PROBLEM OF PARTICLE PHYSICS

Energy projectors

1. The 4 solutions ($n = 1, 2, 3, 4$) of the *Dirac* equation, for a spin 1/2 particle, are the 4-component spinors [with subscript $j = 1, 2, 3, 4$]: $\psi_j^{(n)} = u_j^{(n)} \, e^{-\frac{i}{\hbar} p^\alpha x_\alpha}$, where the *Lorentz* product $p^\alpha x_\alpha$ involves the particle 4-momentum p^α and 4-vector [1] coordinates x^α.
   (a) Calculate $\partial_k \, \psi_j^{(n)}$ where $\partial_k = \frac{\partial}{\partial x^k}$.
   (b) Is the quantity $\bar{u}^{(n)} \gamma^\mu \gamma^\nu u^{(n)}$ Hermitian? (γ^μ denotes the 4×4 *Dirac* matrices)
2. Show that $\not{p}\not{p} = m^2 \, \mathbb{1}_{4\times4}$, where $\not{p} \hat{=} \gamma_\mu p^\mu$ and m is the considered particle mass.
3. Let us introduce the operators, $\Lambda_\pm = \frac{1}{2m}(m \, \mathbb{1} \pm \not{p})$. The goal here is to demonstrate that $\Lambda_\pm$ satisfy the projector properties.
   (a) Calculate the sum $\Lambda_+ + \Lambda_-$.
   (b) Calculate the product $\Lambda_+ \times \Lambda_-$.
   (c) Calculate $(\Lambda_+)^2$ and then $(\Lambda_-)^2$.
4. Within the *Dirac-Pauli* representation, $u_j^{(1,2)}$ are the Hamiltonian eigenstates with positive energy, whereas $u_j^{(3,4)}$ have negative eigenvalues. Justify that any 4-component spinor of the *Hilbert* space can be developed accordingly to $\sum_{n=1}^{4} c_n \, u_j^{(n)}$ where c_n are complex numbers.
5. The aim of this question is to determine the effect the Λ_+ projector. For this purpose, calculate its action on a generic spinor:

$$\sum_{j=1}^{4} \Lambda_{+ij} \left(\sum_{n=1}^{4} c_n \, u_j^{(n)} \right) .$$

   Make use of the spinor ortho-normalisation condition, $\sum_{i=1}^{4} \bar{u}_i^{(n)} \, u_i^{(m)} = 2 \, m \, \delta_{nm}$ (with $n, m = 1, 2, 3, 4$), and the completeness relation, $m \, \mathbb{1}_{ij} + \not{p}_{ij} = \sum_{n=1}^{2} u_i^{(n)} \bar{u}_j^{(n)}$.
6. Based on Questions 3 and 5, find out the Λ_- projector effect.

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[1] α being a *Lorentz* index: $\alpha = 0, 1, 2, 3$.