PROBLEM OF PARTICLE PHYSICS

Chirality

I) THE HELICITY

1. *Dirac* **equation.-** The solutions of the *Dirac* equation read as, $\psi = u\, e^{-\frac{i}{\hbar}p_\alpha x^\alpha}$, where the *Lorentz* [1] product $p_\alpha x^\alpha$ involves the 4-momentum $p^\alpha = (E/c, \vec{p})$ and the coordinate [2] 4-vector $x^\alpha = (c\,t, \vec{x})$. The u spinor satisfies the equation:

$$(\gamma^\mu p_\mu - m\,c)\,u = 0 \tag{1}$$

   where γ^μ denotes the *Dirac* matrices and m the elementary fermion mass. How many components does the spinor u possess generically? In turn, the fundamental relation (1) represents the equation of motion for which kind of particle?

2. Equation (1) induces some conditions on the two spinors, u_1, u_2, entering $u \equiv (u_1, u_2)^t$. By expressing the *Dirac* matrices within the *Dirac-Pauli* representation, find those conditions in terms of the *Pauli* matrices σ^i $[i = 1, 2, 3]$.

3. **Solutions.-** Show that the conditions obtained at the previous question lead to two spinor solutions of the following form,

$$u^{(n)} = N \begin{pmatrix} \chi^{(n)} \\ \frac{c\,\vec{p}.\vec{\sigma}}{(E+mc^2)}\,\chi^{(n)} \end{pmatrix}, \text{ with } n = 1, 2\,. \tag{2}$$

   These two solutions are eigenstates of the Hamiltonian with the degenerate eigenvalue $E > 0$. How many other solutions exist for the u spinor? What are their energy eigenvalues?

4. Provide the explicit expressions for the two solutions $u^{(1,2)}$, from relation (2), in the canonical case,

$$\chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

   Use the *Pauli* matrix expressions [3] and the natural unit system where the light velocity c is equal to unity [4].

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[1] We consider the usual metric, $g^{\alpha\beta} = diagonal(+1, -1, -1, -1)$.

[2] α being a *Lorentz* index: $\alpha = 0, 1, 2, 3$.

[3] *Pauli* matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

[4] From now on, we will take $c = 1$.

5. **Spin.-** We introduce the helicity operator,

$$\hat{h} \hat{=} \frac{\hbar}{2} \begin{pmatrix} \frac{\vec{p}}{||\vec{p}||}.\vec{\sigma} & 0 \\ 0 & \frac{\vec{p}}{||\vec{p}||}.\vec{\sigma} \end{pmatrix} .$$

   (a) In the simplifying axis choice where the entire $\vec{p}$ momentum is along the third (z) axis, show that $\chi^{(n)}$ are eigenstates of $\vec{p}.\vec{\sigma}/||\vec{p}||$ and give their eigenvalues.

   (b) Deduce, from question (5a), the respective eigenvalues of the two $u^{(1,2)}$ spinors with respect to the helicity observable. What does represent physically the helicity?

6. BONUS QUESTION: The chirality operator is defined as the 4×4 matrix, $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Express γ^5 as a function of the *Pauli* matrices, within the *Dirac-Pauli* representation. Then calculate the product $\sigma^1\sigma^2\sigma^3$. Finally, deduce that,

$$\gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} . \tag{3}$$

7. Compute $(\vec{p}.\vec{\sigma})^2$ and express it in term of the identity operator.

8. **Relativistic limit.-** Let us consider in this question the relativistic limit $||\vec{p}|| \gg m$.

   (a) Express the relativistic energy E as a function of $||\vec{p}||$ only (in the case $E > 0$).

   (b) Using this energy result together with question (7) and equation (3), express $\gamma^5 u^{(n)}$ in term of $\hat{h}\, u^{(n)}$, where the spinors are given by relation (2).

   (c) Deduce the eigenstates and eigenvalues of $\frac{\hbar}{2}\gamma^5$.

   (d) Based on question (8c), compare the properties of the operators $\frac{\hbar}{2}\gamma^5$ and $\hat{h}$. Conclude physically. Are the chirality and helicity always *Lorentz* invariant?

II) THE PARITY

1. **Transformation.-** The *Lorentz* matrix for the Parity transformation on the spatial coordinates clearly reads as,

$$\Lambda^{\mu}_{\cdot\,\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{4}$$

   Express [5] the transformed 4-vector x'^{α} in terms of this $\Lambda^{\mu}_{\cdot\,\nu}$ matrix and the initial x^{α}. Deduce the new coordinates t', x', y', z' as functions of t, x, y, z.

2. Using the covariant relation, $\Lambda^{\mu}_{\cdot\,\nu}\, \gamma^{\nu} = P^{-1}\gamma^{\mu}\, P$, where P is the Parity operator acting on the spinorial *Hilbert* space, express the quantities $P^{-1}\gamma^0 P$ and $P^{-1}\gamma^k P$ $[k = 1, 2, 3]$ in terms of *Dirac* matrices.

[5] Choosing of course all the *Lorentz* indices in a consistent way.

3. Show that $P = \gamma^0$ indeed fulfils the two properties derived in the previous question.
4. **Bilinear terms.-** We denote by ψ a general solution of the *Dirac* equation and $\bar{\psi} \hat{=} \psi^\dagger \gamma^0$.
   (a) Give the scalar term $\bar{\psi}'\psi'$, transformed under parity, in terms of ψ. Use question (3).
   (b) Same question for the transformed pseudo-scalar term $\bar{\psi}'\gamma^5\psi'$. Simplify the result.
   (c) Same question for the components $\bar{\psi}'\gamma^0\psi'$ and $\bar{\psi}'\gamma^k\psi'$ $[k = 1, 2, 3]$ of the vector term.
   (d) Same question for the parts $\bar{\psi}'\gamma^5\gamma^0\psi'$ and $\bar{\psi}'\gamma^5\gamma^k\psi'$ of the axial vector term.
   (e) Justify the terminology of those 4 bilinear terms.
   (f) Comment about the chirality and parity.

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