

Circuits of thermodynamic devices in stationary non-equilibrium

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We introduce a non-linear theory of thermodynamic circuits in non-equilibrium stationary states. The non-equilibrium conductance matrix of a composite device is obtained from the ones of its sub-devices. This generalizes to thermodynamic devices the concept of equivalent impedance defined in electronics. An abstract example and the serial connection of two thermoelectric generators (TEG) with constant thermoelectric coefficients are considered. Interestingly, a current-dependent electrical resistance emerges from this connection.

Introduction— Dividing a problem into several pieces often simplifies its resolution. Combining this approach with a graphical representation produces circuits made of various sub-circuits. Circuits conveniently summarize the conservation of physical quantities such as energy, momentum, charge, or chemical species. Different kinds of circuits or graphs have emerged: bond graphs in engineering science [1], Feynman diagrams in particle physics [2], electric circuits in electrokinetic [3, 4] or hyper-graphs of chemical reactions in chemistry [5]. Electric power and signal processing represent the paramount application of circuit theory, although only one conserved quantity is usually considered. For a single (or multiple but decoupled) potential(s), the problem has long since been solved. Many works were devoted to coupled potentials although without considering arbitrary boundary conditions [6–10]. Neumann (fixed current) or Dirichlet (fixed potential) conditions are often assumed but never mixed boundary conditions where both current and potential achieve non-prescribed stationary values. In this case, real integrated balances at the boundaries, such as entropy balance, are central in determining the stationary state. A non-linear theory dealing with mixed boundary conditions and several conserved quantities coupled through complex circuits is appealing. It will have deep consequences in many scientific fields, such as biology, chemistry, and engineering.

The treatment of conservation laws has been systematized within stochastic thermodynamics in Ref. [11, 12]. This breakthrough allows the foundation of a circuit theory mixing various thermodynamic systems. For instance, in the framework of chemical reaction networks, this approach has led to an effective circuit description of otherwise complex chemical reaction networks [13, 14]. Then, in principle, the current–concentration characteristics of each chemical module yield the stationary currents exchanged with the environment or other modules. However, getting rid of all internal degrees of freedom and going beyond multivariable functions connecting currents and forces within global characteristics call for more operational methods valid for any thermodynamic system.

Non-equilibrium conductance matrices accurately

characterize thermodynamic devices while providing the current–force relations [15, 16]. They are the multidimensional generalization of the scalar current–force characteristic of dipoles (e.g. current–tension for an electric dipole). As such, they extend to vectors the concept of impedance defined as the scalar ratio between a force and a current. Conductance matrices generalize to non-equilibrium stationary states Onsager’s response matrices of linear irreversible thermodynamics [17–19]. It is possible to compute them when precise modeling is available, as in stochastic thermodynamics, or from integrating a local response, as for our model of TEGs. In this letter, the non-equilibrium conductance matrix of a composite thermodynamic system is determined using those of its two subsystems’ non-equilibrium conductance and their conservation laws. We first explain how current conservation at the interface allows to integrate the internal degrees of freedom, i.e. the local potentials at the connection; second, we generalize the law of impedance addition in a way that matches their matrix dimension thanks to conservation laws within each subsystem.

Devices connection and internal degrees of freedom— Fig. 1 shows three devices represented by boxes with sets of pins $\mathcal{P}^{(m)} = \{1, 2, \dots, |\mathcal{P}^{(m)}|\}$ for $m = 1, 2, 3$. We denote $|\mathcal{P}^{(m)}|$ the total number of pins of device m . Without loss of generality, we focus on connecting device 1 to device 2 to create device 3. Connecting more devices follows from a sequence of pairwise connections. We split the pins of each device into left and right disjoint sets $\mathcal{P}^{(m)} = \mathcal{P}_l^{(m)} \cup \mathcal{P}_r^{(m)}$ to connect the right pins of device 1 to the left pins of device 2 as in Fig. 1. For device 3, these connected pins are internal: local potentials there are functions of the external ones. The local potential vector $\mathbf{a}^{(m)}$, of component $a_p^{(m)}$ (e.g., temperature, pressure, chemical potential) at pin $p \in \mathcal{P}^{(m)}$ can be set by another device or by reservoirs of various conjugated extensive quantity (e.g. heat, volume, chemical species). By convention, the physical current vector $\mathbf{i}^{(m)}$ has a positive p ’th component $i_p^{(m)}$ when a positive amount of the corresponding extensive quantity is received by the device m through

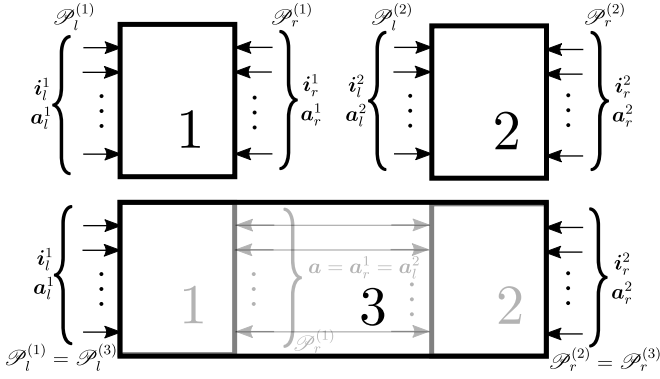


FIG. 1. (Top) Separated devices 1 and 2 with their various pins $\mathcal{P}^{(m)} = \mathcal{P}_l^{(m)} \cup \mathcal{P}_r^{(m)}$ where currents $\mathbf{i}^{(m)}$ and conjugated local potentials $\mathbf{a}^{(m)}$ takes well defined values. (Bottom) Merged device 3 made out of 1 and 2, with equal local potentials \mathbf{a} on the wire connecting 1 and 2. Current conservation at connection implies $\mathbf{i}_r^1 + \mathbf{i}_l^2 = 0$.

pin p . Given the left/right splitting of pins, the column vector for physical currents and local potentials write $\mathbf{i}^{(m)} = (\mathbf{i}_l^{(m)}, \mathbf{i}_r^{(m)})^T$ and $\mathbf{a}^{(m)} = (\mathbf{a}_l^{(m)}, \mathbf{a}_r^{(m)})^T$ respectively, the sub-vectors with index $\chi = l, r$ including only components on the χ side. The superscript T denotes transposition. We notice that $\mathbf{i}^{(3)} = (\mathbf{i}_l^{(1)}, \mathbf{i}_r^{(2)})^T$ by definition. Given $\mathbf{a}^{(m)}$, we assume that device m reaches a unique stationary state with constant physical currents $\mathbf{i}^{(m)}$ that are non-linear functions of $\mathbf{a}^{(m)}$. The Entropy Production Rate (EPR) for device m in this stationary state reads $\sigma^{(m)} = \mathbf{a}^{(m)T} \mathbf{i}^{(m)}$. Assuming no dissipation at the interface, the EPR of device 3 is

$$\sigma^{(3)} = \mathbf{a}_l^{(1)T} \mathbf{i}_l^{(1)} + \mathbf{a}_r^{(2)T} \mathbf{i}_r^{(2)} = \sigma^{(1)} + \sigma^{(2)}. \quad (1)$$

The last equality requires identical local potentials on the connection pins $\mathbf{a}_r^{(1)} = \mathbf{a}_l^{(2)} \equiv \mathbf{a}$ given the current conservation at the interface

$$\mathbf{i}_r^{(1)} + \mathbf{i}_l^{(2)} = 0. \quad (2)$$

Eq. (2) is a system of $|\mathcal{P}_r^{(1)}| = |\mathcal{P}_l^{(2)}|$ equations that must be solved to determine the internal local potentials. This provides \mathbf{a} as a function of the external potentials $\mathbf{a}_l^{(1)}$ and $\mathbf{a}_r^{(2)}$ eliminating all internal degrees of freedom.

Non-equilibrium conductance for serial connection— We focus on conductance matrices relating *independent* currents to their conjugated forces. Physical currents $\mathbf{i}^{(m)}$ are linearly dependent due to the set $\mathcal{L}^{(m)} = \{\ell_1^{(m)}, \ell_2^{(m)}, \dots\}$ of conservation laws. These linear dependencies read $\ell^{(m)} \mathbf{i}^{(m)} = \mathbf{0}$ denoting $\ell^{(m)}$ the matrix whose k th line is given by $\ell_k^{(m)}$. The row vectors in $\mathcal{L}^{(m)}$ have $|\mathcal{P}^{(m)}|$ components and are linearly independent. Given the left/right partition of $\mathcal{P}^{(m)}$ for $m = 1, 2$, the matrix of conservation laws splits into two submatrices

such that

$$\ell^{(m)} \mathbf{i}^{(m)} = \begin{pmatrix} \ell_l^{(m)} & \ell_r^{(m)} \end{pmatrix} \begin{pmatrix} \mathbf{i}_l^{(m)} \\ \mathbf{i}_r^{(m)} \end{pmatrix} = \mathbf{0}. \quad (3)$$

This will be useful in the following to relate internal and external currents. Graphically, each conservation law of devices $m = 1, 2$ can be represented as a tree graph, see Fig. 2 and 3. We assume that after connection, the conservation laws of device 3 are also tree graphs, i.e. with no internal loops. Making loops is possible in the end by equating the local potential of two external pins, effectively connecting them and fixing the gauge freedom with a reservoir. Using the conservation laws, we select a subset of pins $\mathcal{J}^{(m)} \subset \mathcal{P}^{(m)}$ for which currents $i_p^{(m)}$ with $p \in \mathcal{J}^{(m)}$ are independent. This defines the basis of fundamental currents [12]. We denote in capital letters the vector of fundamental currents $\mathbf{I}^{(m)} = (i_p^{(m)} | p \in \mathcal{J}^{(m)})^T$ whose dimension is $|\mathcal{J}^{(m)}| = |\mathcal{P}^{(m)}| - |\mathcal{L}^{(m)}|$. The selection matrix $\mathbf{S}^{(m)}$ relates the physical and fundamental current vectors and defines the vector of fundamental forces $\mathbf{A}^{(m)}$:

$$\mathbf{i}^{(m)} = \mathbf{S}^{(m)} \mathbf{I}^{(m)}, \quad \mathbf{A}^{(m)} = \mathbf{S}^{(m)T} \mathbf{a}^{(m)}, \quad (4)$$

such that the EPR satisfies

$$\sigma^{(m)} = \mathbf{a}^{(m)T} \mathbf{i}^{(m)} = \mathbf{a}^{(m)T} \mathbf{S}^{(m)} \mathbf{I}^{(m)} = \mathbf{A}^{(m)T} \mathbf{I}^{(m)}. \quad (5)$$

We remark that by construction $\ell^{(m)} \mathbf{S}^{(m)} = 0$ and the $|\mathcal{J}^{(m)}|$ columns of the selection matrix are independent. The non-linear characteristic of each device provides the fundamental currents $\mathbf{I}^{(m)}$ as a function of the forces $\mathbf{A}^{(m)}$

$$\mathbf{I}^{(m)} = \mathbf{G}^{(m)} \mathbf{A}^{(m)}, \quad (6)$$

where $\mathbf{G}^{(m)} = \mathbf{G}^{(m)}(\mathbf{A}^{(m)})$ is a non-equilibrium conductance matrix. It is symmetric, force-dependent, and positive definite at non-equilibrium stationary state [15]. In this state, the numerical values of the entries of matrices $\mathbf{G}^{(1)}$ and $\mathbf{G}^{(2)}$ are known for the fundamental forces given by Eq. (4) evaluated at $\mathbf{a}^{(1)} = (\mathbf{a}_l^{(1)}, \mathbf{a})^T$ and $\mathbf{a}^{(2)} = (\mathbf{a}, \mathbf{a}_r^{(2)})^T$ respectively. Then, Eqs. (1) and (5–6) lead to the relation between conductance matrices

$$\mathbf{I}^{(3)T} \mathbf{G}^{(3)-1} \mathbf{I}^{(3)} = \sum_{m=1,2} \mathbf{I}^{(m)T} \mathbf{G}^{(m)-1} \mathbf{I}^{(m)}. \quad (7)$$

This defines the inverse of the non-equilibrium conductance matrix for device 3 as

$$\mathbf{G}^{(3)-1} = \sum_{m=1}^2 \mathbf{\Pi}^{(m,3)T} \mathbf{G}^{(m)-1} \mathbf{\Pi}^{(m,3)}, \quad (8)$$

provided that a matrix $\mathbf{\Pi}^{(m,3)}$ exists for which $\mathbf{I}^{(m)} = \mathbf{\Pi}^{(m,3)} \mathbf{I}^{(3)}$ for $m = 1, 2$. Eq. (8) is our main result: it

generalizes the concept of equivalent impedance for the serial connection of thermodynamic devices.

In the following, we determine $\mathbf{\Pi}^{(m,3)}$ in two steps. We look for $\boldsymbol{\pi}^{(m,3)}$ satisfying $\mathbf{i}^{(m)} = \boldsymbol{\pi}^{(m,3)} \mathbf{i}^{(3)}$ and related to the former matrix by

$$\mathbf{\Pi}^{(m,3)} = \mathbf{S}^{(m)+} \boldsymbol{\pi}^{(m,3)} \mathbf{S}^{(3)}. \quad (9)$$

We use $\mathbf{S}^{(m)+} = [\mathbf{S}^{(m)T} \mathbf{S}^{(m)}]^{-1} \mathbf{S}^{(m)T}$ as the pseudo-inverse of $\mathbf{S}^{(m)}$ for $m = 1, 2$. These matrices are known since they determine the fundamental currents in which the conductance matrices $\mathbf{G}^{(1)}$ and $\mathbf{G}^{(2)}$ are given. As a first step, we combine Eqs. (2-3) into

$$\mathbf{L}_i \mathbf{i}_r^{(1)} = \mathbf{L}_e \mathbf{i}^{(3)} \text{ with } \mathbf{L}_i \equiv \begin{bmatrix} -\boldsymbol{\ell}_r^{(1)} \\ \boldsymbol{\ell}_l^{(2)} \end{bmatrix}, \mathbf{L}_e \equiv \begin{bmatrix} \boldsymbol{\ell}_l^{(1)} & 0 \\ 0 & \boldsymbol{\ell}_r^{(2)} \end{bmatrix}. \quad (10)$$

We remark that \mathbf{L}_i is full column rank since it is an incidence matrix of a tree graph whose vertices are the conservation laws and whose edges are the connection pins [20]. Then, we define $\boldsymbol{\pi} = \mathbf{L}_i^+ \mathbf{L}_e$ to obtain $\mathbf{i}_r^{(1)} = \boldsymbol{\pi} \mathbf{i}^{(3)} = -\mathbf{i}_l^{(2)}$ and conclude our first step with

$$\boldsymbol{\pi}^{(1,3)} = \begin{bmatrix} \mathbb{1} & 0 \\ \boldsymbol{\pi} & \end{bmatrix} \text{ and } \boldsymbol{\pi}^{(2,3)} = \begin{bmatrix} -\boldsymbol{\pi} \\ 0 & \mathbb{1} \end{bmatrix}, \quad (11)$$

since $\mathbf{i}_l^{(1)} = [\mathbb{1} \ 0] \mathbf{i}^{(3)}$ and $\mathbf{i}_r^{(2)} = [0 \ \mathbb{1}] \mathbf{i}^{(3)}$ by definition. Second, the column of $\mathbf{S}^{(3)}$ defines a basis of $\ker(\boldsymbol{\ell}^{(3)})$ albeit the conservation laws of the third device remain to determine. This can be done by computing the left null eigenvectors of \mathbf{L}_i , gathered as lines of a matrix \mathbf{v} . The rank-nullity theorem gives the number of lines of \mathbf{v} :

$$\dim(\text{coker } \mathbf{L}_i) = |\mathcal{L}^{(1)}| + |\mathcal{L}^{(2)}| - |\mathcal{P}_r^{(1)}|, \quad (12)$$

since \mathbf{L}_i has $|\mathcal{L}^{(1)}| + |\mathcal{L}^{(2)}|$ lines, $|\mathcal{P}_r^{(1)}|$ columns and rank $|\mathcal{P}_r^{(1)}|$. Given our constraints on the devices' connection, Eq. (12) also gives the cardinal of $\mathcal{L}^{(3)}$. Then, from $\mathbf{v} \mathbf{L}_i = 0$ and Eq. (10), we find the conservation laws of the third device $\boldsymbol{\ell}^{(3)} = \mathbf{v} \mathbf{L}_e$. A basis of its kernel leads to $\mathbf{S}^{(3)}$ and finally to $\mathbf{\Pi}^{(m,3)}$.

Serial connection of two TEG— We illustrate our general method on the serial association of two TEGs. We chose the fundamental current and force vectors as

$$\mathbf{I}^{(m)} = \begin{pmatrix} i_{El}^{(m)} \\ i_{Nl}^{(m)} \end{pmatrix}, \quad \mathbf{A}^{(m)} = \begin{pmatrix} \frac{1}{T_r^{(m)}} - \frac{1}{T_l^{(m)}} \\ \frac{\mu_l^{(m)}}{T_l^{(m)}} - \frac{\mu_r^{(m)}}{T_r^{(m)}} \end{pmatrix}, \quad (13)$$

where $i_{El}^{(m)}$ and $i_{Nl}^{(m)}$ are respectively the energy and electric currents entering device m from the left. The magnitude of the electric charge of an electron is denoted $e > 0$. The temperature and electrochemical potential on the $\chi = l, r$ side of device m are denoted $T_\chi^{(m)}$ and $\mu_\chi^{(m)}$ respectively. The current-force characteristic of Eq. (6)

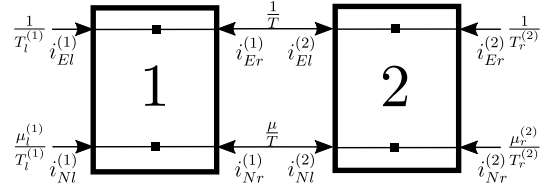


FIG. 2. Serial connection of two TEG. The conservation of energy and matter implies for each device the following relation: $i_{El}^{(m)} + i_{Er}^{(m)} = 0$ and $i_{Nl}^{(m)} + i_{Nr}^{(m)} = 0$. For $m = 1, 2$, $S_N^{(m)}, K^{(m)}, R^{(m)}$ are independent of the intensive parameters α_p .

holds with the conductance matrix [21, 22]

$$\mathbf{G}^{(m)} = \frac{T_l^{(m)} T_r^{(m)}}{e^2 R^{(m)} \bar{T}^{(m)}} \begin{bmatrix} e^2 K^{(m)} R^{(m)} \bar{T}^{(m)} + H^{(m)2} & H^{(m)} \\ H^{(m)} & 1 \end{bmatrix}. \quad (14)$$

We denote $R^{(m)}$ the electrical resistance, $K^{(m)}$ the thermal conductivity and $S_N^{(m)}$ the Seebeck coefficient of the m th TEG. The later appears in the coupling factor $H^{(m)} = S_N^{(m)} \bar{T}^{(m)} + \bar{\mu}^{(m)}$ involving the average temperature and electrochemical potential

$$\bar{T}^{(m)} = \frac{T_l^{(m)} + T_r^{(m)}}{2}, \quad \bar{\mu}^{(m)} = \frac{\mu_l^{(m)} + \mu_r^{(m)}}{2}. \quad (15)$$

The conservation of the currents at the interface reads $i_{Er}^{(1)} + i_{El}^{(2)} = 0$ and $i_{Nr}^{(1)} + i_{Nl}^{(2)} = 0$. From this we deduce $T_r^{(1)} = T_l^{(2)} \equiv T$ and $\mu_r^{(1)} = \mu_l^{(2)} \equiv \mu$:

$$T = \frac{K^{(1)} T_l^{(1)} + K^{(2)} T_r^{(2)} + \frac{e^2}{2} (R^{(1)} + R^{(2)}) i_{Nl}^{(2)2}}{K^{(1)} + K^{(2)} - \delta_{S_N} i_{Nl}^{(2)}} \quad (16)$$

$$\mu = R_{\parallel} \left(\frac{\mu_l^{(1)} - S_N^{(1)} \Delta T^{(1)}}{R^{(1)}} + \frac{\mu_r^{(2)} + S_N^{(2)} \Delta T^{(2)}}{R^{(2)}} \right) \quad (17)$$

where $\delta_{S_N} = S_N^{(2)} - S_N^{(1)}$, $1/R_{\parallel} = 1/R^{(1)} + 1/R^{(2)}$ and $\Delta T^{(m)} = T_r^{(m)} - T_l^{(m)}$. Finally, Eq. (8) becomes $\mathbf{G}^{(3)-1} = \mathbf{G}^{(1)-1} + \mathbf{G}^{(2)-1}$. The outcome of this equation is in fact also given by Eq. (14) for $m = 3$ using the following new parameters

$$\frac{1}{K^{(3)}} = \sum_{m=1}^2 \frac{1}{K^{(m)}}, \quad \frac{S_N^{(3)}}{K^{(3)}} = \sum_{m=1}^2 \frac{S_N^{(m)}}{K^{(m)}}, \quad (18)$$

$$\frac{H^{(3)}}{K^{(3)}} = \sum_{m=1}^2 \frac{H^{(m)}}{K^{(m)}}, \quad (19)$$

and the electrical resistance

$$R^{(3)} = \left[1 - \frac{\delta_{S_N} i_{Nl}^{(2)}}{2(K^{(1)} + K^{(2)})} \right] \sum_{m=1}^2 R^{(m)} + \frac{\delta_{S_N}^2 T}{e^2 (K^{(1)} + K^{(2)})}. \quad (20)$$

The detailed derivation of these results will be published in a forthcoming publication. Interestingly, $R^{(3)}$ depends on the matter current and is not simply the sum of the sub-devices resistances (excepted when $\delta_{S_N} = 0$) as already noticed for TEGs under mixed boundary conditions [23, 24].

Example of dimension matching for conductances—

We now illustrate the construction of an equivalent non-equilibrium conductance for the devices of Fig. 3. The first and second devices have 4 and 7 pins, 1 and 3 conservation laws leading to 3 and 4 fundamental currents respectively. Their connection via $\mathbf{i}_r^{(1)} = (i_3, i_4)^T$ produces a third device with 7 pins, 2 conservation laws and 5 fundamental currents. Contrarily to the serial connection of TEGs, the conductance matrices' dimensions are all different calling for a dimensional matching. We assume that the local potentials at the interface have already been determined and that the conductance matrix $\mathbf{G}^{(m)}$ for $m = 1, 2$ are known for fundamental currents $\mathbf{I}^{(1)} = (i_2, i_3, i_4)^T$ and $\mathbf{I}^{(2)} = (i_5, i_6, i_7, i_9)^T$ associated to

$$\mathbf{S}^{(1)} = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{S}^{(2)} = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (21)$$

Eq. (4) leads to physical currents $\mathbf{i}^{(1)} = (i_1, \dots, i_4)^T$ and $\mathbf{i}^{(2)} = (i_3, \dots, i_9)^T$ as expected given the matrices

$$\ell^{(1)} = [1 \ 1 \ | \ 1 \ 1], \quad \ell^{(2)} = \begin{bmatrix} 1 & 0 & | & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & | & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & | & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad (22)$$

for the conservation laws $\ell^{(m)} \mathbf{i}^{(m)} = \mathbf{0}$ depicted on Fig. 3. Vertical bars emphasize the bloc decomposition

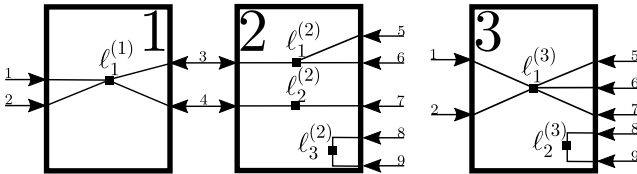


FIG. 3. Example of the serial connection of two devices. For device $m = 1$ the pins are divided into $\mathcal{P}_l^{(1)} = \{1, 2\}$ and $\mathcal{P}_r^{(1)} = \{3, 4\}$. For device $m = 2$ the pins are divided into $\mathcal{P}_l^{(2)} = \{3, 4\}$ and $\mathcal{P}_r^{(2)} = \{5, 6, 7, 8, 9\}$.

of Eq. (3). Applying Eq. (10) to our example, we find

$$\mathbf{L}_e = \begin{bmatrix} 1 & 2 & 5 & 6 & 7 & 8 & 9 \\ 1 & 1 & | & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & | & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & | & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & | & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{L}_i = \begin{bmatrix} 3 & 4 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad (23)$$

where we indicate the pin index on top of each column. As expected, the matrix \mathbf{L}_i is full column rank with pseudo inverse

$$\mathbf{L}_i^+ = \frac{1}{3} \begin{pmatrix} -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{pmatrix}. \quad (24)$$

Matrices $\boldsymbol{\pi}^{(m,3)}$ for $m = 1, 2$ appearing in Eq. (11) read

$$\boldsymbol{\pi}^{(1,3)} = \begin{bmatrix} \mathbb{1}_2 & \mathbb{0}_{2 \times 5} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 \end{bmatrix}, \quad (25)$$

$$\boldsymbol{\pi}^{(2,3)} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & 0 & 0 \\ \mathbb{0}_{5 \times 2} & \mathbb{1}_5 \end{bmatrix}. \quad (26)$$

We denote by $\mathbb{1}_n$ the identity square matrix of dimension n and by $\mathbb{0}_{n \times m}$ the null matrix of dimension $n \times m$. Next, the choice of selection matrix $\mathbf{S}^{(3)}$ determines the fundamental basis in which $\mathbf{G}^{(3)}$ is given. The column vectors of $\mathbf{S}^{(3)}$ realize a basis of $\ker(\ell^{(3)})$. Let's first determine $\ell^{(3)}$ by finding the left null eigenvectors of \mathbf{L}_i that we gather in the lines of matrix

$$\mathbf{v} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (27)$$

Then, the matrix of conservation laws arises from

$$\ell^{(3)} = \mathbf{v} \mathbf{L}_i = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}. \quad (28)$$

The vector of physical currents $\mathbf{i}^{(3)} = (i_1, i_2, i_5, i_6, i_7, i_8, i_9)^T$ is obtained from the product of the selection matrix

$$\mathbf{S}^{(3)} = \begin{bmatrix} -1 & -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (29)$$

with the vector of fundamental currents $\mathbf{I}^{(3)} = (i_2, i_5, i_6, i_7, i_9)^T$. Finally, the matrices $\boldsymbol{\Pi}^{(m,3)}$ for $m = 1, 2$ and then $\mathbf{G}^{(3)}$ follow from Eqs. (8–9).

Conclusion— The connection of thermodynamic devices, converting physical quantities of any kind, opens many possibilities that must be further explored. This

requires theoretical works to extend the notion of non-equilibrium conductance, for instance from stationary states to periodic steady states, or from graphs to hypergraphs. Current fluctuations within composite devices are also of great interest, for instance in relation to relaxation toward stationary non-equilibrium. On a side more inspired by electronics, impedance adaptation could be generalized to allow maximal transmission across the circuits.

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