QUANTUM FIELD THEORY IN CURVED SPACE-TIME

Tutorials (n'4)

1. **The Riemannian metric.-** Demonstrate the following relations (the prime symbol holds for a different frame).

   (a)
   $$g^{\mu\nu} = g^{\nu\mu}$$

   (b)
   $$g'^{\mu\nu} = g^{\alpha\beta}\,\frac{\partial x'^{\mu}}{\partial x^{\alpha}}\,\frac{\partial x'^{\nu}}{\partial x^{\beta}}$$

   (c)
   $$g'^{\mu\nu} g'_{\nu\rho} = \delta^{\mu}_{\rho}$$

2. **Stress-energy tensor.-** Calculate
   $$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}(\phi, g^{\mu\nu}))}{\delta g_{\mu\nu}}\,,$$
   for the following scalar field Lagrangian,
   $$\mathcal{L}(\phi, g^{\mu\nu}) = \frac{1}{2} g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - \frac{1}{2} m^2\phi^2\,.$$

3. **Scalar product.-** Show that,
   $$(f_p, f_{p'}) \hat{=} i\int d^3x \left(f_p^*\,\partial_0 f_{p'} - (\partial_0 f_p^*)\,f_{p'}\right) \;=\; \delta^{(3)}(\vec{p} - \vec{p}\,')$$
   and $(f_p, f^*_{p'}) = 0$ for Klein-Gordon equation solutions $f_p(x^\mu)$.

4. **Bogoliubov coefficients.-** Demonstrate that the operator $\hat{\tilde{a}}_p$, defined by,
   $$\hat{\tilde{a}}_p = \alpha_p\,\hat{a}_p + \beta^*_{-p}\,\hat{a}^\dagger_{-p}\,,$$
   satisfies the canonical commutation relations, the operator $\hat{a}_p$ being the standard creation operator of QFT (flat space-time).

5. **Wronskian.-** Show that the Wronskian,

$$W = \mathrm{Det}\begin{pmatrix} \psi_p & \psi_p^* \\ \partial_\tau \psi_p & \partial_\tau \psi_p^* \end{pmatrix} = \psi_p\, \partial_\tau \psi_p^* - (\partial_\tau \psi_p)\, \psi_p^* \,,$$

involving the solution $\psi_p(\tau)$ of the extended Klein-Gordon time-equation, is time independent. Then, using the known solutions in the limits $\tau \to \pm\infty$, deduce the following relation on the Bogoliubov coefficients,

$$|\alpha_p|^2 - |\beta_p|^2 = 1 \,.$$

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