) eigenvectors ³ in terms of $u_j^{(n)}$ and give their energy eigenvalues as a function of E, specifying its sign in each case.

4. The rotation.

- (a) Give a set of four *Dirac* spinors being common eigenvectors of \hat{h} , \hat{P}^k and \hat{H} (that can be embedded into a 4-momentum operator \hat{P}^{α}). Can they constitute an orthonormal-basis of the whole *Hilbert* space? Show that this common basis is unique, by considering the eigenvalues/eigenvectors correspondences; how is called such a set of operators $\{\hat{h}, \hat{P}^{\alpha}\}$?
- (b) Based on Question n'2e, compare the obtained eigenvalues and eigenvectors $\left(u_{j}^{(n)}e^{-\frac{i}{\hbar}p_{\alpha}x^{\alpha}}\right)$ of the observables $\hat{h}, \hat{P}^{\alpha}$ with the case ⁴ of a momentum along the third spatial axis, $\vec{p} = (0, 0, p)$. Then make a physical comment.
- (c) In the case $\vec{p} = (0, 0, p)$, the first spinor is given by,

$$u_j^{(1)}|_{p_z} = \begin{pmatrix} 1 \\ 0 \\ \frac{p}{E+m} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{pmatrix} .$$

Calculate the effect on this spinor of the matrix,

$$U(\theta) = \begin{pmatrix} U_{2\times 2}(\theta) & 0\\ 0 & U_{2\times 2}(\theta) \end{pmatrix} \text{ with } U_{2\times 2}(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2)\\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.$$

Compare the result with the $u_j^{(1)}$ spinor found in Questions n'2b, 2e and 2f. Deduce the precise physical interpretation of the matrix $U(\theta)$. Show that

$$U_{2\times 2}(\theta) = \cos(\theta/2) - i\sigma^2 \sin(\theta/2)$$

One can then demonstrate that $U_{2\times 2}(\theta) = e^{-i\theta \frac{\sigma^2}{2}}$. What is the name in group theory of the operator $\frac{\sigma^2}{2}$?

³ In the *Dirac-Pauli* representation.

⁴ Studied during the N.P. lectures.