QUANTUM FIELD THEORY

Tutorials (n'1)

1. **Rigid body coordinates.-** Let us consider the one-dimensional Lagrangian,

$$L^{1D} = \frac{1}{2} m\dot{x}^2(t) - V(x) \ .$$

m denotes a point-like system mass, x its coordinate and V a potential energy. We use the notation $\dot{x} = \frac{dx(t)}{dt}$ for the time derivative.

(a) Show that the *Euler-Lagrange* equations for the Lagrangian L^{1D} are nothing else but the second *Newton*'s law.
Applying the course, we have $r = 1$, $q = x$ so that,

$$\frac{d}{dt}\left(\frac{\partial L^{1D}(x(t), \dot{x}(t), t)}{\partial \dot{x}(t)}\right) = \frac{\partial L^{1D}(x(t), \dot{x}(t), t)}{\partial x(t)} \ ,$$

$$\frac{d}{dt}(m\dot{x}) = m \underbrace{\ddot{x}}_{a(x)} = \underbrace{-V'(x)}_{F_x} \ .$$

Note that $[\mathcal{A}] = [\int dt\, L^{1D}] = [E]\, T = [\hbar]$ (as in QFT).

(b) Calculate the conjugate momentum $p^{1D}(t)$ and then the Hamiltonian H^{1D} for the Lagrangian L^{1D}.

$$p^{1D}(t) \mathrel{\widehat{=}} \frac{\partial L^{1D}(x(t), \dot{x}(t), t)}{\partial \dot{x}(t)} = m\dot{x} \ .$$

We recover the canonical momentum.

$$H^{1D} = m\dot{x} \times \dot{x} - \left(\frac{1}{2} m\dot{x}^2 - V(x)\right) = \frac{1}{2} m\dot{x}^2 + V(x) = \frac{(p^{1D})^2}{2m} + V(x) \ .$$

Let us notice here the generic form (for rigid coordinates as $q = x$), $H = T + V$, where T constitutes the kinetic energy and V the potential energy.

(c) Apply the *Hamilton-Jacobi* equations to the Lagrangian L^{1D}.

$$\frac{\partial H}{\partial p^{1D}} = \dot{x}, \ \ \frac{\partial H}{\partial x} = -\dot{p}^{1D} \ \ \Leftrightarrow \ \ \frac{p^{1D}}{m} = \dot{x}, \ \ V'(x) = -\dot{p}^{1D} = -m\ddot{x} \ .$$

2. *Poisson* **brackets.-** Calculate the following *Poisson* brackets, where p_s represents generically the conjugate momentum of the variable q_s and H the Hamiltonian. For this purpose, make use of the *Hamilton-Jacobi* equations.

   (a) $[q_r,\, p_s]_P$.

   (b) $[q_r,\, H]_P$.

   (c) $[p_r,\, H]_P$.

$$\begin{aligned}
(a) \quad &\Rightarrow \quad [q_r, p_s]_P = \frac{\partial q_r}{\partial q_w}\frac{\partial p_s}{\partial p_w} - \frac{\partial q_r}{\partial p_u}\frac{\partial p_s}{\partial q_u} = \delta_r^w \delta_s^w = \delta_{rs} \\
(b) \quad &\Rightarrow \quad [q_r, H]_P = \frac{\partial q_r}{\partial q_s}\frac{\partial H}{\partial p_s} - \frac{\partial q_r}{\partial p_{s'}}\frac{\partial H}{\partial q_{s'}} = \delta_r^s \dot{q}_s = \dot{q}_r \\
(c) \quad &\Rightarrow \quad [p_r, H]_P = \frac{\partial p_r}{\partial q_s}\frac{\partial H}{\partial p_s} - \frac{\partial p_r}{\partial p_{s'}}\frac{\partial H}{\partial q_{s'}} = -\delta_r^s(-\dot{p}_s) = \dot{p}_r
\end{aligned} \tag{1}$$

The *Kronecker* symbol product is performed by writing explicitly the sum over w (non-vanishing contribution for $w = r = s$ in this sum) and can also be seen as a matrix product. For instance, one has generally for the two following independent variables, $\frac{\partial p_r}{\partial q_s} = 0$.

3. **Relativistic quantum theory.-** We consider the following Lagrangian density for a spinless complex (scalar) field ϕ,

$$\mathcal{L}_\phi = (\partial_\mu \phi)^* \partial^\mu \phi - m^2 \phi^* \phi \ ,$$

   where $\partial_\mu = \frac{\partial}{\partial x^\mu}$ is the 4-vector derivative and m a mass parameter.

   (a) Calculate the conjugate momenta π_ϕ and π_{ϕ^*} respectively for the fields ϕ and its complex conjugate ϕ^*.

   (b) Calculate the Hamiltonian density $\mathcal{H}_\phi$ for the associated Lagrangian density $\mathcal{L}_\phi$.

4. **Gauge field.-** We consider the following Lagrangian density for the real electromagnetic field of the spin-one photon A^μ (μ being a *Lorentz* index),

$$\mathcal{L}_{A^\mu} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu \ ,$$

   where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength and j_μ represents a charge distribution.

   (a) Calculate the conjugate momenta π_{A^μ} of the fields A^μ, with special care for π_{A^0}.

   (b) Calculate the Hamiltonian density $\mathcal{H}_{A^\mu}$ for the corresponding Lagrangian density $\mathcal{L}_{A^\mu}$.

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