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Particle Physics : Introducing the Standard Model

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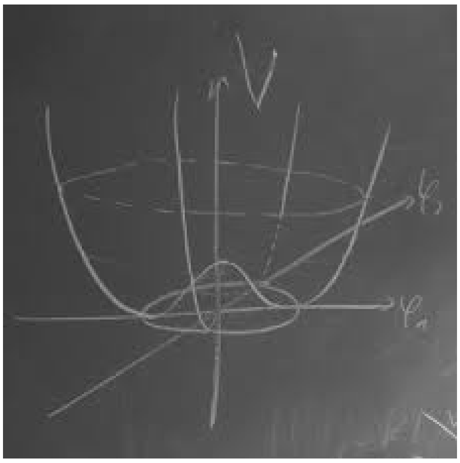
Abstract

The present notes provide exclusively a synthesis of the formalism presented in the lecture series “Particle Physics : Introducing the Standard Model” [part of the Minor Course “Nuclear and Particle physics”] at the 1st year Master *General Physics* of the Paris-Saclay University. The spinors and Dirac matrices are defined and described in details. Their Lorentz transformations are discussed in order to build Lorentz invariant Lagrangians. The Dirac equation is studied and solved explicitly. The operators of Helicity and Charge Conjugation are presented. The Higgs mechanism generating masses to fermions is explained together with the Goldstone theorem.



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1 The *Dirac* equation

1.1 The original form

Historically, in order to avoid the problem of negative energy, P.A.M.Dirac has searched for a linear quantum relativistic equation, as an alternative to the Klein-Gordon equation,

$$\partial_t \rho + \vec{\nabla}.\vec{j} = 0 \Leftarrow \boxed{(\partial_\mu \partial^\mu + \frac{m^2c^2}{\hbar^2})\Phi(x^\nu) = 0} \tag{1}$$

where Φ is the wave function and ρ the probability density. The wanted linear dependence in ∂_t together with the covariance form (∂^μ form) lead to a linearity in the space derivatives as well ($\hat{\vec{P}} = -i\hbar\vec{\nabla}$).

The general form is:

$$\boxed{\hat{H}\Psi = i\hbar\ \partial_t \Psi(\vec{x}, t)}\ \text{(Schrödinger Equation)}, \tag{2}$$

with

$$\boxed{\hat{H} = \vec{\alpha}.\hat{\vec{P}} + \beta m}$$

In particular, there are no constant terms to satisfy the conditions (linear equations in ∂_t):

$$\begin{aligned} \hat{H}^2\Psi &= (\hat{\vec{P}}^2c^2 + m^2c^4)\Psi \\ \Leftrightarrow\ E^2 &= (\hat{\vec{P}}^2c^2 + m^2c^4) \quad \text{(relativistic energy).} \end{aligned} \tag{3}$$

Remark: There arises a global consistency of the theory as the Schrödinger equation together with the relativistic energy expression give rise to the Klein-Gordon equation (for a spinless particle $s = 0$) or the Dirac equation ($s = \frac{1}{2}$).

Determination of α, β: From Eq.(3), we have:

$$\begin{aligned} H^2 &= (\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \beta m)(\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \beta m) \\ &= \sum_{i=1}^{3} (\alpha_i P_i)^2 + \sum_{i \neq j,\, i,j \in \{1,2,3\}} [\alpha_i P_i.\alpha_j P_j + \alpha_j P_j.\alpha_i P_i] + \sum_{i=1}^{3} [\alpha_i P_i \beta + \beta \alpha_i P_i].m + (\beta m)^2 \\ &= \sum_{i=1}^{3} (\alpha_i P_i)^2 + \sum_{i \neq j,\, i,j \in \{1,2,3\}} (\alpha_i \alpha_j + \alpha_j \alpha_i) P_i P_j + \sum_{i=1}^{3} [\alpha_i \beta + \beta \alpha_i] P_i m + \beta^2 m^2 \end{aligned}$$

so that $\{\alpha_i, \alpha_j\} = \{\alpha_i, \beta\} = 0$ and $\alpha_i^2 = \beta^2 = 1$.

Four properties are induced:

- $H^\dagger = \hat{P}_i^\dagger \alpha_i^\dagger + m\beta^\dagger = H = \hat{P}_i \alpha_i + \beta m$.
- $\beta|\beta\rangle = b|\beta\rangle$, so $\beta^2|\beta\rangle = b(\beta|\beta\rangle) \Leftrightarrow \beta^2|\beta\rangle = b^2\, |\beta\rangle \Leftrightarrow b = \pm 1$.
- Anti-commutativity conditions.
- $Tr[\alpha_i] = Tr[\beta] = 0$.

  Example: One can consider $\Psi(\vec{x}, t) = N\, \langle \vec{x}\,|\,\vec{p}(t)\rangle \otimes |\eta_{spin}\rangle$.

The lowest dimensionality matrices satisfying those properties are 4×4 matrices.

$\Rightarrow$ structure describing spin $\frac{1}{2}$ particle (and its anti-particle):

$\boxed{s = \frac{1}{2},\ -\frac{1}{2} \leqslant s_z \leqslant \frac{1}{2}}$

$$\begin{aligned}
\hat{\vec{S}}^2 |\, s,\, s_z\rangle &= s(s+1)\hbar^2\ |s, s_z\rangle = \frac{3}{2}\hbar^2\ \left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle \\
\hat{S}_z\, |\, s,\, s_z\rangle &= s_z\hbar\ \ |\, s,\, s_z\rangle = \pm\frac{\hbar}{2}\ \left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle \\
\hat{S}_i &= \frac{\sigma_i\, \hbar}{2}\ \text{(Pauli matrices)}
\end{aligned}$$

The Pauli matrices:

$$\begin{cases}
\sigma_x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{cases}$$

The choice of $(\vec{\alpha}, \beta)$ matrices is not unique.

Examples:

- The Dirac-Pauli representation:

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix},\ \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$

- The Weyl representation:

$$\alpha_i = \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix},\ \beta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

Physics results are independent of this conventional frame.

1.2 The covariant form

The β general form:

$$\begin{aligned} \beta.\,(\alpha_i \hat{P}_i c + \beta m c^2)\psi &= i\,\hbar c\,\partial_t\,\beta\,\psi(x^\nu) \\ \left[\beta\alpha_i\left(-i\hbar\frac{\partial}{\partial x^i}\right)c + m\frac{c^2}{\hbar} - i\hbar c\beta\frac{\partial}{c\partial_t}\right]\psi &= 0 \\ \left(i\gamma^\mu\frac{\partial}{\partial x^\mu} - \frac{m}{\hbar c}\right)\psi &= 0, \end{aligned} \tag{4}$$

where $\gamma^\mu = \left\{\beta,\, \beta\alpha^i\right\}$. The Dirac matrices γ^μ will be shown to be 4-vectors indeed.

$$\boxed{\sum_{k=1}^{4}\left\{i\,\gamma^\mu_{jk}\partial_\mu - m\,\delta_{jk}\right\}\psi_k(x^\nu) = 0}$$

Dirac matrix properties:

$$\boxed{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}} \tag{5}$$

Exercise: Demonstrate this property.

- $\mu = \nu = 0$: $(\gamma^0)^2 + (\gamma^0)^2 = 2.\mathbb{1}$ true as $\beta^2 = \mathbb{1}$.
- $\mu = 1, \nu = 0$: $\gamma^1\gamma^0 + \gamma^0\gamma^1 = 0$, $\beta(\alpha^1\beta) + \beta(\beta\alpha^1) = 0$.
- $\mu = 2, \nu = 3$: $(\beta\alpha^2)(\beta\alpha^3) + (\beta\alpha^3)(\beta\alpha^2) = 0$, $-\alpha^2\alpha^3 - \alpha^3\alpha^2 = 0$.
- $\mu = \nu = 3$: $\beta\alpha^3.\beta\alpha^3 + \beta\alpha^3.\beta\alpha^3 = 2(-1)$.

We have also that

$$\boxed{\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0} \tag{6}$$

Indeed, if $\mu = 0$, $(\gamma^0)^\dagger = \gamma^0\gamma^0\gamma^0$.
If $\mu = i \neq 0$, $(\gamma^i)^\dagger = \gamma^0\gamma^i\gamma^0 = -(\gamma^0)^2\gamma^i$.

There are other useful properties:

- $(\gamma^0)^\dagger = \gamma^0$, $(\gamma^0)^2 = \mathbb{1}$ as $\gamma^0 = \beta$ Hermitian (see the 4 properties above);
- $(\gamma^i)^\dagger = -\gamma^i$ as $(\gamma^i)^\dagger = (\beta\alpha^i)^\dagger = \alpha^i\beta = -\beta\alpha^i$;
- $\left(\gamma^i\right)^2 = -\mathbb{1}$ since $\gamma^i\gamma^i = \beta\alpha^i\beta\alpha^i = -\beta^2(\alpha^i)^2 = -\mathbb{1}$.

1.3 The Hermitian form

As one could take the complex conjugate of the Klein-Gordon equation, we take here the Hermitian conjugate of Eq.(4),

$$\begin{aligned} i\gamma^0\,\partial_0\psi \,+\, i\gamma^i\,\partial_i\psi - m\psi &= 0 \\ -i\,\partial_0\psi^\dagger(\gamma^0)^\dagger \,+\, -i\,\partial_i\psi^\dagger(\gamma^i)^\dagger \,-\, m\psi^\dagger &= 0 \\ -i\,\partial_0\psi^\dagger\gamma^0\gamma^0 \,+\, i\,\partial_i\psi^\dagger\gamma^i\gamma^0 \,-\, m\psi^\dagger\gamma^0 &= 0 \\ i(\partial_\mu\overline{\psi})\,\gamma^\mu + m\overline{\psi} &= 0. \end{aligned}$$

Hence,

$$\boxed{i(\partial_\mu\overline{\psi})\,\gamma^\mu + m\overline{\psi} = 0.} \tag{7}$$

As for the Klein-Gordon equation, $\overline{\psi}\times$[Eq.(4)] + [Eq.(7)]$\times\psi$ gives:

$$\begin{aligned} \overline{\psi}\,i\gamma^\mu\,\partial_\mu\psi \,-\, m\overline{\psi}\psi \,+\, i(\partial_\mu\overline{\psi})\,\gamma^\mu\,\psi \,+\, m\overline{\psi}\psi &= 0+0 \\ \partial_\mu(\overline{\psi}\gamma^\mu\psi) &= 0 \end{aligned}$$

Defining $j^\mu = \overline{\psi}\gamma^\mu\psi$, we have $\partial_\mu j^\mu = 0$ (continuity equation). $\vec{j}$ is the flux density while $j^0 \equiv \rho$ is the probability density (of spatial presence).

$$\rho = \overline{\psi}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi = \sum_{k=1}^{4}\psi_k^*\psi_k = \langle\eta_s|\eta_s\rangle\,|\,\langle\vec{x}\,|\,\psi(t)\rangle\,|^2 \equiv \frac{dP}{d^3x}(t) > 0$$

One can define the electron charge current: $\boxed{j^\mu = -e\overline{\psi}\gamma^\mu\psi}$.

Eq.(7) is to be compared to the Klein-Gordon equation for complex conjugate field ϕ^*.

1.4 The Lagrangian description

For a free particle, with spin $\frac{1}{2}$ and mass m, within a quantum relativistic framework, the Lagrangian reads as,

$$\boxed{\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi \,-\, m\overline{\psi}\psi}\,.$$

- Euler-Lagrange equation for the field ψ:

$$\frac{\partial\mathcal{L}}{\partial\psi} = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right].$$

Exercise: Write this equation.
$-m\overline{\psi} = \partial_\mu\left[i\overline{\psi}\gamma^\mu\right] \iff Eq.(7).$

- Euler-Lagrange equation for the field $\overline{\psi}$:

$$\frac{\partial\mathcal{L}}{\partial\overline{\psi}} = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi})}\right].$$

  Exercise: Write this equation.
  $i\gamma^\mu\partial_\mu\psi - m\psi = 0 \iff Eq.(4)$.

Exercise: Show that $\mathcal{L}$ is Hermitian.

$\left(\overline{\psi}\psi\right)^\dagger = \psi^\dagger\left(\psi^\dagger\gamma^0\right)^\dagger = \psi^\dagger\left(\gamma^0\right)^\dagger\psi = \overline{\psi}\psi \ni \mathcal{L}$.
$\left(\overline{\psi}i\gamma^\mu\partial_\mu\psi\right)^\dagger = \partial_\mu\overline{\psi}\gamma^\mu\psi(-i)$, which gives, by integration by part, $i\overline{\psi}\gamma^\mu\partial_\mu\psi \ni \mathcal{L}$.

1.5 Solutions

1.5.1 Wave function part

Exercise: Show that each of the four components of the spinor ψ_i satisfies the Klein-Gordon equation.

Let us apply $\gamma^\alpha\partial_\alpha$ on the Dirac equation Eq.(4):

$$\begin{aligned}
\gamma^\alpha\partial_\alpha i\gamma^\mu\partial_\mu\psi - m\gamma^\alpha\partial_\alpha\psi &= 0\\
\frac{i}{2}\left[\gamma^\alpha\gamma^\mu\partial_\alpha\partial_\mu + \gamma^\mu\gamma^\alpha\partial_\mu\partial_\alpha\right] &= \frac{i}{2}\left[\gamma^\alpha\gamma^\mu + \gamma^\mu\gamma^\alpha\right]\partial_\alpha\partial_\mu\\
\left(ig^{\alpha\mu}\partial_\alpha\partial_\mu + im^2\right)\psi &= 0\\
\left(g^{\alpha\mu}\partial_\alpha\partial_\mu + m^2\right)\psi_i &= 0
\end{aligned}$$

Therefore,

$$\psi_j = u_j\,\mathrm{e}^{-\frac{i}{\hbar}p_\mu x^\mu}, \tag{8}$$

so one can identify $u_i \equiv |\eta_{spin}\rangle$ and $\langle\vec{x}|\psi(t)\rangle \equiv \mathrm{e}^{-\frac{i}{\hbar}p_\mu x^\mu} = \mathrm{e}^{-\frac{i}{\hbar}Et+\frac{i}{\hbar}\vec{p}\cdot\vec{x}}$.

Exercise: Of which operator(s) ψ_{sol} is eigenstate?

We have $\hat{P}_i\,\psi = -i\hbar\,\partial_i\left(u\,\mathrm{e}^{-\frac{i}{\hbar}Et}\,\mathrm{e}^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}}\right) = -i\hbar u\,\mathrm{e}^{-\frac{i}{\hbar}Et}\,\partial_i\,\mathrm{e}^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}} = p_i\psi$, so
$\hat{P}_i\hat{P}_i\psi = p_i\left(\hat{P}_i\psi\right) = p_i^2\psi$, then $\hat{\vec{P}}^2\psi = \vec{p}^{\,2}\psi$ and finally, by Eq.(3), $H^2\psi = E^2\psi$.

Remark: $\hat{P}_i\psi = p_i\psi \Rightarrow \hat{P}_i u\,\langle\vec{x}\,|\,\psi(t)\rangle = p_i u\,\langle\vec{x}\,|\,\psi(t)\rangle \Leftrightarrow \hat{P}_i\,|\,\psi(t) >= p_i\,|\,\psi(t)\rangle$.

ψ should satisfy as well the linear form of Schrödinger equation:

$$H\psi = i\hbar\partial_t\psi \Leftrightarrow H\psi = i\hbar\partial_t\left(u\,\mathrm{e}^{-\frac{i}{\hbar}p_\mu x^\mu}\right) = i\hbar u\left(-\frac{iE}{\hbar}\right)\mathrm{e}^{-\frac{i}{\hbar}p_\mu x^\mu} = E\psi\ . \tag{9}$$

1.5.2 Determination of the u-spinor

From Eq.(8) and Eq.(4), we have:

$$\begin{aligned}
i\,\gamma^{\mu}\,\frac{\partial}{\partial x^{\mu}} u\, \mathrm{e}^{-\frac{i}{\hbar}p_{\nu}x^{\nu}} \;-\frac{m}{\hbar/c}\psi &= 0 \\
i\,\gamma^{\mu}\,u\left(-\frac{i}{\hbar}\,p^{\nu}\frac{\partial x^{\nu}}{\partial x^{\mu}}\right)\mathrm{e}^{-\frac{i}{\hbar}p^{\nu}x^{\nu}} \;-\frac{m}{\hbar/c}\,u\,\mathrm{e}^{-\frac{i}{\hbar}p_{\nu}x^{\nu}} &= 0 \\
(\gamma^{\mu}p_{\mu}\;-\;m\,c)\;u &= 0 \\
(\not{p}\;-\;m\,c)\;u &= 0 \qquad (10)
\end{aligned}$$

Then, Eq.(3) and Eq.(9) give:

$$\begin{aligned}
\left(\vec{\alpha}\cdot\hat{\vec{P}}\,+\,\beta m\right)\psi &= E\psi \\
\alpha_i\;(-i\,\hbar\,\partial_i)\;u\,\mathrm{e}^{-\frac{i}{\hbar}p^{\nu}x^{\nu}} \;+\beta m\;u\,\mathrm{e}^{-\frac{i}{\hbar}p^{\nu}x^{\nu}} &= E u\,\mathrm{e}^{-\frac{i}{\hbar}p^{\nu}x^{\nu}} \\
\alpha_i\;\left(-i\hbar\left[\frac{i}{\hbar}p^{i}\right]\right)\;u\,\mathrm{e}^{-\frac{i}{\hbar}p_{\nu}x^{\nu}} \;+\beta m\;u\,\mathrm{e}^{-\frac{i}{\hbar}p^{\nu}x^{\nu}} &= E u\,\mathrm{e}^{-\frac{i}{\hbar}p_{\nu}x^{\nu}} \\
(\vec{\alpha}\cdot\vec{p}\,+\,\beta m)\;u &= E u
\end{aligned}$$

There are four independent solutions for the spinor 'u', two with $E > 0$, two with $E < 0$.

In the Dirac-Pauli representation...

ⓐ Let us assume that $\vec{p} = \vec{0}$ (in Dirac equation).

From Eq.(11), we have $Hu \;=\; \beta m u \;=\; E u \Leftrightarrow \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & -m & 0 \\ 0 & 0 & 0 & -m \end{pmatrix} u = Eu.$

The eigenvalues of H are m and $-m$ (degeneracy of degree 2). The eigenvectors are

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\;\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\;\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\;\begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

ⓑ Now, let us assume that $\vec{p} \neq \vec{0}$.

From Eq.(11) and the Dirac-Pauli equation, we have

$$\begin{aligned}
Hu \;=\; Eu \;&=\; \left\{p_1\begin{pmatrix}0 & \sigma_1\\ \sigma_1 & 0\end{pmatrix} + p_2\begin{pmatrix}0 & \sigma_2\\ \sigma_2 & 0\end{pmatrix} + p_3\begin{pmatrix}0 & \sigma_3\\ \sigma_3 & 0\end{pmatrix} + \begin{pmatrix}m & 0\\ 0 & -m\end{pmatrix}\right\} u \\
&=\; \begin{pmatrix}m & \vec{p}\cdot\vec{\sigma}\\ \vec{p}\cdot\vec{\sigma} & -m\end{pmatrix}\begin{pmatrix}u_a\\ u_b\end{pmatrix}
\end{aligned}$$

Then, we have:

$$\left\{\begin{aligned} \vec{p}\cdot\vec{\sigma}u_b &= u_a\,(E-m) \\ \vec{p}\cdot\vec{\sigma}u_a &= u_b\,(E+m) \end{aligned}\right. \qquad (11)$$

$$\Leftrightarrow\quad \left\{\begin{aligned} (\vec{p}\cdot\vec{\sigma})^2\,u_b &= \vec{p}\cdot\vec{\sigma}u_a(E-m) \;=\; u_b(E+m)(E-m) \;=\; u_b\left(E^2-m^2\right) \\ (\vec{p}\cdot\vec{\sigma})^2\,u_a &= \vec{p}\cdot\vec{\sigma}u_b(E+m) \;=\; u_a(E-m)(E+m) \;=\; u_a\left(E^2-m^2\right) \end{aligned}\right.$$

Remark: The combination of the two lines of Eq.(11) gives no new information, so only one of those two equations fixes the solution 'u'.

We have:

$$\begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix} \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix} = \begin{pmatrix} p_3^2 + (p_1^2 + p_2^2) & 0 \\ 0 & (p_1^2 + p_2^2) + p_3^2 \end{pmatrix}$$

Choosing $u_a^{(n)} = V^{(n)}$ with $V^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $V^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ then Eq.(11) gives $u_b^{(n)} = \frac{1}{E+m}\vec{p}\cdot\vec{\sigma}V^{(n)}$.

Eigenstates of H (see Eq.(3) and Eq.(9)): $u^{(n)} = N \begin{pmatrix} V^{(n)} \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m}V^{(n)} \end{pmatrix}$ associated to eigenvalue $E > 0$.

Now if $u_b^{(n)}$ is chosen to be $V^{(n)}$, then Eq.(11) gives $u_a^{(n)} = \frac{1}{E-m}\vec{p}\cdot\vec{\sigma}V^{(n)}$.

Eigenstates of H: $u^{(n+2)} = N \begin{pmatrix} -\frac{\vec{p}\cdot\vec{\sigma}}{|E|+m}V^{(n)} \\ V^{(n)} \end{pmatrix}$ associated to $E < 0$.

Having $H = H^\dagger$, there exist an orthonormal basis made of eigenstates. We have:

$$(u^{(1)})^\dagger\, u^{(2)} = N_1^* N_2 \left(p_3\,[p_1 - ip_2] + [p_1 - ip_2]\,[-p_3]\right) = 0$$

and

$$(u^{(1)})^\dagger u^{(3)} = N_1^* N_3 \left(\frac{-p_3}{|E|+m} + \frac{p_3}{|E|+m}\right) = 0.$$

Similarly, one can easily prove that $\forall n,\, m \in \{1,2,3,4\}$ and $n \neq m$, $(u^{(n)})^\dagger u^{(m)} = 0$.

The explicit spinor solutions are:

$$u^{(1)} = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{1}{|E|+m}\begin{pmatrix} p_3 \\ p_1 + i\,p_2 \end{pmatrix} \end{pmatrix} \rightarrow +\frac{\hbar}{2}$$

$$u^{(2)} = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{1}{|E|+m}\begin{pmatrix} p_1 - i\,p_2 \\ -p_3 \end{pmatrix} \end{pmatrix} \rightarrow -\frac{\hbar}{2}$$

$$u^{(3)} = N_3 \begin{pmatrix} \frac{-1}{|E|+m}\begin{pmatrix} p_3 \\ p_1 + i\,p_2 \end{pmatrix} \\ 1 \\ 0 \end{pmatrix} \rightarrow +\frac{\hbar}{2}$$

$$u^{(4)} = N_4 \begin{pmatrix} \frac{-1}{|E|+m}\begin{pmatrix} p_1 - i\,p_2 \\ -p_3 \end{pmatrix} \\ 0 \\ 1 \end{pmatrix} \rightarrow -\frac{\hbar}{2}$$

Remark: The orthonormal basis is not unique, there are other possible choices for the eigenvectors i.e. H is not a complete set of commuting observables.

1.5.3 Solutions as spin states

Let us introduce the Helicity observable,

$$h \equiv \vec{G}\cdot\frac{\vec{p}}{\|\vec{p}\|} = \begin{pmatrix} \frac{\vec{\sigma}}{2} & 0 \\ 0 & \frac{\vec{\sigma}}{2} \end{pmatrix}\cdot\frac{\vec{p}}{\|\vec{p}\|}\hbar = \begin{pmatrix} \frac{\vec{\sigma}}{2}\cdot\vec{p} & 0 \\ 0 & \frac{\vec{\sigma}}{2}\cdot\vec{p} \end{pmatrix}\frac{\hbar}{\|\vec{p}\|}$$

with

$$\hat{S}_p = \|\hat{\vec{S}}\|\cos\theta = \|\hat{\vec{S}}\|\cos\theta\frac{\|\vec{p}\|}{\|\vec{p}\|} = \frac{\hat{\vec{S}}}{\|\vec{p}\|} = \frac{\hbar}{2}\vec{\sigma}\cdot\frac{\vec{p}}{\|\vec{p}\|}.$$

$\{h, H, \vec{P}\}$ forms a complete set of commuting operators.

$$H\vec{G}\cdot\frac{p}{\|\vec{p}\|} = \frac{1}{\|\vec{p}\|}\begin{pmatrix} m & \vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -m \end{pmatrix}\begin{pmatrix} \vec{\sigma}\cdot\vec{p} & 0 \\ \vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix} = \frac{1}{\|\vec{p}\|}\begin{pmatrix} m\vec{\sigma}\cdot\vec{p} & (\vec{\sigma}\cdot\vec{p})^2 \\ (\vec{\sigma}\cdot\vec{p})^2 & -m\vec{\sigma}\cdot\vec{p} \end{pmatrix}$$

$$\vec{G}\cdot\frac{\vec{p}}{\|\vec{p}\|}H = \frac{1}{\|\vec{p}\|}\begin{pmatrix} \vec{\sigma}\cdot\vec{p} & 0 \\ 0 & \vec{\sigma}\cdot\vec{p} \end{pmatrix}\begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix} = \frac{1}{\|\vec{p}\|}\begin{pmatrix} m\vec{\sigma}\cdot\vec{p} & (\vec{\sigma}\cdot\vec{p})^2 \\ (\vec{\sigma}\cdot\vec{p})^2 & -m\vec{\sigma}\cdot\vec{p} \end{pmatrix},$$

so $\left[H, \vec{G}.\frac{\vec{p}}{\|\vec{p}\|}\right] = 0$. Furthermore, $\left[H, \vec{P}\right] = 0$ and $\left[\vec{P}, \vec{G}.\frac{\vec{p}}{\|\vec{p}\|}\right] = 0$.

Completeness: the set of eigenvalues fixes the associated eigenstate uniquely.

Example:

$\vec{p}$ along the (Oz) axis: $\vec{p} = (0, 0, p)$.

$$\hbar\frac{\vec{\sigma}}{2}\cdot\frac{\vec{p}}{\|\vec{p}\|}V^{(n)} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\vec{p}}{\|\vec{p}\|}V^{(n)} = \pm\frac{1}{2}\hbar V^{(n)}.$$

Let $n \in \{1, 2\}$. Then,

$$\vec{G}\cdot\frac{\vec{p}}{\|\vec{p}\|}u^{(n)} = \begin{pmatrix} \hat{S}_p & 0 \\ 0 & \hat{S}_p \end{pmatrix} N_1 \begin{pmatrix} V^{(n)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}V^{(n)} \end{pmatrix} = N_1 \begin{pmatrix} \hat{S}_p V^{(n)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\hat{S}_p V^{(n)} \end{pmatrix}$$

$$= \pm\frac{\hbar}{2}N_1 \begin{pmatrix} V^{(n)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}V^{(n)} \end{pmatrix}.$$

Similarly, if $n \in \{3, 4\}$, we have: $hu^{(n)} = \pm\frac{\hbar}{2}u^{(n)}$.

The generic eigenvalues of h or S_p are those of $\frac{\hbar}{2\|\vec{p}\|}\vec{\sigma}.\vec{p}$, so $\pm\frac{\hbar}{2}$. Indeed,

$$0 = \det[\vec{p}\cdot\vec{\sigma} - \lambda\mathbb{1}] = (p_3 - \lambda)(-p_3 - \lambda) - [p_1^2 - (ip_2)^2] = (-\lambda)^2 - p_3^2 - p_1^2 - p_2^2,$$

i.e. $\vec{p}^{\,2} = \lambda^2$ so $\lambda = \pm\|\vec{p}\|$.

Anti-particle solutions

The first two solutions correspond to $E > 0$, electron (particle): $u^{(1,2)}\,\mathrm{e}^{-ip^\mu \frac{x_\mu}{\hbar}}$.
The last two solutions correspond to $E < 0$, positron (anti-particle): $u^{(3,4)}\,\mathrm{e}^{-ip^\mu \frac{x_\mu}{\hbar}}$.

As for scalar field current: one has to re-interpret the negative energy, to avoid physical drawbacks,

$$\begin{aligned}\Psi^{(3,4)} &= u^{(3,4)}_{(p)}\,\mathrm{e}^{-ip\cdot x} \Leftrightarrow u^{(3,4)}(-\vec{p})\,\mathrm{e}^{-i(-p)\cdot x} &\equiv\; & v^{(2,1)}(\vec{p})\,\mathrm{e}^{-i(-E)t}\,\mathrm{e}^{i(-\vec{p})\cdot\vec{x}} \\ & &=\; & v^{(1,2)}(\vec{p})\,\mathrm{e}^{ip\cdot x} = \Psi_c^{(1,2)} \;.\end{aligned}$$

Which equation satisfies the spinor $v(\vec{p})$?

$$Eq.(10):\; (\not{p} - m)u^{(3,4)}(\vec{p}) = 0 \Leftrightarrow (-\not{p} - m)u^{(3,4)}(-\vec{p}) \Leftrightarrow (\not{p} + m)v(\vec{p}) = 0 \qquad (12)$$

Exercise: Show that the global angular momentum $\vec{J} = \vec{L} + \vec{S}$ (orbital+spinorial) is conserved, which confirms the spin structure introduced by Dirac equation.

Comment: Conservation involves a classical quantity, $\frac{d}{dt}\left\langle \vec{L} + \vec{S}\right\rangle_{|\Psi(t)\rangle} = 0$ in fact. It is true if $\left[H, \vec{J}\right] = \vec{0}$ as,

$$\frac{d}{dt}\left\langle \vec{J}\right\rangle = \frac{1}{i\hbar}\left\langle\left[\vec{J}, H\right]\right\rangle + \left\langle\frac{\partial}{\partial t}\vec{J}\right\rangle$$

$\Leftrightarrow \left[H, \mathbb{1} - i\theta_i J^i\right] = 0 \Leftrightarrow \left[H, \exp -i\theta_i J^i\right] \simeq 0 \Leftrightarrow H$ invariant by τ. Indeed, $H = H' = \tau H \tau^{-1}$.

$$\begin{aligned}[H, L_1] &= \left[\vec{\alpha}\cdot\vec{P} + \beta m, \vec{R}\times\vec{P}|_1\right] = \vec{\alpha}\cdot\left[\vec{P}, R_2P_3 - R_3P_2\right] \\ &= \alpha_2\left[P_2, R_2P_3 - R_3P_2\right] + \alpha_3\left[P_3, R_2P_3 - R_3P_2\right] \\ &= \alpha_2\left(R_2\left[P_2,P_3\right] + \left[P_2,R_2\right]P_1\right) - \alpha_3\left(R_3\left[P_3,P_2\right] + \left[P_3,R_3\right]P_2\right) \\ &= \alpha_2(-i\hbar)P_3c - \alpha_3(-i\hbar)P_2c = -i(-\alpha_3P_2 + \alpha_2P_2) = -i\vec{\alpha}\times\vec{P}|_1\end{aligned}$$

Let us choose the Weyl representation (only in this exercise, to illustrate another representation) to calculate

$$[H, S_1] = \left[\vec{\alpha}\cdot\vec{P} + \beta m, S_1\right] \;.$$

- We have:

$$m\beta S_1 = m\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}\begin{pmatrix}S_1 & 0\\ 0 & S_1\end{pmatrix} = m\frac{\hbar}{2}\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_1\end{pmatrix} = m\frac{\hbar}{2}\begin{pmatrix}0 & \sigma_1\\ \sigma_1 & 0\end{pmatrix}$$

and, $S_1 m\beta = m\frac{\hbar}{2}\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = m\frac{\hbar}{2}\begin{pmatrix}0 & \sigma_1\\ \sigma_1 & 0\end{pmatrix}$ so $[\beta m, S_1] = 0$.

- We have:

$$\vec{\alpha}\cdot\vec{P}S_1 = \begin{pmatrix} -\vec{P}\cdot\vec{\sigma} & 0 \\ 0 & \vec{P}\cdot\vec{\sigma} \end{pmatrix}\begin{pmatrix} S_1 & 0 \\ 0 & S_1 \end{pmatrix} = \begin{pmatrix} -\vec{P}\cdot\vec{\sigma}\sigma_1 & 0 \\ 0 & \vec{P}\cdot\vec{\sigma}\sigma_1 \end{pmatrix}\frac{\hbar}{2}$$

$$S_1\vec{\alpha}\cdot\vec{P} = \begin{pmatrix} S_1 & 0 \\ 0 & S_1 \end{pmatrix}\begin{pmatrix} -\vec{P}\cdot\vec{\sigma} & 0 \\ 0 & P\sigma \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} -\sigma_1\vec{P}\cdot\vec{\sigma} & 0 \\ 0 & \sigma_1\vec{P}\cdot\vec{\sigma} \end{pmatrix}$$

$$\begin{aligned}
\left[\vec{\alpha}\cdot\vec{P}, S_1\right] &= \vec{\alpha}\cdot\vec{P}S_1 - S_1\vec{\alpha}\cdot\vec{P} = \begin{pmatrix} -\vec{P}\cdot\vec{\sigma}\sigma_1 + \sigma_1\vec{P}\cdot\vec{\alpha} & 0 \\ 0 & \vec{P}\cdot\vec{\sigma}\sigma_1 - \sigma_1\vec{P}\cdot\vec{\sigma} \end{pmatrix}\frac{\hbar}{2} \\
&= \frac{\hbar}{2}\begin{pmatrix} \left[\sigma_1, \vec{P}\cdot\vec{\sigma}\right] & 0 \\ 0 & -\left[\sigma_1, \vec{P}\cdot\vec{\sigma}\right] \end{pmatrix} \\
&= \frac{\hbar}{2}\begin{pmatrix} iP_2\ \epsilon^{1,2,3}2\sigma_3 + iP_3\ \epsilon^{1,3,2}2\sigma_2 & 0 \\ 0 & -(iP_2\ \epsilon^{1,2,3}2\sigma_3 + iP_3\ \epsilon^{1,3,2}2\sigma_2) \end{pmatrix} \\
&= i\hbar\begin{pmatrix} P_2\sigma_3 - P_3\sigma_2 & 0 \\ 0 & -P_3\sigma_2 \end{pmatrix} \\
&= i\hbar(-P_2\alpha_3 + P_3\alpha_2) \\
&= i\vec{\alpha}\times\vec{P}|_1
\end{aligned}$$

Hence $[H, L_1 + S_1] = -i\vec{\alpha}\times\vec{P}|_1 + i\vec{\alpha}\times\vec{P}|_1 = 0$. After generalisation, $\left[H, \vec{J}\right] = \vec{0}$.

2 Charge conjugation operator

The Lagrangian of Quantum ElectroDynamics (QED) reads as:

$$\boxed{\mathcal{L}_{QED} = i\overline{\psi}\gamma^\mu D_\mu\psi - m\overline{\psi}\psi - \frac{1}{4\mu_0}\ F_{\mu\nu}F^{\mu\nu}}$$

$$D_\mu = \partial_\mu + i\ \frac{q}{\hbar}\ A_\mu, \quad \text{Dimensions: } [\mathcal{L}] = [E]^4, [\psi] = [E]^{\frac{3}{2}}, [A] = [E]$$

$$\text{Gauge transformations} \left| \begin{array}{lcl} A^\mu & \mapsto & A^\mu - \partial_\mu\lambda(x) \\ \psi & \mapsto & e^{iq\lambda(x)}\ \psi \end{array} \right.$$

<u>Exercise</u> : Show that the QED Lagrangian is gauge invariant.

$F_{\mu\nu}$ invariant & $\overline{\psi}\psi$ too as $\overline{\psi} = \psi^\dagger\gamma^0 = (\psi^*)^t\gamma^0 \mapsto (e^{-iq\lambda}\psi^*)^t\gamma^0 = e^{-iq\lambda}\overline{\psi}$.

For $q = -e$,

$$\begin{aligned}
D_\mu\psi &\mapsto (\partial_\mu - ieA_\mu + ie\partial_\mu\lambda)e^{-ie\lambda}\psi \\
&= \cancel{-ie\partial_\mu\lambda e^{-ie\lambda}}\psi + e^{-ie\lambda}\partial_\mu\psi - ie\ A_\mu e^{-ie\lambda}\psi + \cancel{ie\partial_\mu\lambda e^{-ie\lambda}}\psi \\
&= e^{-ie\lambda}(\partial_\mu - ieA_\mu)\psi \\
&= e^{-ie\lambda}\ D_\mu\ \psi\ .
\end{aligned}$$

Euler-Lagrange equation with a new term for ψ :

$$\begin{aligned}
\frac{\partial}{\partial\overline{\psi}}\Big[i\overline{\psi}\gamma^\mu(iqA_\mu)\psi\Big] &= e\gamma^\mu A_\mu\psi \\
[i\gamma^\mu\partial_\mu\psi - m\psi] + e\gamma^\mu A_\mu\psi &= 0 \\
i\gamma^\mu(\partial_\mu - ieA_\mu)\psi - m\psi &= 0 \\
(i\gamma^\mu D_\mu\psi)^\dagger\gamma_0 - m\overline{\psi} &= 0 \\
i\overline{\psi}D^*_\mu\gamma^0\gamma^{\mu\dagger}\gamma^0 - m\overline{\psi} &= 0
\end{aligned} \tag{13}$$

Let us check:

$$\begin{aligned}
-m\overline{\psi} + e\overline{\psi}\gamma^\mu A_\mu &= i\ \partial_\mu\overline{\psi}\gamma^\mu \\
i(\partial_\mu + ieA_\mu)\overline{\psi}\gamma^\mu + m\overline{\psi} &= 0
\end{aligned}$$

A version of Eq.(13) should exist for the anti-particle state ψ_c :

$$i\gamma^\mu(\partial_\mu + ieA_\mu)\psi_c - m\psi_c = 0 \tag{14}$$

This equation defines ψ_c ; let's find it via Eq.(13), by complex conjugating it,

$$-i\gamma^{\mu^*}(\partial_\mu + ieA_\mu)\psi^* - m\psi^* = 0$$

By using $-C\gamma^0\gamma^{\mu^*} = \gamma^\mu C\gamma^0$,

$$\begin{aligned}
i\Big[-C\gamma^0\gamma^{\mu^*}\Big](\partial_\mu + ieA_\mu)\psi^* - mC\gamma^0\psi^* &= 0 \\
\Big\{i\gamma^\mu(\partial_\mu + ieA_\mu) - m\Big\}C\gamma^0\psi^* &= 0
\end{aligned}$$

So the identification of Eq.(13) and Eq.(14) gives rise to $\psi_C = C\overline{\psi}^t = C\gamma^0\psi^*$

Let us check that in the Dirac-Pauli representation, C satisfies,

$$\begin{aligned}
C\gamma^0 = i\gamma^2 = i\beta\alpha^2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \\
&= i\begin{pmatrix} & & & -i \\ & & i & \\ & i & & \\ -i & & & \end{pmatrix} = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}
\end{aligned} \tag{15}$$

Action of C on $\psi^{(1)}$:

$$\psi_c^{(1)} = C\gamma^0\psi^{(1)} = i\gamma^2 \left(N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{1}{E+m}\binom{p_3}{p_1+ip_2} \end{pmatrix} e^{-ip.x} \right)$$

$$= \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix} N_1^* \begin{pmatrix} 1 \\ 0 \\ \frac{1}{E+m}\binom{p_3}{p_1+ip_2} \end{pmatrix} e^{ip.x}$$

$$= N_1^* \begin{pmatrix} \frac{1}{E+m}\binom{p_1-ip_2}{-p_3} \\ 0 \\ 1 \end{pmatrix} e^{ip.x}$$

$$= v^{(1)}(\vec{p})\, e^{ip.x}$$

The spinor solutions are, $\{\psi^{(1)}, \psi^{(2)}, \psi^{(3)}, \psi^{(4)}\}$, with,

$$\begin{aligned} (\psi_c)_c &= (C\gamma^0\psi^*)_c \\ &= C\gamma^0(C\gamma^0\psi^*)^* \\ &= (C\gamma^0)^2\psi \\ &= \psi \end{aligned}$$

Exercise : Show the following useful formula's in this representation.

$$\boxed{C^{-1}\gamma^\mu C = (-\gamma^\mu)^t}$$

$$(15)\times(C\gamma^0)^{-1} : -C\gamma^0\gamma^{\mu^*}(C\gamma^0)^{-1} = \gamma^\mu$$

$$(C\gamma^0)^{-1} = \gamma^{0-1}C^{-1} \text{ as } C\gamma^0(\underbrace{\gamma^{0-1}}_{\gamma^0 \text{ as } \gamma^0\gamma^0=1} C^{-1}) = 1$$

$$\text{so, } -C \underbrace{\gamma^0\gamma^{\mu^*}\gamma^0}_{\gamma^{\mu t} \text{ since } Eq.(6)^*:\ \gamma^{\mu t}=\gamma^{0*}\gamma^{\mu^*}\underbrace{\gamma^{0*}}_{\gamma^0}} C^{-1} = \gamma^\mu$$

$$\boxed{C = -C^t} \text{ and } \boxed{-C = C^{-1}}$$

$$\begin{aligned} C^{-1}\gamma^0 C &= -\gamma^{0(t)} \Leftrightarrow \gamma^0 C = -C\gamma^0 = -(C\gamma^0)^t = -\gamma^{0(t)}C^t \\ \gamma^0 C(C\gamma^0) &= -C\gamma^0(C\gamma^0) \\ \gamma^0 C^2\gamma^0 &= -1 \\ C^2 = -\gamma^0\gamma^0 &= -1 \\ (-C)(C) &= 1 \end{aligned}$$

$$\boxed{C^\dagger = C^{-1}}$$

$$(C\gamma^0)^\dagger \underbrace{=}_{Eq.(15)} C\gamma^0 \text{ ie } \gamma^{0\dagger}C^\dagger = C\gamma^0$$

$$CC^\dagger = C\underbrace{\gamma^0(\gamma^0}_{=\mathbb{1}}C^\dagger) = C\gamma^0(C\gamma^0) \underbrace{=}_{Eq.(15)} \mathbb{1}$$

From which one can easily deduce,

$$C^{-1} = \quad C^t \quad \left| \begin{array}{lcl} C & = & -C^t \\ C^{-1} & = & -C \end{array} \right.$$

$$C^\dagger = \quad -C \quad \left| \begin{array}{lcl} C^{-1} & = & -C \\ C^\dagger & = & C^{-1} \end{array} \right.$$

<u>Exercise</u> : Show that $\overline{\psi_c} = -\psi^t C^{-1}$ which will be useful as well when writing covariant terms.

$$\begin{aligned} (\overline{\psi}_C) &= \psi_C^t\gamma^0 \\ &= (C\gamma^0\psi_*)^\dagger\gamma^0 \\ &= \psi^t\gamma^{0(t)}\underbrace{C^\dagger}_{C^{-1}}\gamma^0 \\ &= \psi^t\gamma^0\underbrace{C^{-1}\gamma^0}_{-\gamma^{0(t)}C^{-1}} \\ &= \psi^t\overbrace{\gamma^0\gamma^0}^{=\mathbb{1}}(-C^{-1}) \end{aligned}$$

while, the following quantity makes no sense,

$$\overline{\psi}\Big|_{\mathrm{c}} = [\psi^\dagger\gamma^0]_c = C\gamma^0[\psi^t\gamma^{0*}] \ .$$

3 The spinors and their normalisations

As we have seen,

$$\rho = \overline{\psi}\gamma^0\psi = \psi^\dagger\psi = \frac{d\ P}{d^3x} \quad \Rightarrow \quad \int_V d^3x \ \ \psi^\dagger\psi = \int_V d^3x \ \ \frac{d\ P}{d^3x} = N_{\text{part.}}$$

$$\text{with } \psi : \quad u e^{ip.x} \quad = \quad u\Psi(\vec{x},t)\ , \quad |\Psi(\vec{x},t)|^2 \quad = \quad \frac{d\ P}{d^3x}\ ,$$

so that,

$$u^{(m)\dagger}_{\vec{p}} u^{(m)}_{\vec{p}} \underbrace{\int_{V_{\text{unit}}} d^3x}_{=1} = 2E$$

where V_{unit} is a reference volume. Those relations translate into a more compact form, using the previous orthogonal conditions,

$$\underbrace{u^{(n)^\dagger}}_{||u||^2\geq 0} u^{(m)} \quad = \quad \delta^{nm}\ 2E, \quad \forall m = 1,...,4 \quad (\text{and} \quad \underbrace{u^{(n)}}_{E>0} \perp v^{(m)}\ , \quad \forall\ n,m)$$

$$(\text{I}) \quad u(-\vec{p})^{(n)\dagger} u(-\vec{p})^{(m)} = \delta^{nm}\ 2E$$

$$(\text{II}) \quad v(\vec{p})^{(n)\dagger} v(\vec{p})^{(m)} = \delta^{nm}\ \underbrace{2E}_{E>0}$$

The conditions derived allow one to find out the normalisation factors,

$$\begin{aligned}
\underbrace{u^{(1)^\dagger} u^{(1)}}_{2E(>0)} \quad &= \quad |N_1|^2\ [1 + p_3^2 + p_1^2 - (ip_2)^2]\ \frac{1}{(|E|+m)^2} \\
&= \quad |N_1|^2 \left(\frac{[E^2 + \cancel{m^2} + 2Em] + [E^2 - \cancel{m^2}]}{(E+m)^2} \right) \\
&= \quad |N_1|^2\ \frac{2E\cancel{[E+m]}}{(E+m)^{\cancel{2}}}
\end{aligned}$$

$$\text{so that,} \quad \boxed{N_1 = (E+m)^{\frac{1}{2}}}$$

$$v^{(1)^\dagger} v^{(1)} = |N_4|^2 \Big((|E|+m)^2\ \left[p_1^2 - (ip_2)^2 + p_3^2\right] + 1 \Big) \Rightarrow |N_4| = |N_1| = |N_2| = |N_3|$$

Normalisation conditions on $\overline{u},\ \overline{v}$:

$$\boxed{\overline{u}^{(n)} u^{(m)} = \delta^{nm}\ 2m} \quad [n,m = 1,2]$$

$$\boxed{\overline{v}^{(n)} v^{(m)} = -\delta^{nm}\ 2m} \quad [n,m = 1,2]$$

Exercise: Demonstration.

The spinor $\overline{u}$ requires the Dirac equation to obtain some information on it,

$$(\not{p} - m)u = 0 \ , \qquad u^\dagger(\gamma^{\mu\dagger}p_\mu - m)\underbrace{\gamma^0}_{\times\gamma^0} = 0 \ , \qquad u^\dagger\gamma^0(\gamma^\mu p_\mu - m) = 0$$

$$\boxed{\overline{u}(\not{p} - m) = 0} \qquad \boxed{\overline{v}(\not{p} + m) = 0} \tag{16}$$

This commutation is needed: $\gamma^{\mu\dagger}\gamma^0 = \gamma^0\gamma^\mu$

$$\text{Relation correct} : \begin{cases} \mu = 0 & : \quad \gamma^{0(\dagger)}\gamma^0 = \gamma^0\gamma^0 \\ \mu \neq 0 & : \quad \underbrace{\gamma^{k\dagger}}_{=-\gamma^k}\gamma^0 - \gamma^0\gamma^k = 0 \end{cases}$$

$$\left.\begin{array}{l} Eq.(16)^{(n)} \times \gamma^0 u^{(m)} : \overline{u}(\gamma^\mu p_\mu - m)\gamma^0 u = 0 \\ \overline{u}^{(n)}\gamma^0 \times Eq.(10)^{(m)} : \overline{u}\gamma^0(\gamma^\mu p_\mu - m)u = 0 \end{array}\right| \text{To be added.}$$

$$\underbrace{\overline{u}\gamma^0 p_0\gamma^0 u + \overline{u}\gamma^0\gamma^0 p_0 u}_{-2\overline{u}up_0 - 2(u^\dagger \underbrace{\gamma^0)\gamma^0}_{1} u\ m = 0} + \overline{u}\overbrace{\gamma^k p_k\gamma^0}^{-p_k\gamma^0\gamma^k} u + \overline{u}\gamma^0\gamma^k p_k u - 2\overline{u}\gamma^0 u\ m = 0$$

One thus obtains,

$$\overline{u}^{(n)}u^{(m)} = \frac{m}{\not{E}}\underbrace{u^{(n)\dagger}u^{(m)}}_{2\not{E}\ \delta^{nm}}$$

Completeness relations:

$$\boxed{\sum_{n=1,2} u_i^{(n)}(\vec{p})\ \overline{u}_j^{(n)}(\vec{p}) = \not{p} + m\Big|_{\text{ij}}}$$

$$\boxed{\sum_{n=1,2} v_i^{(n)}(\vec{p})\ \overline{v}_j^{(n)}(\vec{p}) = \not{p} - m\Big|_{\text{ij}}}$$

Demonstration:

$$N\left(1, 0, -\frac{p_3}{E+m}, -\frac{p_1 - ip_2}{E+m}\right) \longleftarrow \overline{u}_j^{(1)} = u^\dagger\gamma^0 = u^\dagger\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$
N \begin{pmatrix} 1 \\ 0 \\ \frac{1}{E+m}\binom{p_3}{p_1+ip_2} \end{pmatrix} \begin{pmatrix} 1 & 0 & -p_3/(E+m) & -(p_1-ip_2)/(E+m) \\ 0 & 0 & 0 & 0 \\ p_3/(E+m) & 0 & -p_3^2/(E+m)^2 & -p_3(p_1-ip_2)/(E+m)^2 \\ (p_1+ip_2)/(E+m) & 0 & -p_3(p_1+ip_2)/(E+m)^2 & -(p_1^2+p_2^2)/(E+m)^2 \end{pmatrix}
$$

$$
N \begin{pmatrix} 1 \\ 0 \\ \frac{1}{E+m}\binom{p_1-ip_2}{-p_3} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -(p_1+ip_2)/(E+m) & +p_3/(E+m) \\ 0 & (p_1-ip_2)/(E+m) & -(p_1^2+p_2^2)/(E+m)^2 & +p_3(p_1-ip_2)/(E+m)^2 \\ 0 & -p_3/(E+m) & +p_3(p_1+ip_2)/(E+m)^2 & -p_3^2/(E+m)^2 \end{pmatrix}
$$

$$
\cancel{N^2} \begin{pmatrix} 1 & 0 & -p_3/\cancel{(E+m)} & -(p_1-ip_2)/\cancel{(E+m)} \\ 0 & 1 & -(p_1+ip_2)/\cancel{(E+m)} & +p_3/\cancel{(E+m)} \\ p_3/\cancel{(E+m)} & (p_1-ip_2)/\cancel{(E+m)} & -\vec{p}^2/(E+m)^{\cancel{2}} & 0 \\ (p_1+ip_2)/(E+m) & -p_3/\cancel{(E+m)} & 0 & -\vec{p}^2/(E+m)^{\cancel{2}} \end{pmatrix}
$$

with,

$$
-\vec{p}^2/(E+m)^{\cancel{2}} : -\frac{E^2-m^2}{E+m} = -\frac{(E-m)\cancel{(E+m)}}{\cancel{(E+m)}} = m - E
$$

To be compared with,

$$
\not{p} + m = \underbrace{p_\mu \gamma^\mu}_{p_0\beta + p_k\beta\alpha^k} + m = \begin{pmatrix} p_0 + m & p_k\sigma^k \\ -p_k\sigma^k & -p_0 + m \end{pmatrix} = \begin{pmatrix} E + m & -p_k\sigma^k \\ +p_k\sigma^k & m - E \end{pmatrix} .
$$

4 Lorentz transformations

4.1 Bilinear terms

Let us make an exhaustive list of the bilinear quantitites $\bar{\psi} f(\gamma^\mu)\psi$ in order to construct Lorentz invariant objects – useful to write realistic Lagrangians for spin-1/2 particles.

For that purpose, we introduce the γ_5 matrix defined accordingly to,

$$\gamma^5 \hat{=} i\gamma^0\gamma^1\gamma^2\gamma^3$$

Exercise: Show that $\boxed{\gamma^{5\dagger} = \gamma^5}$, $\boxed{(\gamma^5)^2 = \mathbb{1}}$ and $\boxed{\gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0}$.

In the Dirac-Pauli representation,

$$\gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \text{ , with } \gamma^k = \beta\ \alpha^k = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}$$

since,

$$\begin{aligned}
\gamma^0\ \gamma^1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \\
\gamma^0\ \gamma^1\ \gamma^2 &= \begin{pmatrix} -\sigma^1\sigma^2 & 0 \\ 0 & \sigma^1\sigma^2 \end{pmatrix} \\
\gamma^0\ \gamma^1\ \gamma^2\ \gamma^3 &= \begin{pmatrix} -\sigma^1\sigma^2 & 0 \\ 0 & \sigma^1\sigma^2 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma^1\sigma^2\sigma^3 \\ -\sigma^1\sigma^2\sigma^3 & 0 \end{pmatrix} \\
\Rightarrow\ \gamma^3 &= \begin{pmatrix} 0 & -i\sigma^1\sigma^2\sigma^3 \\ -i\sigma^1\sigma^2\sigma^3 & 0 \end{pmatrix} = \begin{pmatrix} 0_{2\times2} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \\ \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0_{2\times2} \end{pmatrix}
\end{aligned}$$

as,

$$\sigma^1\sigma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\Rightarrow \sigma^1\ \sigma^2\ \sigma^3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

List of bilinear quantities:

Name	Expression	# of components
scalar	$\overline{\psi}\psi$	1
vector	$\overline{\psi}\gamma^\mu\psi$	4
tensor	$\overline{\psi}\sigma^{\mu\nu}\psi$	6
axial vector	$\overline{\psi}\gamma^5\gamma^\mu\psi$	4
pseudo scalar	$\overline{\psi}\gamma^5\psi$	1

- Increasing order of $\gamma' s$: $\gamma^5\gamma^\mu = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu \Leftrightarrow$ product of 3 γ matrices as $(\gamma^\mu)^2 = \pm\,\gamma^\mu$

- ex : $\gamma^5\gamma^2\gamma^3 = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^2\gamma^3$ is a product of 2 $\gamma' s \Rightarrow$ it is a complete list of possibilities.

4.2 Lorentz invariants

One expects, through a Lorentz boost,

$$i\,\gamma^\mu \frac{\partial}{\partial\, x^\mu}\,\psi(x) - m\,\psi(x) = 0 \qquad \longmapsto \qquad i\,\gamma^\mu \frac{\partial}{\partial\, x'^\mu}\,\psi'(x') - m\,\psi'(x') = 0$$

$$\begin{aligned} \text{with the Lorentz transformation:} \qquad x^\mu &= \Lambda^{\mu\cdot}_{\cdot\nu}\, x^{\nu\prime} \\ \text{and the quantum transformation:} \qquad \hat{X}'^\mu &= T\,\hat{X}^\mu\,T^{-1} \\ \text{from} \qquad \langle\psi'|X'|\psi'\rangle &= \langle\psi|X|\psi\rangle \\ \langle\psi|T^\dagger\,X'\,T|\psi\rangle &= \qquad \text{“Measure Invariance”} \end{aligned}$$

as well as the spinor field transformation:

$$\psi(x^\nu) \;\longmapsto \psi'(x^{\nu\prime}) = T\,\psi(x^\nu)$$

In the frame $\mathcal{F}$,

$$i\,\gamma^\rho \underbrace{\partial_\rho}_{\Lambda^{\mu\cdot}_{\cdot\rho}\partial'_\mu} \quad \psi(x) - m\,\psi(x) = 0 \qquad \text{since}\;\; \partial_{x'^\mu} = \Lambda^{\cdot\nu}_{\mu\cdot}\,\partial_{x^\nu}$$

$$\rightarrow\; \Lambda^{\mu\cdot}_{\cdot\rho}\,\partial'_\mu = \underbrace{\delta^\nu_\rho\,\partial_\nu}_{\partial_\rho}$$

while in the frame $\mathcal{F}'$,

$$iT^{-1}\,\gamma^\mu\,\partial_{x'^\mu}\,\underbrace{[T\psi(x)]}_{\psi'(x')} - m\,\underbrace{T^{-1}\,\psi'(x')}_{\psi(x)} = 0$$

By identification,

$$\boxed{\Lambda^{\mu\cdot}_{\cdot\rho}\,\gamma^\rho = T^{-1}\gamma^\mu T} \quad\Leftrightarrow\quad \underbrace{\Lambda^{\mu\cdot}_{\cdot\rho}(-\beta)\gamma^\rho}_{\text{classical transf. with `}-\beta\text{'}} \quad = \quad \underbrace{T\gamma^\mu T^{-1}(\beta)}_{\text{quantum transf. by definition }(\gamma'^\mu)}$$

<u>Exercise:</u>

For the infinitesimal Lorentz transformation $\Lambda^{\nu\cdot}_{\cdot\mu} = \delta^\nu_\mu + \epsilon^{\nu\cdot}_{\cdot\mu}$

check that the condition $T(\delta^\mu_\rho + \epsilon^{\mu\cdot}_{\cdot\rho})\gamma^\rho = \gamma^\mu T$ is well satisfied by $\boxed{T = \mathbb{1} - \frac{i}{4}\ \sigma_{\alpha\beta}\ \epsilon^{\alpha\beta}}$.

$$
\begin{aligned}
(\cancel{\delta^\mu_\rho} + \epsilon^{\mu\cdot}_{\cdot\rho})\gamma^\rho - \frac{i}{4}\ \sigma_{\alpha\beta}\ \epsilon^{\alpha\beta}(\delta^\mu_\rho + \epsilon^\mu_\rho)\gamma^\rho &= \cancel{\gamma^\mu} - \frac{i}{4}\gamma^\mu\epsilon_{\alpha\beta}\ \epsilon^{\alpha\beta} \\
\epsilon^\mu_\rho\gamma^\rho + \frac{1}{8}[-2\gamma_\beta\gamma_\alpha]\gamma_\nu\epsilon^{\alpha\beta}g^{\mu\nu} &= \frac{1}{8}\gamma^\mu[\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha]\epsilon^{\alpha\beta} \\
&= -\frac{i}{4}\gamma_\nu\gamma_\beta\gamma_\alpha\epsilon^{\alpha\beta}g^{\mu\nu}
\end{aligned}
$$

$$
\begin{aligned}
\epsilon^\mu_\rho\gamma_\rho - \frac{1}{4}(-)2\underbrace{g_{\beta\nu}g^{\overbrace{\nu\mu}^{sym.}}}_{\delta^\mu_\beta}\gamma_\alpha\epsilon^{\alpha\beta} - \frac{1}{4}\gamma_\rho 2\underbrace{g_{\alpha\nu}g^{\nu\mu}}_{\delta^\mu_\alpha}\epsilon^{\alpha\beta} &= 0 \\
\epsilon^{\mu\rho}\gamma_\rho + \underbrace{\frac{1}{2}\gamma_\alpha\epsilon^{\alpha\mu}}_{-\frac{1}{2}\gamma_\alpha\epsilon^{\mu\alpha}} - \frac{1}{2}\gamma_\beta\epsilon^{\mu\beta} &= 0 \\
\epsilon^{\mu\rho}\gamma_\rho - \gamma_\alpha\epsilon^{\mu\alpha} &= 0
\end{aligned}
$$

Exercise:

Show that : $\boxed{T^\dagger\gamma^0 = \gamma^0 T^{-1}}$ at order $\epsilon^{\alpha\beta}$.

Remark : careful treatment of the relation $T^\dagger = T^{-1}$ at $\mathcal{O}(\epsilon)$.

It is equivalent to show that,

$$
\begin{aligned}
T^\dagger\gamma^0 T\gamma^0 &= \mathbb{1} \\
\Leftrightarrow \mathbb{1} &= \left(1 + \frac{1}{8}\underbrace{\underbrace{[\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha]^\dagger}_{\gamma^\dagger_\beta\gamma^\dagger_\alpha - \gamma^\dagger_\alpha\gamma^\dagger_\beta}}_{-(-\gamma^\dagger_\beta\gamma^\dagger_\alpha + \gamma^\dagger_\alpha\gamma^\dagger_\beta)}\epsilon^{\alpha\beta}\right)\ \underbrace{\gamma 0\left(1 + \frac{1}{8}[\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha]\epsilon^{\alpha\beta}\right)\gamma^0}_{1+\frac{1}{8}[\gamma^\dagger_\alpha\gamma^\dagger_\beta - \gamma^\dagger_\beta\gamma^\dagger_\alpha]\epsilon^{\alpha\beta}}
\end{aligned}
$$

(Infinitesimal) Lorentz invariants:

A first invariant is,

$$
\begin{aligned}
\overline{\psi'}\psi' &= (\psi')^\dagger\gamma^0 T\psi \\
&= (T\psi)^\dagger\gamma^0 T\psi \\
&= \psi^\dagger T^\dagger\gamma^0 T\psi \\
&= \overline{\psi}\psi\ .
\end{aligned}
$$

Besides, one can now write,

$$
\overline{\psi'}\gamma^\mu\psi' = \overline{\psi}\,\underbrace{T^{-1}\gamma^\mu T}_{\Lambda^{\mu\cdot}_{\cdot\rho}\gamma^\rho}\,\psi
$$

together with,

$$\partial'_\mu = \Lambda_{\mu .}^{. \sigma} \partial_\sigma = \Lambda^{-1 \sigma .}_{\;\;\;\; . \mu} \partial_\sigma$$

so that another invariant can be built,

$$\overline{\psi'} \partial'_\mu \gamma^\mu \psi' = \overline{\psi} \underbrace{\Lambda^{-1 \sigma .}_{\;\;\;\; . \mu} \Lambda^{\mu .}_{. \rho}}_{\text{matrix } \delta^\sigma_\rho} \gamma^\rho \partial_\sigma \psi = \overline{\psi} \gamma^\rho \partial_\rho \psi \ .$$

5 The Higgs mechanism

5.1 The scalar potential

5.1.1 Free field

In quantum relativistic theory, the complex scalar field can have such a Lagrangian,

$$\mathcal{L}_m = \partial_\mu \phi (\partial^\mu \phi)^* - V, \qquad V(\phi) = m^2 \phi \phi^*$$

$$\text{leading to EOM : } \ (\Box + m^2)\phi = 0 \quad \& \quad (\Box + m^2)\phi^* = 0$$

from the Euler-Lagrange equations,

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \Phi)} \ .$$

The Lagrangian can be rewritten in terms of real fields,

$$\mathcal{L}_m(\phi, \phi^*) \Leftrightarrow \mathcal{L}_m(\phi_1, \phi_2) = \sum_{i=1}^{2} \ \frac{1}{2}\partial_\mu \Phi_i \partial^\mu \phi_i - \frac{m^2}{2}\phi_i^2$$

leading to,

$$\text{EOM : } \boxed{(\Box + m^2)\phi_i = 0}$$

The Hamiltonian density is then,

$$\begin{aligned} \mathcal{H}_m &\ \widehat{=}\ \partial^0 \phi_i \Pi_{\phi_i} - \mathcal{L} \\ &= \sum_{i=1}^{2} \left((\partial^0 \phi_i)^2 - \frac{1}{2}(\partial^0 \phi_i)^2 + \frac{1}{2}\vec{\nabla}\phi_i.\vec{\nabla}\phi_i + \frac{m^2}{2}\phi_i^2 \right) \\ \mathcal{H}_m &= \sum_{i=1}^{2} \left(\frac{1}{2}(\partial^0 \phi_i)^2 + \frac{1}{2}(\vec{\nabla}\phi_i)^2 + \frac{1}{2}m^2 \phi_i^2 \right) \end{aligned}$$

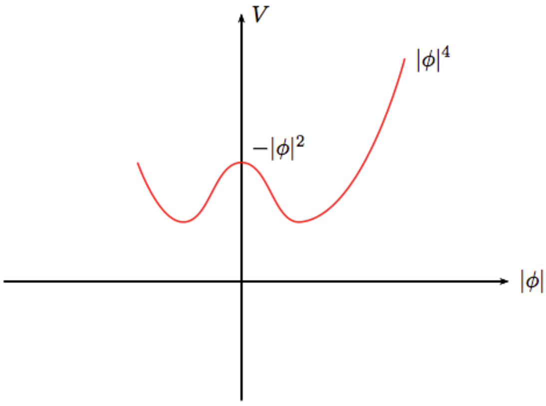
$$\mathcal{H}_m(\phi_i) \geq 0 \quad \text{is minimum at zero for the unique fields} \quad \phi_1^0 = \phi_2^0 = 0 \ .$$

5.1.2 Self interaction

The potential can be modified accordingly to,

$$\begin{aligned} \mathcal{L}_V &= \partial_\mu \phi \partial^\mu \phi^* - V, \qquad V(\phi) = -k^2 |\phi|^2 + \lambda |\phi|^4 \\ \mathcal{L}_V &= \sum_{i=1}^{2} \left(\frac{1}{2}\ \partial_\mu \phi_i \ \partial^\mu \phi_i \right) \ + \ \frac{k^2}{2}(\phi_1^2 + \phi_2^2) \ - \ \frac{\lambda}{4}(\phi_1^2 + \phi_2^2) \end{aligned}$$

$$\frac{\partial \mathcal{L}^{new}}{\partial \phi_i} = -2\frac{\lambda}{4}(\phi_1^2 + \phi_2^2)[2\phi_i] \ , \quad \text{so that EOM: } \ \boxed{(\Box - k^2)\phi_i = -2\lambda \mid \phi \mid^2 \phi_i}$$



The Hamitonian is now,

$$\mathcal{H}_V = \sum_i \left[\frac{1}{2}(\partial^0\phi_i)^2 + \frac{1}{2}\vec{\nabla}\phi_i^2 - \underbrace{\frac{1}{2}k^2\phi_i^2\right] + \frac{1}{4}\underbrace{(\phi_1^2+\phi_2^2)^2}_{(\phi\phi^*)^2 \equiv (|\phi|^2)^2}}_{V(\phi_1,\phi_2)\equiv -k^2|\phi|^2+\lambda\,|\phi|^4}$$

Defining $u = |\phi|^2$ for simplicity, the minimum is at,

$$0 = \frac{\partial V}{\partial u} = (-k^2 + \lambda 2u) \quad \text{for } u^0 = \frac{k^2}{2\lambda}, \quad \phi^0 = \sqrt{\frac{k^2}{2\lambda}}\; e^{i\theta}$$

$$\text{and} \quad \phi^{0*} = \sqrt{\frac{k^2}{2\lambda}}\; e^{-i\theta}, \quad \underbrace{(\phi_1^0 + i\phi_2^0)\frac{1}{\sqrt{2}}}_{\phi^0} = \underbrace{\sqrt{\frac{k^2}{2\lambda}}}_{R^0/\sqrt{2}}\; e^{i\theta}$$

Graphically,

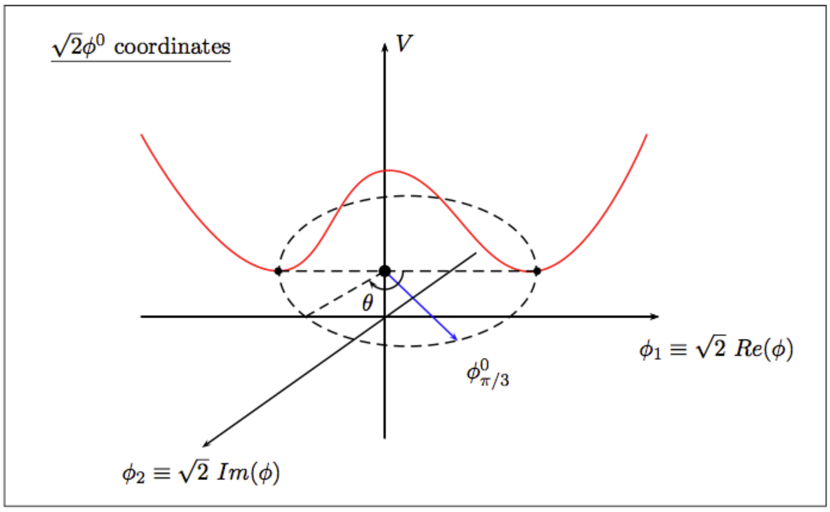
$\star$ for $\theta = \left|\begin{matrix} 0 \\ \pi \end{matrix}\right.$, $\phi_2^0 = 0,\ \phi_1^0 = \left|\begin{matrix} R^0 \\ -R^0 \end{matrix}\right.$ then V minimum

$\star$ case $\phi_2 = 0$ then $|\phi| = \phi_1$, so $V(|\phi|) = V(\phi_1)$

$\star$ $R^2 = \phi_1^2 + \phi_2^2 = 2|\phi|^2$ (invariance under rotation)

The vacuum is infinitely degenerate since $\mathcal{L}_V$ is invariant under the transformation:

$$\tau_\alpha \left[\begin{array}{l} \phi(x) \mapsto \phi'(x) = e^{i\alpha}\phi(x) \\ \\ \phi^*(x) \mapsto \phi^{*\prime}(x) = e^{-i\alpha}\phi^*(x) \end{array}\right.$$



The field $\phi(x^\nu)$ has a fixed vacuum e.g. at $\phi^0_{\frac{\pi}{3}} = \sqrt{\frac{k^2}{\lambda 2}} \exp\{i\frac{\Pi}{3}\}$ and thus **breaks the** τ_α **symmetry spontaneously**, $\phi^0_{\frac{\pi}{3}} \not\mapsto e^{i\alpha}\ \phi^0_{\frac{\pi}{3}}$.

5.2 Goldstone theorem

The relevant solution is $\phi^0(x^\nu) + \overbrace{\underbrace{\delta\phi(x^\nu)}_{<<\ \phi^0}}^{\psi(x^\nu)}$ in analogy with Quantum Field Theory,

$$\langle 0|\phi^0(x) + \widehat{\phi}(x^*)[\hat{a}, \hat{a}^\dagger]|0\rangle = \phi^0 \ .$$

It makes sense formally to consider the $\mathcal{L}(\phi's)$ form for the solution,

$$\phi(x^\nu) = \phi^0 + \underbrace{\psi(x^\nu)}_{(\psi_1 + i\psi_2)\frac{1}{\sqrt{2}}} \qquad \text{with } \psi_{1,2} \neq \phi_{1,2} \quad \text{having minimum} \quad \phi^0_{1,2}$$

$$\mathcal{L}^{SB}_V = \sum_i \frac{1}{2}\partial_\mu\psi_i\partial^\mu\psi_i - V(|\phi|^2)$$

with the potential,

$$\begin{aligned} V(\underbrace{|\phi|^2}_{u}) &= -k^2 u + \lambda u^2 \\ &= \lambda(-2u\ v^2 + u^2) \\ &= \lambda\Big((u - v^2)^2 - v^4\Big) \\ &= \lambda\Big((|\phi|^2 + \sqrt{2}v\ \psi_1)^2 - v^4\Big) \end{aligned}$$

once we have expressed,

$$u = (v+\psi)(v+\psi^*) = v^2 + |\phi|^2 + v(\psi+\psi^*) = v^2 + |\phi|^2 + v\sqrt{2}\ \psi_1$$

where $\phi^0 = ve^{i0}$.

Neglecting higher order terms ($\delta\phi^{3,4,\cdots}$) and the constant v^4 [vacuum energy] not contributing to EOM,

$$\mathcal{L}_V^{SB} \ = \ \frac{1}{2}\partial_\mu\psi_2\partial^\mu\psi_2 \ + \ \frac{1}{2}\partial_\mu\psi_1\partial^\mu\psi_1 \ - \ \underbrace{\overbrace{2\lambda v^2}^{>\,0}}_{\text{mass } 2\sqrt{\lambda}v=\sqrt{2}\ k} \ \psi_1^2\ .$$

The Goldstone theorem predicts indeed that when a Lagrangian is invariant under a continuous transformation which is broken by the vacuum, there exist a massless and real scalar field in the theory: **the Goldstone boson** (here ψ_2). The other (real) scalar field, ψ_1, gets a mass through the spontaneous breaking of the symmetry.

5.3 Yukawa couplings

Let us introduce spinor fields in order to study the impact of the spontaneous symmetry breaking on those.

$$\mathcal{L} = \mathcal{L}_V + \mathcal{L}_\psi^{free} + \mathcal{L}_Y = \mathcal{L}_V + i\sum_{i=1}^{2}\ \overline{\Psi}_i\gamma^\mu\partial_\mu\Psi_i - Y\phi\ \overline{\Psi}_1\Psi_2 - Y^*\phi^*\overline{\Psi}_2\Psi_1$$

No new scalar field $\Rightarrow \mathcal{H}$ scalar $= \mathcal{H}_V$ (still) minimised by $\phi^0 = \overbrace{\sqrt{\frac{k^2}{2\lambda}}}^{v}\ e^{i0}$

and no $\Psi^0 + \Psi(x^\nu)$ meaning $\Psi^0 = 0$ as $\mathrm{T}\Psi^0 \neq \Psi^0$ would break Lorentz symmetry.

$\mathcal{L}$ is invariant under the transformation,

$$\tau_\alpha^\psi \left[\begin{array}{lcl} \phi \longmapsto e^{i\alpha}\phi & ; & \phi^* \longmapsto e^{-i\alpha}\phi^* \\ \Psi_1 \longmapsto e^{i\alpha}\Psi_1 & ; & \overline{\Psi}_1 \longmapsto e^{-i\alpha}\overline{\Psi}_1 \\ \Psi_2 \longmapsto \Psi_2 & ; & \overline{\Psi}_2 \longmapsto \overline{\Psi}_2 \end{array}\right.$$

Choosing a transformation absorbing ψ_2,

$$\phi \longmapsto \left[\ \underbrace{\phi^0}_{\text{untransformed}} \ + (\psi_1 + i\psi_2)\frac{1}{\sqrt{2}}\right]e^{i\alpha}\ ,$$

one obtains the Lagrangian, replacing $\phi(x)$ by $[v+\psi_1(x)]/\sqrt{2}$,

$$\mathcal{L}_V \longmapsto \mathcal{L}_V^{SB},\quad \mathcal{L}_\Psi^{free} \longmapsto \mathcal{L}_\Psi^{free}$$

$$\mathcal{L}_Y \longmapsto \underbrace{-Y_V v\bar{\Psi}_1\Psi_2 - Y_V^* v\overline{\Psi}_2\Psi_1}_{\text{mixed mass terms}} \underbrace{-\frac{1}{\sqrt{2}}Y\psi_1\overline{\Psi}_1\Psi_2 - \frac{1}{\sqrt{2}}Y^*\psi_1\overline{\Psi}_2\Psi_1}_{\text{`Higgs-matter' interaction terms}}$$

A more realistic scenario: the Standard Model

- The symmetry is gauged (local transformation): $\tau_\alpha \ \mapsto \ \tau_{\alpha(x)}$.

- The ψ_2 degree of freedom is absorbed into the longitudinal polarisation of the massive gauge bosons (in the so-called London gauge, physical gauge or unitary gauge).

- The group breaking U(1) → X is promoted to $\mathrm{SU(2)_L \times U(1)_Y \to U(1)_Q}$.

- The Higgs boson (h) and the Goldstone bosons (G's) arise from a $\mathrm{SU(2)_L}$ doublet of the kind,

$$H \ = \ \begin{pmatrix} G_1^+ + iG_2^+ \\ \\ \frac{v+h}{\sqrt{2}} + iG^0 \end{pmatrix} \begin{matrix} \longrightarrow \ m_{W^\pm} \\ \\ \longrightarrow \ m_{Z^0} \end{matrix}$$

and the Yukawa coupling of the Lagrangian has now the form,

$$-Y_t \overline{\begin{pmatrix} t_L \\ b_L \end{pmatrix}} \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} t_R \ \ni \ \mathcal{L} \ .$$

The Higgs mechanism applies as well to supraconductivity and is then called the Meissner effect.