PARTICLE PHYSICS

Tutorials (n'1)

1. *Lorentz* **boost.-** We consider the case of a *Lorentz* boost along the (Ox) axis of the frame $\mathcal{F}$.
   (a) Express the 4-coordinates x'^{μ} in a frame $\mathcal{F}'$ in terms of the 4-coordinates x^{μ} in $\mathcal{F}$ [$\mu = 0, 1, 2, 3$ being a *Lorentz* index]. The relative velocity of $\mathcal{F}'$ with respect to $\mathcal{F}$ is noted $\vec{V}'$. Use the expression of the *Lorentz* matrix $\Lambda^{\mu}_{\cdot\nu}$.
   (b) Comment the limiting case $1 \gg |\beta|$ where $\beta = \bar{V}'/c$, c being the speed of light.

2. **Basic covariant calculations.-** We consider the covariant formalism of special relativity [1].
   (a) Show that $A^{\sigma}B_{\sigma} = A_{\sigma}B^{\sigma}$ where A^{σ} and B^{σ} are 4-vectors.
   (b) Calculate $g^{\mu\nu}g_{\mu\nu}$ where $g^{\mu\nu}$ is the metric tensor.

3. **4-momentum.-** The 4-momentum of an elementary particle can be written as $p^{\mu} = m\, v^{\mu}$ with the 4-velocity $v^{\mu} = (\gamma_v c, \gamma_v \vec{v})$, $\vec{v}$ being the velocity of the system (particle), m the particle mass and $\gamma_v = 1/\sqrt{1 - \frac{\vec{v}^2}{c^2}}$.
   (a) Calculate the *Lorentz* product $p^{\mu}p_{\mu}$.
   (b) Comment about the *Lorentz* invariance of the result.

4. **Relativistic energy.-** We study the non-relativistic limit of the global energy.
   (a) Based on the energy expression $E = \gamma_v mc^2$, develop the energy at leading order in the expansion parameter $\beta_v^2 = (\vec{v}/c)^2$.
   (b) Based on the energy expression $E = \sqrt{\vec{p}^2c^2 + m^2c^4}$ and momentum expression $\vec{p} = \gamma_v m\vec{v}$, develop the energy at leading order in $\beta_v^2 = (\vec{v}/c)^2$.
   (c) Compare and comment the two above results.

5. **Inverse** *Lorentz* **transformation.-** Show that if one has the *Lorentz* transformation $A'^{\sigma} = \Lambda^{\sigma}_{\cdot\rho}A^{\rho}$ where A^{σ} is a 4-vector, then one has $A^{\sigma} = (\Lambda^{-1})^{\sigma}_{\cdot\rho}A'^{\rho}$.

6. **Metric tensor.-** Show that the metric tensor $g^{\mu\nu}$ is a *Lorentz* invariant rank-two tensor.

7. *Lorentz* **matrix determinant.-** Based on the previous exercise, show that $\det(\Lambda^{\sigma}_{\cdot\rho}) = \pm 1$.

[1]Throughout the tutorials, we use the Minkowski metric tensor $g^{\mu\nu} = diag(+ - - -)$.

8. **Jacobian.-** Using the previous exercise and seing the *Lorentz* transformation as a change of variables within an integration process, demonstrate that $d^4x = dx^0dx^1dx^2dx^3$ is a *Lorentz* scalar.

9. **4-derivative.-** Demonstrate that $\partial'^{\sigma} = \Lambda^{\sigma}_{.\rho}\partial^{\rho}$ in a *Lorentz* transformation, noting $\partial^{\mu} = \frac{\partial}{\partial x_{\mu}}$. Start form the 4-coordinate transformations and multiply those equalities by a *Lorentz* matrix.

10. **Natural unit system.-** Study the electric charge within the natural unit system using the *Coulomb*'s force.

11. **Probability current.-** Verify that the current $\vec{j} = -\frac{i\hbar}{2m}(\vec{\nabla}\phi\,\phi^{\star} - \phi\,\vec{\nabla}\phi^{\star})$ is well a solution of the continuity equation $\vec{\nabla}.\vec{j} + \frac{\partial|\phi|^2}{\partial t} = 0$ where $\phi(\vec{x},t)$ is the generic wave function. Make use of the *Schrödinger* equation.

12. *Schrödinger* **equation solution.-** We consider the free *Schrödinger* equation.

    (a) Check that $\phi(\vec{x},t) = N\,e^{\frac{i}{\hbar}(\vec{p}.\vec{x}-E\,t)}$ is well solution of the *Schrödinger* equation.

    (b) Show that $f(\vec{x}) = e^{\frac{i}{\hbar}(\vec{p}.\vec{x})}$ is eigenfunction of the Hamiltonian.

    (c) Calculate the associated probability density of location $|\phi(\vec{x},t)|^2$ as well as the probability density flux $\vec{j}$.

13. **Interpretations of the Schrödinger equation.-** Within the non-relativistic framework of quantum mechanics, we consider the following Lagrangian density, involving the wave function (complex scalar field) $\phi(t,\vec{x})$ for a particle of mass m,

$$\mathcal{L} \;=\; \frac{i\hbar}{2}\,(\phi^*\partial_t\phi - \phi\,\partial_t\phi^*) \;-\; \frac{\hbar^2}{2m}\sum_{k=1}^{3}\partial_k\phi\,\partial_k\phi^* \;-\; V(t,\vec{r})\,\phi\,\phi^*\,. \tag{1}$$

$\partial_t = \partial/\partial t$, $\partial_k = \partial/\partial x_k$ [no covariant formalism] are respectively the time and space partial derivatives, the exponent * stands for the complex conjugate and V is some energy potential.

   (a) To find out the equation of motion, apply the Euler-Lagrange equation,

$$\frac{\partial\mathcal{L}}{\partial\phi} \;=\; \partial_t\,\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} \;+\; \sum_{j=1}^{3}\partial_j\,\frac{\partial\mathcal{L}}{\partial(\partial_j\phi)} \tag{2}$$

   to the Lagrangian (**??**). Comment on the obtained equation.

   (b) Calculate the following quantity, by using Equation (**??**),

$$\mathcal{Q} \;=\; \phi\,\frac{\partial\mathcal{L}}{\partial\phi} \;+\; \partial_t\phi\,\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} \;+\; \sum_{k=1}^{3}\partial_k\phi\,\frac{\partial\mathcal{L}}{\partial(\partial_k\phi)} \tag{3}$$

   and compare the resulting $\mathcal{Q}$ with the Lagrangian $\mathcal{L}$ itself. Same question for the Quantity (**??**) with the replacement $\phi \to \phi^*$ [but same $\mathcal{L}$].

(c) Let us now define the two new objects,

$$R \;=\; -\frac{i}{\hbar}\,\left\{\phi\,\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\;-\;\phi^*\,\frac{\partial\mathcal{L}}{\partial(\partial_t\phi^*)}\right\}\;,\;C_j\;=\;-\frac{i}{\hbar}\,\left\{\phi\,\frac{\partial\mathcal{L}}{\partial(\partial_j\phi)}\;-\;\phi^*\,\frac{\partial\mathcal{L}}{\partial(\partial_j\phi^*)}\right\}\;. \tag{4}$$

Calculate the combination $i\hbar(\partial_t R + \partial_j C_j)$ only by using Equations (**??**), (**??**) and the previous question (without calculating explicitly R and C_j through the $\mathcal{L}$ definition) [2]. Interpret physically the result as well as R and C_j.

(d) Calculate both R and C_j by injecting the Lagrangian (**??**) into Equalities (**??**). Give C_j as an imaginary part.

[2] Noticing in Equation (**??**), that some terms arise by replacing ϕ with ϕ^*, might help to have more compact expressions.