Noncommutative Induced Gauge Theory

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work in coll. with **Axel de Goursac, Raimar Wulkenhaar**

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Motivations

- Attempt to construct possible candidate(s) for renormalisable actions for gauge theories on noncommutative $D = 4$ Moyal "space". The popular noncommutative analog of the Yang-Mills action $\int d^4x(F_{\mu\nu} \star F_{\mu\nu})(x)$ has UV/IR mixing.

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- Pick $S_H(\phi, \phi^\dagger)$, couple it to external A_μ in a gauge invariant way, integrate over matter and get effective action $\Gamma(A)$.
  - Guess possible form(s) for a candidate as a renormalisable gauge action
  - Is there some additional terms that appear in the action, beyond $F_{\mu\nu} \star F_{\mu\nu}$.
  - How does the harmonic term survive in the resulting effective action?
  - Check whether or not $\exists$ some relic of the Langmann-Szabo duality

The general structure

- The structure of the resulting action:

$$S_f \sim \int d^4x \Big(\frac{\alpha}{4g^2} F_{\mu\nu} \star F_{\mu\nu} + \frac{\Omega'}{4g^2} \{\mathcal{A}_\mu, \mathcal{A}_\nu\}_\star^2 + \frac{\kappa}{2} \mathcal{A}_\mu \star \mathcal{A}_\mu \Big)$$

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Noncommutative connections - Basics

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- Moyal algebra $\mathcal{M}$ with Moyal $\star$-product, unital, involutive algebra, assumed to be equiped with a differential calculus based on ∂_μ. (Recall $\mathcal{M}=\mathcal{L}\cap\mathcal{R}$; $\mathcal{L}$ (resp. $\mathcal{R}$): subspace of elements of $\mathcal{S}'(\mathbb{R}^4)$ whose multiplication from right (resp. left) by any Schwarz function is Schwartz). [see e.g Gracia-Bondia, Varilly, J.M.P 1988; Grossmann et al., Ann. Inst. Fourier 1968].

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- In NC geometry, the connections defined from set of sections of vector bundles in ordinary geometry can be generalized to connections on modules over an algebra.
- Let $\mathcal{H}$ be a right $\mathcal{M}$-module with a hermitean structure h. A connection is defined by a linear map from $\mathcal{H}$ to $\mathcal{H}$ verifying a Leibnitz rule:

$$\nabla_\mu : \mathcal{H} \to \mathcal{H}$$

$$\nabla_\mu(m\star f) = \nabla_\mu(m)\star f + m\star\partial_\mu f,\ \forall m\in\mathcal{H},\quad \forall f\in\mathcal{M}$$

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- The connection is further assumed to preserve the hermitian structure h, i.e

$$\partial_\mu h(m_1, m_2) = h(\nabla_\mu m_1, m_2) + h(m_1, \nabla_\mu m_2), \quad \forall m_1, m_2 \in \mathcal{H}$$

(Recall that h is a sesquilinear map from $\mathcal{H}\times\mathcal{H}$ to $\mathcal{M}$ verifying $h(m_1 \star f_1, m_2 \star f_2)=f_1^\dagger \star h(m_1, m_2) \star f_2$, $\forall f_1, f_2 \in \mathcal{M}$, $\forall m_1, m_2 \in \mathcal{H}$)

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$$\nabla^A_\mu(\mathbb{I}) \equiv -iA_\mu$$
(Setting m=$\mathbb{I}$ in the definition of ∇_μ yields $\nabla^A_\mu(\mathbb{I}\star f)$= $\nabla^A_\mu(\mathbb{I})\star f + \partial_\mu f$ $\equiv \partial_\mu f - iA_\mu \star f$). This can serve as defining a noncommutative analog of the gauge potential $A_\mu \in \mathcal{M}$.

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- Gauge transformations are defined by the automorphisms of the module $\mathcal{M}$ preserving the hermitian structure h: $\gamma\in Aut_h(\mathcal{M})$. One has
$$\gamma(f)=\gamma(\mathbb{I}\star f)=\gamma(\mathbb{I})\star f\ ,\quad \forall f\in\mathcal{M}$$
$$h\big(\gamma(f_1),\gamma(f_2)\big)=h(f_1,f_2)\quad \forall f_1,f_2\in\mathcal{M}$$
The latter relation implies
$$\gamma(\mathbb{I})^\dagger\star\gamma(\mathbb{I})=\mathbb{I}$$
so that the gauge transformations are determined by $\gamma(\mathbb{I})\in\mathcal{U}(\mathcal{M})$, where $\mathcal{U}(\mathcal{M})$ is the group of unitary elements of $\mathcal{M}$.

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- Now, we set $\gamma(\mathbb{I}) \equiv g$. Then, the action of the gauge group on any matter field $\phi \in \mathcal{M}$ is
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  (Combine $\gamma(\phi) = \gamma(\mathbb{I} \star \phi) = g \star \phi$ with $\nabla_\mu^A(\phi) = \partial_\mu \phi - iA_\mu \star \phi$ and $(\nabla_\mu^A)^g \equiv \partial_\mu - iA_\mu^g$)

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- In $\mathcal{M}$, the derivative ∂_μ is an inner derivative, since one has
$$\partial_\mu\phi = [i\xi_\mu, \phi]_\star, \quad \xi_\mu \equiv -\Theta_{\mu\nu}^{-1}x_\nu$$

Gauge-invariant connections

[1] A 1-form ξ such that $df = [\xi, f]$, $\forall f \in \mathcal{M}$ defines a canonical gauge-invariant connection

Gauge-invariant connections

- Inner derivations implies the existence of a (canonical) gauge-invariant connection. Not specific to Moyal. Reflects general theorem [1]of derivation-based noncommutative frameworks valid when the algebra = the module; already occurs within matrix-valued models.
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- Here, this canonical connection is defined by $\xi_\mu {=}{\equiv} -\Theta^{-1}_{\mu\nu}x_\nu$. It verifies
$$\xi^g_\mu = \xi_\mu$$
can be checked from general form of gauge transformations for A_μ combined with $\partial_\mu\phi = [i\xi_\mu, \phi]_\star$. (Other way: ∇^ξ_μ verifies $\nabla^\xi_\mu\phi{=}\ \partial_\mu\phi - i\xi_\mu\star\phi{=} -i\phi\star\xi_\mu$ where the last equality stems from $\partial_\mu\phi = [i\xi_\mu, \phi]_\star$. Then, $(\nabla^\xi_\mu)^g(\phi) = g\star(\nabla^\xi_\mu(g^\dagger\star\phi)) = -i\phi\star\xi_\mu = \nabla^\xi_\mu\phi$ so that $\xi^g_\mu = \xi_\mu$)

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- The above gauge-invariant connection can be used to define the following tensorial form
  $$\nabla^A_\mu - \nabla^\xi_\mu = -i(A_\mu - \xi_\mu) \equiv -i\mathcal{A}_\mu$$
  which coincides with the so called covariant coordinates.

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takes the usual form
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or alternatively in terms of $\mathcal{A}_\mu$
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- The gauge transformations for $\mathcal{A}_\mu$ and $F^A_{\mu\nu}$ are given by
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$$\mathcal{A}_\mu^g = g \star \mathcal{A}_\mu \star g^\dagger, \quad (F_{\mu\nu}^A)^g = g \star F_{\mu\nu}^A \star g^\dagger$$
- Note that the invariant connection defined by ξ_μ is a constant curvature connection since one has
$$F_{\mu\nu}^\xi = \Theta_{\mu\nu}^{-1}$$

Other "gauge transformations"? I

- Other type of transformations considered by H.G and M.W:

$$\phi^U = U \star \phi \star U^\dagger \equiv \alpha(\phi)$$

for any $U \in \mathcal{U}(\mathcal{M})$. $\sim$NC analog of "gauge transformation in the adjoint representation". The corresponding "covariant derivative" is

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  $$A_\mu^U = U \star A_\mu \star U^\dagger + iU \star \partial_\mu U^\dagger$$
- It defines an automorphism α of algebra:
  $$\alpha(\phi_1 \star \phi_2) = \alpha(\phi_1) \star \alpha(\phi_2)$$
  D_μ satisfies a Leibnitz rule
  $$D_\mu(\phi_1 \star \phi_2) = D_\mu(\phi_1) \star \phi_2 + \phi_1 \star D_\mu(\phi_2)$$
  so that D_μ is a derivation. These NC analogs of "gauge transformation in the adjoint representation" can be understood in terms of *actual* NC gauge transformations provided the initial algebra $\mathcal{M}$ is enlarged to $\mathcal{M}$ by $\mathcal{M} \otimes \mathcal{M}^o$, where $\mathcal{M}^o$ is the opposite algebra which amounts to deal with real structure instead of hermitian structure.

Coupling external gauge potential to a scalar model

1 The noncommutative set-up - Main features

2 Coupling external gauge potential to a scalar model
- The minimal coupling prescription
- From coupled scalar action to effective gauge action

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- The action is
  $$S(\phi) = \int d^4x \big(\partial_\mu \phi^\dagger \star \partial_\mu \phi + \Omega^2 (\widetilde{x}_\mu \phi)^\dagger \star (\widetilde{x}_\mu \phi) + m^2 \phi^\dagger \star \phi\big)(x) + S_{int}$$
  Here, ϕ is a complex scalar field with mass m, $\Omega \in [0,1]$ and $\widetilde{x}_\mu = 2\Theta^{-1}_{\mu\nu} x_\nu$.

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- The interaction term S_{int} is
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- $S(\phi)$ restricted to S^O_{int} (κ=0) is renormalisable for any value of Ω. Notice that the action is covariant under the Langmann-Szabo duality.

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- $S(\phi)$ restricted to S^O_{int} ($\kappa{=}0$) is renormalisable for any value of Ω. Notice that the action is covariant under the Langmann-Szabo duality.
- The effect of the inclusion of non-orientable interactions on the renormalisability is not known.

The minimal coupling prescription

- Owing to the special role played by ξ_μ, the minimal coupling prescription can be conveniently written as (de Goursac, JCW, Wulkenhaar, hep-th/0703075)

$$\partial_\mu\phi \mapsto \nabla^A_\mu\phi = \partial_\mu\phi - iA_\mu\star\phi,$$

$$\widetilde{x}_\mu\phi \mapsto -2i\nabla^\xi_\mu\phi + i\nabla^A_\mu\phi = \widetilde{x}_\mu\phi + A_\mu\star\phi.$$

($\nabla^\xi_\mu\phi = \partial_\mu\phi - i\xi_\mu\star\phi$). Prescription consistent with structure of modules over algebra. Roughly, this permits one to introduce covariant derivatives where it is needed in the action.

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From coupled scalar action to effective gauge action

- The resulting gauge invariant coupled action is given by

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- Goals:
  - Guess possible form(s) for a candidate as a renormalisable gauge action
  - Is there some additional terms that appear in the action, beyond the expected $F_{\mu\nu} \star F_{\mu\nu}$.
  - How does the harmonic term survive in the resulting effective action?
  - Check whether or not some relic of the Langmann-Szabo shows up in the effective action

Computation of the one-loop effective action

1. The noncommutative set-up - Main features
2. Coupling external gauge potential to a scalar model
3. **Computation of the one-loop effective action**
   - Defining the effective action
   - Diagramatics
   - The structure of the effective action

The one-loop effective action

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- The effective action is formally obtained through the evaluation of the following functional integral

$$e^{-\Gamma(A)} \equiv \int D\phi D\phi^\dagger e^{-S(\phi,A)} = \int D\phi D\phi^\dagger e^{-S(\phi)} e^{-S_{int}(\phi,A)},$$

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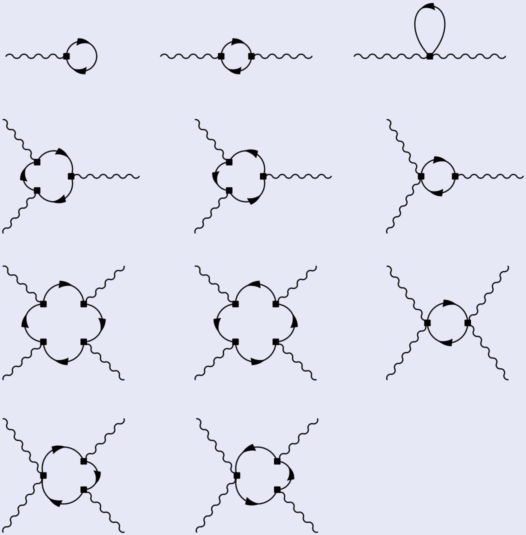
- The effective action $\Gamma_{1loop}(A)$ can be conveniently obtained in the x-space formalism. Compute relevant diagrams using the Mehler-type propagator $C(x,y) \equiv \langle \phi(x)\phi^\dagger(y)\rangle$ (set $\widetilde{\Omega} \equiv 2\frac{\Omega}{\theta}$ and $x \wedge y \equiv 2x_\mu \Theta^{-1}_{\mu\nu} y_\nu$)
$$C(x,y) = \frac{\Omega^2}{\pi^2\theta^2} \int_0^\infty \frac{dt}{\sinh^2(2\widetilde{\Omega}t)} \exp^{(-\frac{\widetilde{\Omega}}{4}\coth(\widetilde{\Omega}t)(x-y)^2 - \frac{\widetilde{\Omega}}{4}\tanh(\widetilde{\Omega}t)(x+y)^2 - m^2 t)}$$
combined with the vertex whose generic expression is
$$\int d^4x (f_1 \star f_2 \star f_3 \star f_4)(x) = \frac{1}{\pi^4\theta^4} \int \prod_{i=1}^4 d^4x_i f_1(x_1) f_2(x_2) f_3(x_3) f_4(x_4)$$
$$\times\, \delta(x_1 - x_2 + x_3 - x_4) e^{-i\sum_{i<j}(-1)^{i+j+1} x_i \wedge x_j}.$$

Diagramatics



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$$\Gamma(A) = \frac{\Omega^2}{4\pi^2(1+\Omega^2)^3}\left(\int d^4u\ (\mathcal{A}_\mu\star\mathcal{A}_\mu - \frac{1}{4}\widetilde{u}^2)\right)\left(\frac{1}{\epsilon} + m^2\ln(\epsilon)\right)$$
$$- \frac{(1-\Omega^2)^4}{192\pi^2(1+\Omega^2)^4}\left(\int d^4u\ F_{\mu\nu}\star F_{\mu\nu}\right)\ln(\epsilon)$$
$$+ \frac{\Omega^4}{8\pi^2(1+\Omega^2)^4}\left(\int d^4u\ (F_{\mu\nu}\star F_{\mu\nu} + \{\mathcal{A}_\mu,\mathcal{A}_\nu\}_\star^2 - \frac{1}{4}(\widetilde{u}^2)^2)\right)\ln(\epsilon) + \dots,$$

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- It is similar to the expression obtained by Grosse and Wohlgenannt from a matrix base approach using heat kernel expansion.
- It involves, beyond the usual expected Yang-Mills contribution $\sim\int d^4x\ F_{\mu\nu}\star F_{\mu\nu}$, additional gauge invariant terms of quadratic and quartic order in $\mathcal{A}_\mu$, $\sim\int d^4x\ \mathcal{A}_\mu\star\mathcal{A}_\mu$ and $\sim\int d^4x\ \{\mathcal{A}_\mu,\mathcal{A}_\nu\}^2_\star$.

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- It involves a mass-type term for the gauge potential A_μ (a bare mass term for a gauge potential is forbidden by gauge invariance in commutative Yang-Mills theories).

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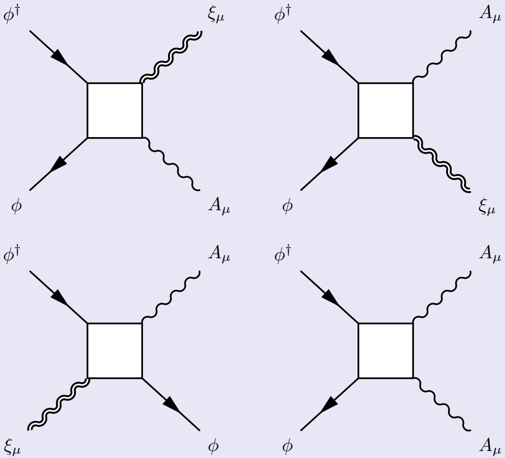
- Conjecture that the following class of actions
$$S \sim \int d^4x \Big(\frac{\alpha}{4g^2} F_{\mu\nu} \star F_{\mu\nu} + \frac{\Omega'}{4g^2} \{\mathcal{A}_\mu, \mathcal{A}_\nu\}^2_\star + \frac{\kappa}{2} \mathcal{A}_\mu \star \mathcal{A}_\mu \Big)$$
involves suitable candidates for renormalisable actions for gauge theory defined on Moyal spaces.

Next step

- Clarify the vacuum problem
- What goes on for IR singularity in the polarisation tensor?

Vertices involving A_μ



Tadpole diagram I

The amplitude for the tadpole diagram is

$$\mathcal{T}_1 = \frac{\Omega^2}{4\pi^6\theta^6} \int d^4x\ d^4u\ d^4z \int_0^\infty \frac{dt\ e^{-tm^2}}{\sinh^2(\widetilde{\Omega}t)\cosh^2(\widetilde{\Omega}t)}\ A_\mu(u)\ e^{-i(u-x)\wedge z}$$
$$\times\ e^{-\frac{\widetilde{\Omega}}{4}(\coth(\widetilde{\Omega}t)z^2+\tanh(\widetilde{\Omega}t)(2x+z)^2}((1-\Omega^2)(2\widetilde{x}_\mu+\widetilde{z}_\mu)-2\widetilde{u}_\mu)$$

Introduce the following 8-dimensional vectors X, J and the 8×8 matrix K defined by

$$X = \begin{pmatrix} x \\ z \end{pmatrix},\quad K = \begin{pmatrix} 4\tanh(\widetilde{\Omega}t)\mathbb{I} & 2\tanh(\widetilde{\Omega}t)\mathbb{I} - 2i\Theta^{-1} \\ 2\tanh(\widetilde{\Omega}t)\mathbb{I} + 2i\Theta^{-1} & (\tanh(\widetilde{\Omega}t)+\coth(\widetilde{\Omega}t))\mathbb{I} \end{pmatrix},\quad J = \begin{pmatrix} 0 \\ i\tilde{u} \end{pmatrix}.$$

This permits one to reexpress the amplitude in a form such that some Gaussian integrals can be easily performed:

$$\mathcal{T}_1 = \frac{\Omega^2}{4\pi^6\theta^6} \int d^4x\ d^4u\ d^4z \int_0^\infty \frac{dt\ e^{-tm^2}}{\sinh^2(\widetilde{\Omega}t)\cosh^2(\widetilde{\Omega}t)}\ A_\mu(u)$$
$$\times\ e^{-\frac{1}{2}X.K.X+J.X}((1-\Omega^2)(2\widetilde{x}_\mu+\widetilde{z}_\mu)-2\widetilde{u}_\mu)$$

By performing the Gaussian integrals on X, we find

$$\mathcal{T}_1 = -\frac{\Omega^4}{\pi^2\theta^2(1+\Omega^2)^3} \int d^4u \int_0^\infty \frac{dt\ e^{-tm^2}}{\sinh^2(\widetilde{\Omega}t)\cosh^2(\widetilde{\Omega}t)}\ A_\mu(u)\widetilde{u}_\mu\ e^{-\frac{2\Omega}{\theta(1+\Omega^2)}\tanh(\widetilde{\Omega}t)u^2}.$$

Tadpole diagram II

Inspection of the behaviour of $\mathcal{T}_1$ for $t \longrightarrow 0$ shows that this latter expression has a quadratic as well as a logarithmic UV divergence. From Taylor expansion:

$$\mathcal{T}_1 = -\frac{\Omega^2}{4\pi^2(1+\Omega^2)^3}\left(\int d^4u\ \widetilde{u}_\mu A_\mu(u)\right)\ \frac{1}{\epsilon}\ -\frac{m^2\Omega^2}{4\pi^2(1+\Omega^2)^3}\left(\int d^4u\ \widetilde{u}_\mu A_\mu(u)\right)\ \ln$$

$$-\frac{\Omega^4}{\pi^2\theta^2(1+\Omega^2)^4}\left(\int d^4u\ u^2\widetilde{u}_\mu A_\mu(u)\right)\ \ln(\epsilon)\ +\ldots,$$

where $\epsilon \longrightarrow 0$ is a cut-off and the ellipses denote finite contributions.

Higher order terms

- The regularisation of the diverging amplitudes is performed in a way that preserves gauge invariance of the most diverging terms. In $D = 4$, these are UV quadratically diverging so that the cut-off ϵ on the various integrals over the Schwinger parameters ($\int_{\epsilon}^{\infty} dt$) must be suitably chosen.
- We find that this can be achieved with $\int_{\epsilon}^{\infty} dt$ for $\mathcal{T}_2''$ while for $\mathcal{T}_2'$ the regularisation must be performed with $\int_{\epsilon/4}^{\infty}$.
- In field-theoretical language, gauge invariance is broken by the naive ϵ-regularisation of the Schwinger integrals and must be restored by adjusting the regularisation scheme. Note that the logarithmically divergent part is insensitive to a finite scaling of the cut-off.

Higher order terms II

- The one-loop effective action can be expressed in terms of heat kernels:

$$\Gamma_{1loop}(\phi,A) = -\frac{1}{2}\int_0^\infty \frac{dt}{t}\,\mathrm{Tr}\big(e^{-tH(\phi,A)} - e^{-tH(0,0)}\big) \quad (2)$$
$$= -\frac{1}{2}\lim_{s\to 0}\Gamma(s)\,\mathrm{Tr}\big(H^{-s}(\phi,A) - H^{-s}(0,0)\big),$$

where $H(\phi,A) = \frac{\delta^2 S(\phi,A)}{\delta\phi\,\delta\phi^\dagger}$. Expanding:

$$H^{-s}(\phi,A) = \Big(1 + a_1(\phi,A)s + a_2(\phi,A)s^2 + \dots\Big)H^{-s}(0,0), \quad (3)$$

we obtain

$$\Gamma_{1loop}(\phi,A) = -\frac{1}{2}\lim_{s\to 0}\mathrm{Tr}\Big(\big(\Gamma(s+1)a_1(\phi,A)+s\Gamma(s+1)a_2(\phi,A)+\dots\big)H^{-s}(0,0)\Big).$$

With $\Gamma(s+1) = 1 - s\gamma + \dots$ we have

$$\Gamma_{1loop}(\phi,A) = -\frac{1}{2}\lim_{s\to 0}\mathrm{Tr}\big(a_1(\phi,A)H^{-s}(0,0)\big)$$
$$-\frac{1}{2}\mathrm{Res}_{s=0}\,\mathrm{Tr}\Big(\big(a_2(\phi,A) - \gamma a_1(\phi,A)\big)H^{-s}(0,0)\Big). \quad (4)$$

The second line is the Wodzicki residue which corresponds to the logarithmically divergent part of the one-loop effective action. The quadratically divergent part $-\frac{1}{2}\lim_{s\to 0}\mathrm{Tr}\big(a_1 H^{-s}(0,0)\big)$ in the action which cannot be gauge-invariant.