PARTICLE PHYSICS

Tutorials (n'2)

1. *Klein-Gordon* **equation solution.-** We consider the free *Klein-Gordon* equation.

   (a) Check that $\phi(\vec{x},t) = N\, e^{\frac{i}{\hbar}(\vec{p}.\vec{x}-E\,t)}$ is well solution of the (free) *Klein-Gordon* equation.

   (b) Show that $f(\vec{x}) = e^{\frac{i}{\hbar}(\vec{p}.\vec{x})}$ is eigenfunction of the squared Hamiltonian.

   (c) Calculate the associated "probability density of location" as well as the probability density flux ($\vec{j}$).

2. **Probability current (relativistic).-** We consider the 4-current $j^\mu = i(\phi^\star \partial^\mu \phi - \phi \partial^\mu \phi^\star)$, where $\phi(\vec{x},t)$ is the generic wave function.

   (a) Verify that this current contains well the "probability density of location" as well as the probability density flux ($\vec{j}$) for the (free) *Klein-Gordon* equation.

   (b) Demonstrate that the covariant form of the continuity equation is $\partial_\mu j^\mu = 0$.

   (c) For the free solution $\phi(\vec{x},t) = N\, e^{\frac{i}{\hbar}(\vec{p}.\vec{x}-E\,t)}$ of the *Klein-Gordon* equation, express the 4-current j^μ as a function of the 4-momentum p^μ.

3. **Lagrangian for the** *Klein-Gordon* **equation.-** Within the covariant framework of quantum mechanics, we consider the following Lagrangian density, involving the wave function (complex scalar field) $\phi(t,\vec{x})$ for a particle of mass m,

$$\mathcal{L} \;=\; \frac{1}{2}\,(\partial_\mu \phi)(\partial^\mu \phi)^\star \;-\; \frac{m^2 c^2}{2\hbar^2}\,\phi\phi^\star\,. \tag{1}$$

   The exponent * stands for the complex conjugate.

   (a) First, make the dimension analysis of the Lagrangian (**??**).

   (b) Then, in order to find out the equation of motion, apply the following Euler-Lagrange equation,

$$\frac{\partial \mathcal{L}}{\partial \phi} \;=\; \partial_\mu\, \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$$

   to the Lagrangian (**??**).

   (c) Comment on the equation obtained in the previous question and also on the equation for the complex conjugate field.

4. **Electromagnetic transition probability.-** Considering the energy potential, $V(\vec{x},t) = V(\vec{x})\, e^{-i\omega t}$, inspired by the electromagnetic force, calculate the probability for the transition of an initial quantum state $|i\rangle$ into a final state $|f\rangle$, $|i\rangle$ and $|f\rangle$ being free Hamiltonian eigenstates, at first order in $V(\vec{x})$. Interpret the result, based on *Feynman* diagrams (using the energy conservation relation obtained indirectly).

5. *Hodge* **dual of the field strength.-** Show that the covariant relation

$$\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} = 0$$

leads [for $\mu \neq \nu \neq \rho$] to half of the *Maxwell* equations: $\partial_\nu \star F^{\nu\rho} = 0$ ($\star F^{\nu\rho}$ being the *Hodge* dual of $F^{\nu\rho}$), as well as [for $\mu = \nu$] to the anti-symmetry property relation for the field strength and rank-two tensor $F^{\mu\nu}$.

6. **Some covariant electromagnetic equations.-** Let us consider the field strength $F^{\mu\nu}$ and 4-current j^μ.
   (a) Show that the covariant equation $\partial_\mu F^{\mu\nu} = \mu_0 j^\nu$ leads to the *Maxwell* equation $\vec{\nabla} \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j}$ (when the *Lorentz* index ν is equal to the spatial index).
   (b) Show that the covariant equation $\partial_\mu \star F^{\mu\nu} = 0$ leads to the *Maxwell* equation $\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0}$ (when the *Lorentz* index ν is equal to the spatial index).
   (c) Show that the covariant equation $\frac{dp^\mu}{d\tau} = q\, F^{\mu\nu} v_\nu$ leads to the relativistic Lorentz force relation $m\frac{d(\gamma\vec{v})}{d\tau} = q\gamma(\vec{E} + \vec{v} \times \vec{B})$ (when the *Lorentz* index μ is equal to the spatial index).

7. **Lagrangian for free** ***Maxwell*** **equation.-** Apply the following Euler-Lagrange equation,

$$\frac{\partial \mathcal{L}}{\partial A_\alpha} = \partial_\beta \frac{\partial \mathcal{L}}{\partial(\partial_\beta A_\alpha)}$$

to the Lagrangian density $\mathcal{L} = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu}$. Then make the dimension analysis of this Lagrangian.

8. **Gauge invariance.-** Show that the term $\frac{1}{2}(D_\mu \phi)(D^\mu \phi)^\star$, where $D^\mu = \partial^\mu + iqA^\mu$ is the covariant derivative, is gauge invariant.

9. **Lagrangian for spinless QED.-** Within the covariant framework of quantum mechanics, we consider the following Lagrangian density, involving the wave function (complex scalar field) $\phi(t,\vec{x})$ for a particle of mass m,

$$\mathcal{L} = \frac{1}{2}\,(D_\mu \phi)(D^\mu \phi)^\star - \frac{m^2 c^2}{2\hbar^2}\,\phi\phi^\star - \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu}\,. \tag{2}$$

   (a) Apply the Euler-Lagrange equation $\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$ to the Lagrangian (**??**). Comment on the obtained relation and the equation for the complex conjugate field.
   (b) Apply the Euler-Lagrange equation $\frac{\partial \mathcal{L}}{\partial A_\alpha} = \partial_\beta \frac{\partial \mathcal{L}}{\partial(\partial_\beta A_\alpha)}$ to the Lagrangian (**??**). Comment on the obtained relation.