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Work statistics in stochastically driven systems

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Abstract
We identify the conditions under which a stochastic driving that induces energy changes into a system coupled with a thermal bath can be treated as a work source. When these conditions are met, the work statistics satisfy the Crooks fluctuation theorem traditionally derived for deterministic drivings. We illustrate this fact by calculating and comparing the work statistics for a two-level system driven respectively by a stochastic and a deterministic piecewise constant protocol.

Keywords: fluctuation theorem, random processes, thermodynamics, work statistics

1. Introduction

Stochastic thermodynamics allows the identification of thermodynamic quantities at the level of single stochastic trajectories for systems described by Markov processes satisfying local detailed balance (LDB) [1–5]. This theory is particularly relevant to describe small systems subjected to significant fluctuations compared with their average behavior. It has been experimentally validated thanks to remarkable progress in experimental techniques, making it possible to measure and manipulate systems at the sub-micron level. Among these achievements are the nonequilibrium versions of the fluctuation-dissipation theorem [6–15], the connection between thermodynamics and information theory [16–24], and—perhaps most important—the formulation and verification of the so-called fluctuation theorem (and its variants) [25–34]. According to this theorem, stochastic positive entropy production is exponentially more likely to be observed than the corresponding negative entropy production. On average, this implies a positive entropy production in agreement with the second law. Depending on the type of setup, entropy production can be related to different physical observables [35]. A particularly important setup consists of a system in contact with a single heat bath and driven by an external work source. The latter is modeled by a deterministic time dependence of the system energies. Along a system trajectory, the energy changes that occur while the system remains in a given energy state are treated as work, whereas the energy changes that occur when the system jumps from one energy state to another due to the bath are treated as heat. In agreement with traditional thermodynamics, these definitions imply that the work source itself does not contribute to the dissipation, whereas heat does. In these setups, entropy production can be expressed as the work W minus the free energy difference ΔF between the initial and the final equilibrium state. The resulting fluctuation theorem for the work statistics in these setups has been used to evaluate free energy differences [36, 37].

In this paper, we analyse the thermodynamic implication of considering stochastic rather than deterministic drivings of system energies. Contrary to deterministic drivings, stochastic drivings are different along each system trajectory and therefore effectively introduce new degrees of freedom in the description of a system. The resulting dynamics occurs in the joint space of the system and of the driving. We emphasize that for the setups we consider, the stochastically driven system is still in contact with a bath with a well-defined temperature (introduced via LDB). This should be distinguished from systems where the only source of noise is a non-thermal stochastic driving (without LDB). In such situations, the source of noise cannot be decomposed into a non-thermal and a thermal noise, as is necessary to properly identify the physical heat. The resulting fluctuation theorem for entropy production becomes a mathematical identity with little connection to thermodynamics. Because the injected power (energy per unit time injected by the noisy source into the system) does not relate to entropy production, it is therefore not surprising that its statistics have been shown to deviate from the fluctuation theorem [38–42].

Our motivation is practical as well as conceptual. From a practical perspective, a deterministic driving is never perfect and will always be accompanied by uncontrollable small random fluctuations. It is therefore important to understand how these fluctuations may affect the work statistics and the fluctuation theorem in the context of thermal systems. From a conceptual perspective, it is clear that the mathematical entropy production in the joint space of the driven system and the stochastic driving will always satisfy a fluctuation theorem. This was recently shown, for example, in [43] by considering Ornstein–Uhlenbeck processes. However, it is not at all clear whether the statistics of the energy injected into the system by the stochastic driving can fully capture the dissipation of the system dynamics, as is the case for deterministic drivings. We show that the conditions for this to occur are rather restrictive because the stochastic driving must evolve reversibly. In this case, the injected energy can be considered as work, and its statistics satisfy the Crooks fluctuation theorem [30].

We study two models satisfying this condition: a two-level Markov process coupled with another independent two-level Markov process and a one-dimensional Ornstein–Uhlenbeck process coupled with another independent one-dimensional Ornstein–Uhlenbeck process. The latter model has been used to show that the 'work' statistics for the stochastically driven colloidal system [44, 45] do not display fluctuation theorem symmetry. In these references, 'work' is defined as the time integral of the stochastic force times the velocity and does not correspond to the Jarzynski work, which (when subtracting the nonequilibrium free energy change) we show satisfies the fluctuation theorem symmetry.

The outline of the paper is as follows. In section 2, we identify the general conditions under which a stochastic driving behaves as a work source. In this case, we show that the work statistics satisfy the Crooks fluctuation theorem. In section 3, we compute and compare the large deviation properties of the work statistics of a two-level system driven by a piecewise constant deterministic driving [46] with those of the same system subjected to a stochastic work source. Conclusions are drawn in section 4. Finally, using the results of [45], we show in appendix C that a stochastically driven overdamped colloidal particle in a harmonic trap does satisfy the Crooks fluctuation theorem when the work is properly identified.

2. Stochastic driving as a work source

We consider stationary Markovian dynamics on a bipartite joint system made of a system with states σ and an independent energy source with states h. The bipartite property means that transitions involving a simultaneous change in σ and h are not allowed. The rates $\omega_{\sigma',\sigma}(h)$ that describe system jumps from σ to σ' satisfy local detailed balance

$$\ln \frac{\omega_{\sigma',\sigma}(h)}{\omega_{\sigma,\sigma'}(h)} = -\beta [E(\sigma', h) - E(\sigma, h)] \tag{1}$$

whereas the rates describing the energy source $\omega_{h,h'}$ do not depend on σ and do not necessarily satisfy local detailed balance. We introduced the inverse temperature $\beta = 1/T$ ($k_b = 1$). Such dynamics can be viewed as the limit of a global dynamics satisfying local detailed balance when the energy scale involved during the source transitions is very large compared with the system energy scale (see appendix A). The energy changes due to transitions between σ states at fixed h are caused by the bath and are thus treated as heat, whereas the energy changes due to transitions between h states at fixed σ are caused by the energy source and may be treated as work, as shown hereafter.

We now turn to energy exchanges described at the level of single trajectories. To set the notation, we denote the joint system, the system, and the energy source trajectories during a time interval $[0, t]$ respectively by $[\sigma, h]$, $[\sigma]$, and $[h]$. Similarly, the time-reversed trajectories are $[\bar{\sigma}, \bar{h}]$, $[\bar{\sigma}]$, and $[\bar{h}]$. The trajectory probabilities $\mathcal{P}[\sigma, h]$ can naturally be expressed as $\mathcal{P}[\sigma, h] = \mathcal{P}[\sigma|h]\mathcal{P}[h]$. Thanks to the independence of the work source with respect to the system, the probability $\mathcal{P}[h]$ of a trajectory $[h]$ is that of the Markovian dynamics of the source solely determined by the rates $\omega_{h,h'}$. $\mathcal{P}[\sigma|h]$ is the conditional probability of a system trajectory $[\sigma]$ subjected to a given trajectory $[h]$ of the energy source, and $\mathcal{P}_{\sigma_0}[\sigma|h]$ is the conditional probability of a system trajectory $[\sigma]$ given the initial state σ_0 and the trajectory $[h]$. Therefore, $\mathcal{P}[\sigma|h] = p(\sigma_0|h_0)\mathcal{P}_{\sigma_0}[\sigma|h]$, where $p(\sigma_0|h_0)$ is the probability of being in the initial state σ_0 for a given initial h_0. Using the local detailed balance (1), the heat entering the system from the bath for a given trajectory $[h]$ can be expressed as

$$Q[\sigma|h] = -\beta^{-1} \ln \frac{\mathcal{P}_{\sigma_0}\left[\sigma\middle|h\right]}{\mathcal{P}_{\bar{\sigma}_0}\left[\bar{\sigma}\middle|\bar{h}\right]} \tag{2}$$

and the entropy production of the system as

$$\Delta_{\rm i} S[\sigma|h] = \ln \frac{\mathcal{P}[\sigma|h]}{\mathcal{P}[\bar{\sigma}|\bar{h}]} = -\beta Q[\sigma|h] + \Delta S[\sigma|h] \tag{3}$$

where $\Delta S[\sigma|h] = \ln p(\sigma_0|h_0)/p(\sigma_t|h_t)$ is the change in the Shannon entropy of the system after a time t. Using energy conservation along a system trajectory, the energy provided by the energy source to the system reads $W[\sigma|h] = \Delta E[\sigma, h] - Q[\sigma|h]$, where $\Delta E[\sigma, h] = E(\sigma_t, h_t) - E(\sigma_0, h_0)$. Therefore, the entropy production becomes

$$\Delta_{\rm i} S[\sigma|h] = \beta\left(W[\sigma|h] - \Delta F[\sigma|h]\right) \tag{4}$$

where $\Delta F[\sigma|h] = \Delta E[\sigma, h] - \beta^{-1}\Delta S[\sigma|h]$ is the change in the nonequilibrium free energy of the system. The entropy production in (3) and (4) is identical to that of a system subjected to a deterministic driving $[h]$ made of sudden jumps [2, 47–49]. However, because the energy source is stochastic and produces a statistical ensemble of drivings $[h]$, the entropy production of the energy source $\Delta_{\rm i} S_{\rm sd} = \ln \mathcal{P}[h]/\mathcal{P}[\bar{h}]$ gives rise to an additional contribution to the joint system entropy production

$$\Delta_{\rm i} S[\sigma, h] = \Delta_{\rm i} S[\sigma|h] + \ln \frac{\mathcal{P}[h]}{\mathcal{P}[\bar{h}]} \tag{5}$$

After ensemble averaging (5), we get

$$\left\langle \Delta_{\rm i} S[\sigma, h] \right\rangle = \sum_{[h]} \mathcal{P}[h] \sum_{[\sigma]} \mathcal{P}[\sigma|h] \Delta_{\rm i} S[\sigma|h] + \sum_{[h]} \mathcal{P}[h] \ln \frac{\mathcal{P}[h]}{\mathcal{P}[\bar{h}]} \geqslant 0. \tag{6}$$

Because a work source is not supposed to give rise to any entropy production, our energy source is a work source only when this additional term vanishes and thus does not affect the system entropy balance. This happens either when the driving is deterministic or when the energy source h evolves reversibly, i.e., when $\mathcal{P}[h] = \mathcal{P}[\bar{h}]$. In the latter case, from a system perspective, the trajectory $[h]$ of the energy source is perceived as a time-dependent stochastic driving. We note that even in the presence of a dissipative energy source, the non-negative first term on the right-hand side of (6) still provides a lower bound to the entropy production of the joint system. The non-negative second term constitutes in turn a lower bound for the dissipation of the energy source because the trajectories $[h]$ may provide only a coarse-grained description of the energy source dynamics.

When the long time limit is considered and for systems with a finite state space, the contribution $\Delta F[\sigma|h]$ to the entropy production is not extensive in time and thus vanishes in the large deviation sense. The steady state fluctuation theorem for the entropy production therefore reads

$$I\left(w + \Sigma_{\rm sd}\right) - I\left(-w - \Sigma_{\rm sd}\right) = -w - \Sigma_{\rm sd} \tag{7}$$

where $w = W/t$ is the energy per unit of time provided by the energy source to the system, $\Sigma_{\rm sd} = \Delta_{\rm i} S_{\rm sd}/t$ is the rate of entropy production due to the energy source, and $I(w + \Sigma_{\rm sd}) = \lim_{t\to\infty}(-1/t) \ln P(\Delta_{\rm i} S = tw + t\Sigma_{\rm sd})$ is the large deviation function for the total entropy production. It is only when $\Sigma_{\rm sd} = 0$ (e.g. when the driving is deterministic or when the energy source h evolves reversibly) that the energy source behaves as a work source, and that the Crooks fluctuation theorem is recovered.

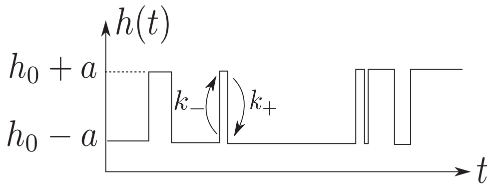

Figure 1. Representation of the work source stochastic dynamics.

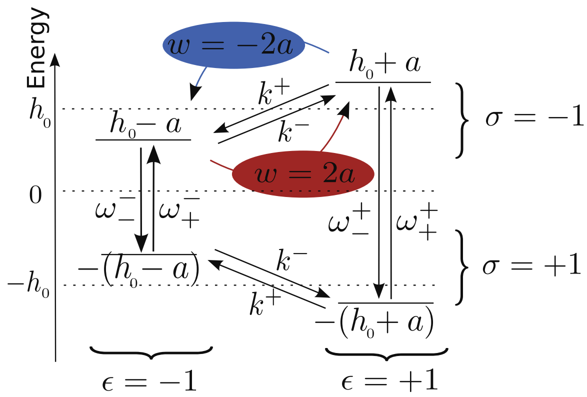

Figure 2. Representation of the energy levels of the two-level system: on the left (respectively right) the work source state is $\varepsilon = -1$ (respectively $\varepsilon = +1$). The upper (respectively lower) part of the figure corresponds to $\sigma = -1$ (respectively $\sigma = 1$). For $\sigma = -1$, a decrease in h will decrease the energy of the system, resulting in a negative work contribution $w = -2a$, whereas an increase in h will increase the energy of the system and result in a positive work contribution $w = 2a$.

The stochastic drivings used for the models presented hereafter all satisfy this condition and qualify as work sources.

3. Modulated two-level system

In this section, we compare the work statistics of a two-level system driven by a stochastic (reversible) work source with those of the same system driven by a deterministic work source.

3.1. Stochastic work source

We consider a two-level system $\sigma = \pm 1$ coupled with a heat bath at temperature T and interacting with a stochastic energy source with two states $\varepsilon = \pm 1$. The Poisson rate to leave the source state ε is denoted k^ε and does not depend on σ. This immediately implies that the stationary dynamics of the energy source is reversible and can thus be considered as a work source. We denote by $h(t) = h_0 + \varepsilon(t)a$ the driving produced by the work source whose state at time t is given by $\varepsilon(t)$. This driving is illustrated in figure 1. The energy of the joint system is $E(\sigma, h) = -\sigma h$ in a unit of $k_B T$ ($\beta = 1$ from now on). This corresponds to the four energy levels $E(\sigma, h_0 + \varepsilon a) = -\sigma(h_0 + \varepsilon a)$ depicted in figure 2. For a given source state ε, the rate describing a transition from system state σ to $-\sigma$ is given by $\omega_\sigma^\varepsilon = \omega(h_0 + \varepsilon a)\mathrm{e}^{-\sigma(h_0+\varepsilon a)}$. This rate satisfies the local detailed balance condition and includes as special cases

- Arrhenius rates, $\omega(h) = \Gamma$
- Fermi rates, $\omega(h) = \Gamma/(2\cosh(h))$
- Bose rates, $\omega(h) = \Gamma/(2|\sinh(h)|)$

where Γ is a positive constant that sets the time scale ($\Gamma = 1$ in all figures).

The dynamics of the joint system is described by a stationary four-state model. Each state is specified by the pair $(\sigma, \varepsilon) = \theta$. The work-generating function is defined as $G_{\theta,\mu}(t) = \langle e^{\mu W(t)}\delta_{\theta,\theta(t)}\rangle$, where δ is the Kronecker delta and $\langle..\rangle$ denotes an average over all possible values of the work $W(t) = -\int_0^t dt'\dot{h}(t')\sigma(t')$ and of the states $\theta(t)$ at time t. Its evolution is ruled by

$$\partial_t G_{\theta,\mu} = \sum_{\theta'} M^{\mu}_{\theta,\theta'} G_{\theta',\mu} \tag{8}$$

where

$$\boldsymbol{M}^{\mu} = \begin{bmatrix} -k^+ - \omega_+^+ & \omega_-^+ & k^- e^{-2a\mu} & 0 \\ \omega_+^+ & -k^+ - \omega_-^+ & 0 & k^- e^{2a\mu} \\ k^+ e^{2a\mu} & 0 & -k^- - \omega_+^- & \omega_-^- \\ 0 & k^+ e^{-2a\mu} & \omega_+^- & -k^- - \omega_-^- \end{bmatrix}. \tag{9}$$

We note that whereas the joint system is ruled by autonomous steady state dynamics in the long time limit, the two-level system continuously undergoes random energy switches from the work source and tries to relax toward the new corresponding equilibrium state. The work statistics in the long time limit are characterized by ϕ_μ, the largest eigenvalue of $\boldsymbol{M}_\mu$. This function ϕ_μ is called the asymptotic cumulant generating function for the work

$$G_\mu(t) = \sum_\theta G_{\theta,\mu}(t) \asymp \exp\left[t\phi_\mu\right] \tag{10}$$

Its analytical expression can be obtained by using the Ferrari method to solve the following characteristic equation

$$\det\left(\boldsymbol{M}^{\mu} - \phi\boldsymbol{I}\right) = \phi^4 + c_3\phi^3 + c_2\phi^2 + c_1\phi^1 + c_0(\mu) = 0 \tag{11}$$

where $\boldsymbol{I}$ is the identity matrix and

$$c_3 = 2\left(k^+ + k^-\right) + \omega^+ + \omega^- > 0$$

$$c_2 = \Omega\left[\cosh\left(2h_0\right) + \cosh(2a)\right] + \left(k^+ + k^-\right)^2 + \omega^+\left(2k^- + k^+\right) + \omega^-\left(2k^+ + k^-\right) > 0$$

$$c_1 = \left(k^- + k^+\right)\left\{k^-\omega^+ + k^+\omega^- + \Omega\left[\cosh\left(2h_0\right) + \cosh(2a)\right]\right\} > 0$$

$$c_0(\mu) = -k^+k^-\Omega\left[\cosh(2a(1 + 2\mu)) - \cosh(2a)\right] \leqslant 0 \tag{12}$$

with $\Omega = 2\omega(h_0 + a)\omega(h_0 - a) > 0$ and $\omega^\pm = \omega_+^\pm + \omega_-^\pm$. Note that the last inequality in (12) holds for any real μ. The characteristic polynomial can always be factorized into a product of two second-degree polynomials with real coefficients. The cumulant-generating function is then the largest solution among the solutions of the two second-degree polynomials, with the choice between the two polynomials depending on the parameters. We do not explicitly provide the lengthy analytical solution to equation (11). Simpler expressions can be obtained in the fast and slow modulation limit or for low and high amplitudes of the field by pertubatively expanding

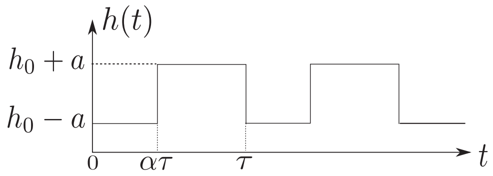

Figure 3. Representation of the periodic work source dynamics.

Table 1. Cumulant-generating function $\tau\phi_\mu$ of the work (stochastic work source) during a time interval τ in the limits of fast ($\Gamma\tau \ll 1$) and slow modulation ($\Gamma\tau \gg 1$), for low amplitudes of modulation ($a \ll 1$) or for large energy gaps ($h_0 \gg 1$). We use (19) to evaluate $k^\pm$ which allows comparison with the deterministic work source (see table 2).

$\Gamma\tau \ll 1$	$\frac{\tau}{2}\sqrt{\left[\alpha\omega^- + (1-\alpha)\omega^+\right]^2 + 4\alpha(1-\alpha)\omega^+\omega^- \frac{\cosh 2a(2\mu+1) - \cosh 2a}{\cosh 2a + \cosh 2h_0}} - \frac{\tau}{2}\alpha\omega^- - \frac{\tau}{2}(1-\alpha)\omega^+$	
$\Gamma\tau \gg 1$	$\frac{-1}{2\alpha(1-\alpha)} + \frac{1}{2}\sqrt{\frac{1}{\alpha^2(1-\alpha)^2} + \frac{4\left[\cosh 2a(2\mu+1) - \cosh 2a\right]}{\alpha(1-\alpha)(\cosh 2a + \cosh 2h_0)}}.$	
$a \ll 1$	$\mu(1+\mu)\langle w\rangle = \frac{8a^2\mu(1+\mu)}{1 + \cosh 2h_0 + \left[\alpha(1-\alpha)\omega(h_0)\tau\right]^{-1}\cosh h_0}$	
	Arrhenius rates	Fermi and Bose rates
$h_0 \gg a$	$\frac{\cosh 2a(2\mu+1) - \cosh 2a}{\cosh 2h_0}$	$\frac{\cosh 2a(2\mu+1) - \cosh 2a}{\left\{1 + [\alpha(1-\alpha)\Gamma\tau]^{-1}\right\}\cosh 2h_0}$

the function ϕ and the coefficients c to various orders. The characteristic polynomial obtained in this way decomposes into several equations for each order. The lowest-order solutions are summarized in table 1.

Note that the only coefficient of the characteristic polynomial containing a μ dependence is the one of zero degree in ϕ. This implies that the lowest-order solutions will always contain the variable μ as expected. We also remark that if the ith order solution is independent of μ, it must vanish because $\phi_{\mu=0}$ is zero by definition. This is helpful for simplifying the calculations leading to table 1. We also note that the results in table 1 rely on the hypothesis that μ is chosen inside an interval that depends on the expansion parameter. They are valid for any types of rates except in the large field expansion ($h_0 \gg 1$) where the function $\omega(h)$ has, at large h, a leading role to determine the order of the various coefficients.

3.2. Periodic work source

We consider now the same two-level system as before but driven by the deterministic and periodic (of period τ) work source depicted in figure 3. The period fraction during which the work source is in the lower (respectively higher) state is denoted by α (respectively $1-\alpha$). The large deviation function and the cumulant-generating function of the work statistics are derived in [46]. We briefly summarize the derivation. The work-generating function $G_{\sigma,\mu}(t) = \langle \mathrm{e}^{\mu W(t)}\delta_{\sigma,\sigma(t)}\rangle$ evolves according to

$$\partial_t G_{\sigma,\mu} = \sum_{\sigma'=\pm 1} L^{\mu}_{\sigma,\sigma'}(h(t))G_{\sigma',\mu} \tag{13}$$

where $L^{\mu}_{\sigma,\sigma'}(h) = -\sigma\sigma'\omega(h)\mathrm{e}^{-\sigma' h} - \dot{h}\mu\sigma\delta_{\sigma,\sigma'}$. The asymptotic work cumulant generating function is given by the logarithm of the highest eigenvalue λ_μ of the following propagator:

$$\boldsymbol{Q} = \overrightarrow{\exp} \int_0^{\tau} \boldsymbol{L}^{\mu}(h(t))\mathrm{d}t \tag{14}$$

where $\overrightarrow{\exp}$ stands for the time-ordered exponential. If $g_{\sigma,\mu}$ denotes the components of the eigenvector associated with the eigenvalue λ_μ and $g_\mu = \sum_\sigma g_{\sigma,\mu}$ denotes the sum of its components, after n periods we get

$$G_\mu(n\tau) = \sum_{\sigma,\sigma'} \left(\boldsymbol{Q}^n\right)_{\sigma,\sigma'} g_{\sigma',\mu} = \left(\lambda_\mu\right)^n g_\mu \tag{15}$$

which leads to the asymptotic cumulant generating function

$$\tilde{\phi}_\mu = \lim_{n\to\infty} \frac{1}{n} \ln G_\mu(n\tau) = \ln \lambda_\mu. \tag{16}$$

The propagator over one period $\boldsymbol{Q}$ can be decomposed into the norm-conserving evolutions over $\alpha\tau$ and $(1 - \alpha)\tau$ interspersed with the propagation over the two time steps coinciding with the change of the work source state h. In the end, the work cumulant generating function is found to be

$$\tilde{\phi}_\mu = \ln\Bigg(\frac{A}{2}\cosh 2a(2\mu + 1) + \frac{B}{2} + \frac{1}{2}\sqrt{\left[A\cosh 2a(2\mu + 1) + B\right]^2 - 4\mathrm{e}^{-(1-\alpha)\tau\omega^+ - \alpha\tau\omega^-}}\Bigg) \tag{17}$$

where

$$A = \frac{\left(1 - \mathrm{e}^{-(1-\alpha)\tau\omega^+}\right)\left(1 - \mathrm{e}^{-\alpha\tau\omega^-}\right)}{\cosh 2a + \cosh 2h_0}$$

$$B = \frac{\left(1 + \mathrm{e}^{-(1-\alpha)\tau\omega^+ - \alpha\tau\omega^-}\right)\cosh 2h_0}{\cosh 2a + \cosh 2h_0} + \frac{\left(\mathrm{e}^{-(1-\alpha)\tau\omega^+} + \mathrm{e}^{-\alpha\tau\omega^-}\right)\cosh 2a}{\cosh 2a + \cosh 2h_0} \tag{18}$$

We defined $\omega^\pm = 2\omega(h_0 \pm a)\cosh(h_0 \pm a)$ as in the stochastic work source case. The expansion of $\tilde{\phi}_\mu$ in the limit of fast and slow modulation and in the limit of low amplitude a and large h_0 is given in table 2.

3.3. Thermodynamics and average behavior

We now turn to the analysis and comparison of the work statistics generated by the stochastic and periodic work sources. In both cases, we compare the work accumulated during a time τ. Because $\tilde{\phi}$ describes the statistics of work per period τ for the periodic work source and because ϕ describes the statistics of work per unit time for the stochastic work source, to compare the two, $\tilde{\phi}$ and $\tau\phi$ have to be considered. Furthermore, the parameters setting the time scale of the two work sources have to be related via

$$k^+ = \frac{1}{(1 - \alpha)\tau} \quad \text{and} \quad k^- = \frac{1}{\alpha\tau} \tag{19}$$

in order to spend, on average, the same amount of time at the high and low value of h. We keep this convention throughout this paper. We also count time in units of a period τ, i.e., $t = n\tau$.

Table 2. Cumulant-generating function $\tilde{\phi}_\mu$ of the work (periodic work source) per period in the same limits as in table 1. In the low-amplitude limit, we have defined $\omega^0 = 2\omega(h_0)\cosh h_0$.

$\Gamma\tau \ll 1$	$\frac{\tau}{2}\sqrt{\left[\alpha\omega^- + (1-\alpha)\omega^+\right]^2 + 4\alpha(1-\alpha)\omega^+\omega^- \frac{\cosh 2a(2\mu+1) - \cosh 2a}{\cosh 2a + \cosh 2h_0}} - \frac{\tau}{2}\alpha\omega^- - \frac{\tau}{2}(1-\alpha)\omega^+$	
$\Gamma\tau \gg 1$	$\left(e^{-(1-\alpha)\tau\omega^+} + e^{-\alpha\tau\omega^-}\right)\frac{\cosh 2a - \cosh 2a(2\mu+1)}{\cosh 2h_0 + \cosh 2a(2\mu+1)} + \ln\frac{\cosh 2h_0 + \cosh 2a(2\mu+1)}{\cosh 2a + \cosh 2h_0}$	
$a \ll 1$	$\mu(1+\mu)\langle\tilde{w}\rangle = 8a^2\mu(1+\mu)\frac{(1-e^{-(1-\alpha)\tau\omega^0})(1-e^{-\alpha\tau\omega^0})}{(1-e^{-\tau\omega^0})(1+\cosh 2h_0)}$	
	Arrhenius rates	Fermi and Bose rates
$h_0 \gg a$	$\frac{\cosh 2a(2\mu+1) - \cosh 2a}{\cosh 2h_0}$	$\frac{\left(1-e^{-(1-\alpha)\Gamma\tau}\right)\left(1-e^{-\alpha\Gamma\tau}\right)}{1-e^{-\Gamma\tau}}\frac{\cosh 2a(2\mu+1) - \cosh 2a}{\cosh 2h_0}$

For large n, in the sense of large deviation the work $w = W/n$ becomes, minus the heat, $q = Q/n$ because the system internal energy is bounded and thus its change between 0 and t is not extensive in time. Similarly, the entropy production per period becomes equal to the heat flow $-q$ because the system entropy change is not extensive in time. This implies that the work fluctuations fully characterize the large deviation properties of the entropy production and of the heat fluctuations.

For the periodic work source, the first derivative of $\tilde{\phi}_\mu$ at $\mu = 0$ is the average work per period

$$\langle\tilde{w}\rangle = \frac{4a\sinh(2a)\left(1-e^{-(1-\alpha)\tau\omega^+}\right)\left(1-e^{-\alpha\tau\omega^-}\right)}{\left[\cosh(2h_0)+\cosh(2a)\right]\left(1-e^{-\alpha\tau\omega^- -(1-\alpha)\tau\omega^+}\right)}. \tag{20}$$

For the stochastic work source, we show in appendix B that the work received by the two-level system during time τ is

$$\langle w\rangle = 4a\tau k^+k^-\left(\omega_+^-\omega_-^+ - \omega_+^+\omega_-^-\right)/Z \tag{21}$$

with Z a normalization constant in the state probability. Note that, in (20) and (21), the average work is positive as required by the second law. For both types of work source, the average work vanishes for $a \to 0$ (no work source) and for $\tau \to 0$ or $\alpha \to 0$ or 1 (the two-level system has no time to switch state between two work source transitions). For $\Gamma\tau \to \infty$, the two-level system typically relaxes to equilibrium before the next transition in the work source happens and for both types of work source:

$$\langle\tilde{w}\rangle = \langle w\rangle = \sum_{\sigma,\varepsilon}\frac{2a\varepsilon\sigma\omega_{-\sigma}^{\varepsilon}}{\omega_+^{\varepsilon}+\omega_-^{\varepsilon}} = \frac{4a\sinh(2a)}{\cosh(2h_0)+\cosh(2a)} \tag{22}$$

$2a\varepsilon\sigma$ is the work given to the two-level system when the joint system state is (σ, ε) and a transition $\varepsilon \to -\varepsilon$ of the work source occurs. We notice that this average work is independent of any dynamical parameters; see figure 4. This limit is not reversible because the driving contains discontinuities. Indeed, the average work is different from the free energy difference. In figure 4, we present the average work for $\tau = 1$ and $\tau = 100$ as a function of the amplitude of the jump a. We notice that the dynamics with Fermi rates always produce less work than the other rates because the Fermi rates are the smallest. Indeed, an energy exchange between the work source and the two-level system requires a change of system state between two work

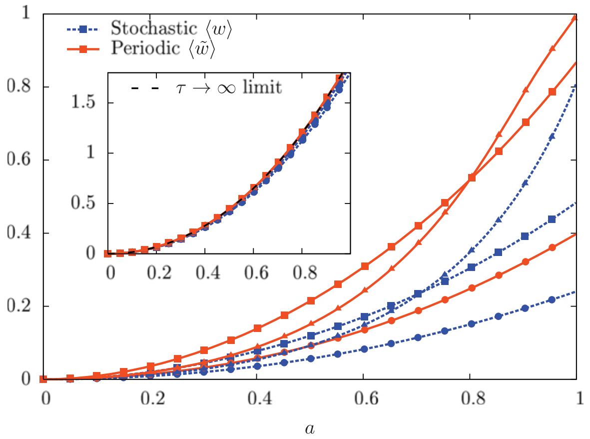

Figure 4. Mean work value versus amplitude parameter a for the periodic (orange solid lines) or for the stochastic (blue dashed lines) work sources. Symbols encode the types of rates: Arrhenius (squares), Bose (triangles), and Fermi (circle). Other parameters are $\alpha = 0.3$, $h_0 = 1$, and $\tau = 1$ (inset: $\tau = 100$) with rates $k^{\pm}$ set from (19).

source transitions. Therefore, the smaller Fermi rates lead to a smaller number of system transitions and thus to a smaller average work contribution. We also see that the average work contribution is always higher for the periodic protocol than for the stochastic one. This is due to the fact that for a Poisson process, the most likely time intervals between work source transitions are the small ones during which the system has less time to change its state and absorb work, on average.

3.4. Fluctuations and statistics of work

When comparing the work probability distributions corresponding to the two types of work source as in figure 5 the most striking feature is the difference in the range of the fluctuations. The stochastically driven system always has a small but finite probability of exchanging a very large amount of work with the work source, and as a result, the support of the large deviation function $I(w) = \max_{\mu}\{\mu w - \tau\phi_{\mu}\}$ is infinite. However, the periodically driven system displays finite support because the work exchanged during a work source transition is $\pm 2a$, thus leading to three possible values for the work per period, $\pm 4a$ and 0, and to the inequality $|w| \leqslant 4a$. The large deviation function of work $\tilde{I}(w) = \max_{\mu}\{\mu w - \tilde{\phi}_{\mu}\}$ is therefore infinite (vanishing work probability) outside that range. At the level of the cumulant-generating function, this implies that $\tilde{\phi}_{\mu}$, the Legendre transform of $\tilde{I}(w)$, has an absolute slope that cannot exceed $4a$.

Both types of work probability distributions satisfy the fluctuation theorem. In fact, any stochastic energy source made of two states is reversible and will thus qualify as a work source. The fluctuation theorem for the periodic work source should in principle relate the work statistics for a forward periodic driving with those of the time-reversed periodic driving. However, here also a periodic driving that jumps once back and forth between two states over a period is invariant under time reversal (up to a time shift that plays no role in the long time limit [46]). We explicitly prove the work fluctuation theorem for the stochastic as well as for the periodic work source by showing that the work cumulant generating function satisfies the relation $\phi_{\mu} = \phi_{-1-\mu}$ [28]. This follows directly from (17) and indirectly from (12) by observing that μ appears in the characteristic polynomial only through the function $\cosh[2a(1 + 2\mu)]$.

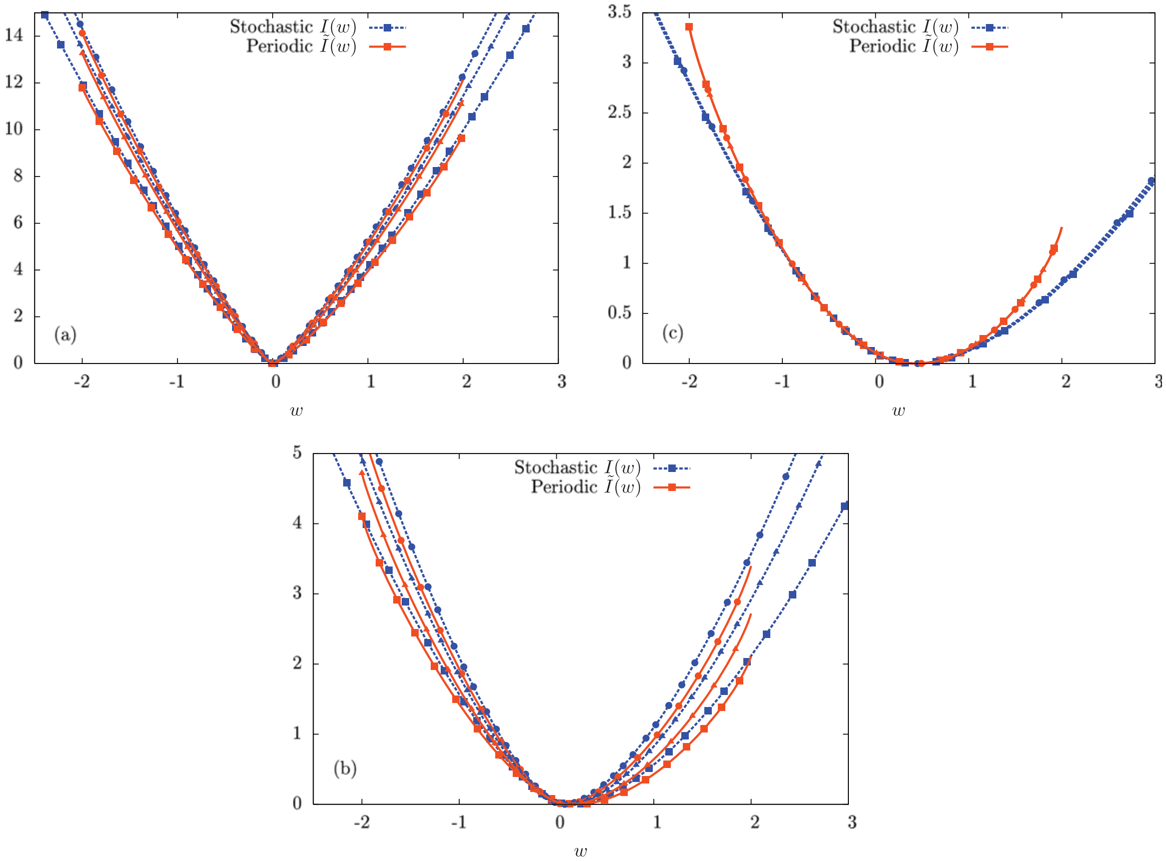

Figure 5. Large deviation functions for the work w produced by the periodic (orange solid lines) and the stochastic (blue dashed lines) work source for (a) fast $\tau = 0.01$, (b) intermediate $\tau = 1$, and (c) slow $\tau = 100$ switching rates compared with the system time scale. Symbols encode different types of rates: Arrhenius (squares), Bose (triangles), and Fermi (circle). Other parameters are $h_0 = 1$, $a = 0.5$, $\alpha = 0.3$.

This function is invariant in the exchange of μ to $-1 - \mu$, and this also holds true for the roots of the polynomial. This symmetry is also explicitly seen in figure 6.

We now turn to the quantitative analysis of the large deviation function for work $I(w)$ obtained by the numerical Legendre transform of the cumulant-generating function corresponding to the two types of work sources.

Figure 5(a) and tables 1 and 2 show that both types of work sources give rise to identical work fluctuations in the limit of fast modulation ($\Gamma\tau \ll 1$). The choice of the rates nevertheless influences the shape of the work distribution, and the work variance decreases as the modulation speeds up. In the opposite limit of slow modulation ($\Gamma\tau \gg 1$), the large deviation function becomes rate independent as seen in figure 5(c). However, the type of work source (stochastic or periodic) still influences the work fluctuations, in particular the large fluctuations away from the common minimum (same expectation value).

In the limit of low field amplitudes (i.e., close to equilibrium), the cumulant-generating function becomes quadratic in μ: $\tau\phi_\mu = \mu(1 + \mu)\langle w\rangle$ and $\tilde{\phi}_\mu = \mu(1 + \mu)\langle\tilde{w}\rangle$, where the mean work values are obtained from (20) and (21) by second-order expansion in a. The average work

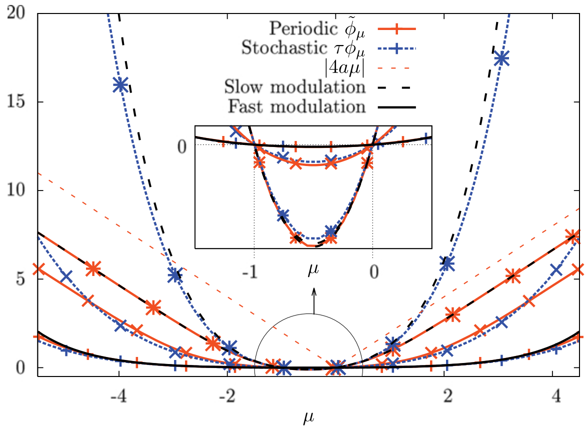

Figure 6. Cumulant-generating function of the work statistics as a function of μ for the periodic (orange solid) and stochastic (blue dashed) work source for fast $\tau = 0.1$ (plus symbol), intermediate $\tau = 1$ (crosses), and slow $\tau = 100$ (stars) switching rates compared with the system time scale. Fermi rates are considered, and $h_0 = 1$, $\alpha = 0.3$. The black dashed line corresponds to $\Gamma\tau \to \infty$ whereas the black dotted line corresponds to $\Gamma\tau \to 0$.

differs for stochastic and periodic work sources and for the different rates. The same observation remains true at the level of work fluctuations.

4. Conclusion

We identified the condition under which a system subjected to a stochastic driving can be seen thermodynamically as a system subjected to a work source; namely, the stochastic driving protocol has to be statistically reversible or, in other words, its entropy production has to vanish. Under this assumption, the statistics of dissipated work and entropy production in the system are identical and a work fluctuation theorem is satisfied. We then compared the exact work statistics of a two-level system driven by a stochastic two-state work source with those of a periodic two-state work source, the two sources spending the same fraction of time in their upper and lower states. We found that the work fluctuations are unbounded in the former case and bounded in the latter. For fast as well as for slow switching rates in the work source, the work fluctuations are quite similar. For low switching amplitudes, the system remains close to equilibrium, where work fluctuations are Gaussian. Finally, for a given amplitude of the jumps in the work source, important values for the work average and variance are obtained when the time scales of the system and of the work source are comparable.

Acknowledgments

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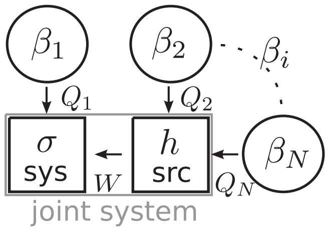

Figure A1. The joint system (σ, h) in contact with several thermal baths. In the limit where h is independent of σ, the evolution of h can be considered as a time-dependent driving for the system σ.

Appendix A. Limit of the energy source

In this appendix, we start from autonomous (no-time-dependent-driving) Markovian dynamics of a bipartite system with rates satisfying local detailed balance and thus described by standard stochastic thermodynamics. We discuss the procedure required to make contact with the description in section 2 where one part of the system becomes a Markovian dynamics that is independent of the second part, while the latter sees the former as an independent stochastic driving (i.e., an energy source).

The energy of the joint system can be expressed as the sum of the bare energy of system σ and h plus an interaction energy, namely $E_{\text{joint}}(\sigma, h) = E^0_{\text{sys}}(\sigma) + E^0_{\text{src}}(h) + E_{\text{int}}(\sigma, h)$. System σ is assumed to be in contact with a single thermal bath at temperature β_1^{-1}, whereas system h is in contact with $N - 1$ baths at temperature β_ν^{-1}, with $\nu = 2, \cdots, N$; see figure A1.

The rates describing transitions between states σ at fixed h satisfy local detailed balance

$$\ln \frac{\omega^{(1)}_{\sigma', \sigma}(h)}{\omega^{(1)}_{\sigma, \sigma'}(h)} = -\beta_1 \Big[E_{\text{int}}(\sigma', h) - E_{\text{int}}(\sigma, h) + E^0_{\text{sys}}(\sigma') - E^0_{\text{sys}}(\sigma) \Big] \tag{A.1}$$

as do the rates describing transitions between states h at fixed σ

$$\ln \frac{\omega^{(\nu)}_{h', h}(\sigma)}{\omega^{(\nu)}_{h, h'}(\sigma)} = -\beta_\nu \Big[E_{\text{int}}(\sigma, h') - E_{\text{int}}(\sigma, h) + E^0_{\text{src}}(h') - E^0_{\text{src}}(h) \Big]. \tag{A.2}$$

We now assume that the energy scales involved during transitions in h are much larger than any other energy scale involved in the σ dynamics. As a result, the rates $\omega^{(\nu)}_{h', h}(\sigma)$ can be assumed to be independent in σ, and equation (A.2) becomes

$$\ln \frac{\omega^{(\nu)}_{h', h}}{\omega^{(\nu)}_{h, h'}} = \beta_\nu \Big[E^0_{\text{src}}(h') - E^0_{\text{src}}(h) \Big]. \tag{A.3}$$

In this limit, the system h follows a dynamics that is independent of the dynamics of σ. However, the converse is not true, and the dynamics of σ still depends on h. The energies of σ read $E_{\text{sys}}(\sigma, h) = E^0_{\text{sys}}(\sigma) + E_{\text{int}}(\sigma, h)$, and the energy balance for σ reads

$$\left\langle \dot{E}_{\text{sys}} \right\rangle = \left\langle \dot{Q}_1 \right\rangle + \left\langle \dot{E}^{\text{h}}_{\text{int}} \right\rangle \tag{A.4}$$

where the heat flow entering system σ from bath $\nu = 1$ is given by

$$\left\langle \dot{Q}_1 \right\rangle = \sum_{\sigma,\sigma',h} \omega^{(1)}_{\sigma',\sigma}(h) p(\sigma, h) \left[E_{\text{int}}(\sigma', h) - E_{\text{int}}(\sigma, h) + E^0_{\text{sys}}(\sigma') - E^0_{\text{sys}}(\sigma) \right] \tag{A.5}$$

whereas the energy received by system σ from system h reads

$$\left\langle \dot{E}^{\text{h}}_{\text{int}} \right\rangle = \sum_{\sigma,h',h,\nu} \omega^{(\nu)}_{h',h}(\sigma) p(\sigma, h) \left[E_{\text{int}}(\sigma, h') - E_{\text{int}}(\sigma, h) \right]. \tag{A.6}$$

We emphasize that the level of description in section 2, i.e., of system σ driven by a stochastic driving force h, disregards the true dissipation that is required to fuel h and that would be expressed in terms of the detailed rates (A.3). It captures only the coarse-grained dissipation expressed in terms of the global rates (i.e., rates summed over all the reservoirs $\nu = 2, \cdots, N$) that describe transitions between states h.

Appendix B. Steady state probability with stochastic driving

We derive the average work done by the stochastic work source of figure 1 in our two-level system. To do so, we calculate the steady probability current for the transitions leading to work exchanges. The steady state probability p^ε_σ of finding the joint system in a state (σ, ε) can be obtained from the spanning tree formula [50] and reads

$$p^+_- = \left(k^- k^- \omega^+_+ + k^+ k^- \omega^-_+ + \omega^+_+ k^- \omega^-_+ + \omega^-_- k^- \omega^+_+ \right) / Z$$

$$p^-_- = \left(k^+ \omega^-_+ k^+ + \omega^+_+ k^+ \omega^-_+ + \omega^+_- k^+ \omega^-_+ + k^- \omega^+_+ k^+ \right) / Z$$

$$p^+_+ = \left(\omega^+_- k^- \omega^-_- + \omega^+_- k^- k^- + \omega^-_+ k^- \omega^+_- + k^+ \omega^-_- k^- \right) / Z$$

$$p^-_+ = \left(\omega^+_- k^+ \omega^-_- + k^+ k^+ \omega^-_- + k^- \omega^+_- k^+ + \omega^+_+ k^+ \omega^-_- \right) / Z$$

Z is a normalization constant such that $p^+_+ + p^-_- + p^+_- + p^-_+ = 1$. The average work accumulated during τ is then

$$\langle w \rangle = 2a\tau \left(p^+_+ k^+ - p^-_+ k^- \right) + 2a\tau \left(p^-_- k^- - p^+_- k^+ \right) \tag{B.1}$$

Appendix C. Stochastically driven colloidal particle: the Gaussian driving case

We consider an overdamped colloidal particle in contact with a bath at temperature T and evolving in a one-dimensional harmonic trap whose position follows an Ornstein–Uhlenbeck process. Because this process plays the role of a reversible work source acting on the colloidal particle, our results from section 2 indicate that a work fluctuation theorem should hold. However, this model has been studied experimentally in [44] and theoretically in [45, 51], and the authors found that the work fluctuation theorem is satisfied only in a certain range of parameters. The reason for these violations is that work is defined in these references as the time integral of the velocity times the stochastic force, and this work definition differs from the Jarzynski work by a boundary term. We now use the results of [45] (trying to keep their notation) to show that, as we predicted, the Crooks work fluctuation theorem is always valid when the Jarzynski work is considered.

We denote by $x = x(t)$ the position of the particle (i.e., the system) and by $y = y(t)$ the position of the trap (i.e., the work source). The stochastic differential equations of motion are

$$\dot{x} = -\frac{x - y}{\tau_\gamma} + \xi \tag{C.1}$$

$$\dot{y} = -\frac{y}{\tau_0} + \zeta \tag{C.2}$$

where $\xi = \xi(t)$ and $\zeta = \zeta(t)$ are two uncorrelated Gaussian white noises averaging to zero and with correlation $\langle \xi(t)\xi(s) \rangle = 2D\delta(t - s)$ and $\langle \zeta(t)\zeta(s) \rangle = 2A\delta(t - s)$. Two time scales are introduced in (C.1) and (C.2): the relaxation time in the harmonic trap $\tau_\gamma = \gamma/k$ with γ the friction coefficient and k the stiffness of the trap, and the relaxation time of the y correlation, i.e., $\langle y(t)y(s) \rangle = A\tau_0 \exp(-|t - s|/\tau_0)$. We note that the Einstein relation $D = T/\gamma$ (which plays the role of the local detailed balance condition in equation (1) when considering continuous models) is verified for the motion of x. The motion of the trap reaches an equilibrium (Gaussian) state that can be characterized by an effective temperature proportional to $A\tau_0$. We define $\delta = \tau_0/\tau_\gamma$, the ratio of the two time scales in the model; and $\theta = A/D$, the ratio of the diffusion coefficients.

The entropy production in the time interval $[0, t]$ contains three contributions. The first is the variation of the system entropy

$$\Delta S[x|y] = -\ln \frac{p_{\rm st}\big(x(t)\big|y(t)\big)}{p_{\rm st}\big(x(0)\big|y(0)\big)} \tag{C.3}$$

where $p_{\rm st}(x|y)$ is the stationary probability of x given y. The second is the entropy production in the bath $-Q[x|y]/T = W[x|y]/T - \Delta E[x, y]/T$, which can be expressed as the difference between the Jarzynski work divided by T

$$W[x|y]/T = \frac{k}{2T}\int_0^t \mathrm{d}t'\dot{y}(t') \circ \frac{\mathrm{d}}{\mathrm{d}y}[x(t') - y(t')]^2 \tag{C.4}$$

where $\circ$ denotes the Stratonovich product, and the variation of system internal energy divided by T

$$\Delta E[x, y]/T = \frac{1}{2D\tau_\gamma}\left\{[x(t) - y(t)]^2 - [x(0) - y(0)]^2\right\}. \tag{C.5}$$

We remark here that the Jarzynski work could be infinite for some rare events because the amplitude of change of the position x is in principle infinite. The third part in the entropy production is the work source entropy production

$$\Delta_{\rm i} S_{\rm sd}[y] = -\frac{1}{2A\tau_0}\Big[y(t)^2 - y(0)^2\Big] - \ln \frac{p_{\rm st}(y(t))}{p_{\rm st}(y(0))} = 0 \tag{C.6}$$

which vanishes because the driving is a Gaussian process always relaxing to an effective equilibrium. Introducing the final state vector $U = (x(t), y(t))^T$, the initial state vector $U_0 = (x(0), y(0))^T$ and the stationary probability distribution

$$p_{\rm st}(U) = \frac{1}{2\pi\sqrt{\det \boldsymbol{H}_1}} \exp\left[\frac{-1}{2} U^T \boldsymbol{H}_1^{-1} U\right] \tag{C.7}$$

with

$$\boldsymbol{H}_1 = \frac{D\tau_0}{\delta(1+\delta)} \begin{pmatrix} 1+\delta+\theta\delta^2 & \theta\delta^2 \\ \theta\delta^2 & \theta\delta+\theta\delta^2 \end{pmatrix} \tag{C.8}$$

the entropy production becomes

$$\Delta_{\rm i} S[x, y] = \frac{k}{T} \int_0^t {\rm d}t' y(t') \circ \dot{x}(t') - \frac{1}{2} U^T \left(-\boldsymbol{H}_1^{-1} + \boldsymbol{R}\right) U - \frac{1}{2} U_0^T \left(\boldsymbol{H}_1^{-1} - \boldsymbol{R}\right) U_0. \tag{C.9}$$

We defined the matrix

$$\boldsymbol{R} = \frac{1}{D\theta\tau_0} \begin{pmatrix} \delta\theta & 0 \\ 0 & 1 \end{pmatrix}. \tag{C.10}$$

In [45], the generating function of the first term on the right-hand side of (C.9) was obtained for a given initial and final state. The entropy production modifies this quantity by a boundary term that depends only on U_0 and U. Using the result of [45], we obtain the following expression for the generating function of the entropy production (which is equal to the dissipated work) at large time t:

$$\left\langle {\rm e}^{\mu \Delta_{\rm i} S} \right\rangle = g_\mu {\rm e}^{t\phi_\mu} \tag{C.11}$$

with

$$\phi_\mu = \frac{1+\delta}{2\tau_0} [1 - \nu(\mu)] \tag{C.12}$$

$$\nu_\mu = \sqrt{1 - \frac{4\theta\delta^2 \mu(1+\mu)}{(1+\delta)^2}}. \tag{C.13}$$

The exponential pre-factor g_μ is important if the generating function has non-analyticities in the region where the saddle approximation is performed to find the large deviation function. Using again the results of [45], we find the following exponential pre-factor

$$g_\mu = \frac{4\nu_\mu}{\left(1+\nu_\mu\right)^2}. \tag{C.14}$$

The work fluctuation theorem follows from (C.12) because $g_{-1-\mu} = g_\mu$ and $\phi_{-1-\mu} = \phi_\mu$. As for the two-level system, this theorem is satisfied because the stochastic driving is reversible; see (C.6).

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