UPC theory

Samuel Wallon

Laboratoire de Physique des 2 Infinis Irène Joliot-Curie
IJCLab
CNRS / Université Paris Saclay
Orsay

UPC 2025

9th June 2025

UPC: using hadron and heavy ions as a source of photons

Basic ideas

- strong interaction has a short interaction range $\sim$ 1 fm
- electromagnetic interaction as an infinite interaction range
- when two hadrons or ions collide, if they do not overlap, only electromagnetic interaction remains.
  We are thus looking for large impact parameters.
  These processes are named Ultra Peripheral Collisions.
- if Z is very large (eg: lead $Z = 82$), an heavy ion can be a very intense and high energetic source of photon

Key message:
a hadron collider (LHC, RHIC) can be used as a single or double photon collider

UPC: using a nucleon or a heavy ion as a source of real photons

Key idea: at very high energy, the nucleon or ion is a source of almost real photon.

- in the target frame (e.g.: in the case of Ap scattering, most probably, p is the target, and A is the source of photons), the projectile, source of photon, is highly boosted
- it emits photons which:
    - propagate almost in the direction of the emitter
    - are almost real, looking like a free plane wave
- this can be shown using classical methods (see part 1), from the structure of the electromagnetic field radiated by a fast moving source.

  E. Fermi 1924; E. J. Williams and K. F. von Weizsäcker 1934

  M. and G. Nordheims et al. 1937

  This is most easily obtained in the impact parameter space.

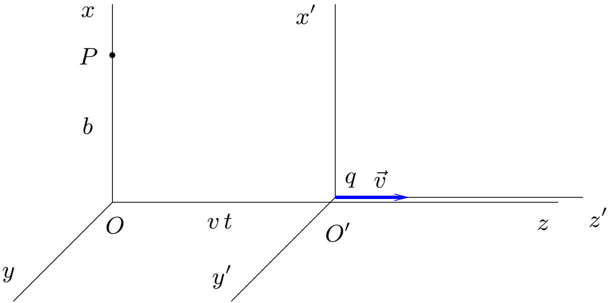
- this can be shown based on a quantum field treatment, focusing on the dominance of the almost collinear photon field, due to the collinear pole of the photon propagator when the nucleon or ion source has a negligible mass wrt its energy (see part 2).

  Dalitz, Yennie 1957; Curtis 1956, D. and P. Kesslers 1956; I.Ya. Pomeranchuk and I.M. Shmushkevich 1961; V. Gribov 1961

  This is most easily obtained in the momentum space.

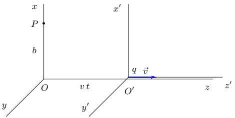
Field of a highly boosted charge

- Consider a charge q moving at velocity $\vec{v}$ in frame K
  K = observer frame, located at P
- Let K' be rest frame of the charge.
  O : origin of K
  O' : origin of K'
  Impact parameter b, i.e. $\overrightarrow{OP} = \vec{b} = b\,\vec{u}_x$.



Field of a highly boosted field

Frame K': static charge



Electromagnetic fields in K', at P

$$\vec{B}' = 0 \,,$$

$$\vec{E}' = \frac{q}{4\pi} \frac{\vec{b} - \vec{v}t'}{\|\vec{b} - \vec{v}t'\|^3} = \frac{q}{4\pi} \frac{\vec{b} - \vec{v}t'}{r'^3} \quad i.e. \quad \begin{cases} E'_x = \dfrac{qb}{4\pi r'^3} \,, \\ E'_y = 0 \,, \\ E'_z = -\dfrac{qvt'}{4\pi r'^3} \,, \end{cases} \quad \text{with} \quad r' = \sqrt{b^2 + (vt')^2}.$$

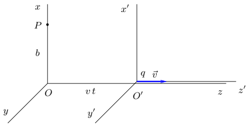
Heaviside-Lorentz unit, with $c = 1$

Since $t' = \gamma(t - v\,z) = \gamma t$ because $z = 0$ for the observer P, $\vec{E}'$ reads

$$\begin{cases} E'_x = \dfrac{q}{4\pi} \dfrac{b}{(b^2 + v^2\gamma^2 t^2)^{3/2}} \,, \\ E'_y = 0 \,, \\ E'_z = -\dfrac{q}{4\pi} \dfrac{v\gamma t}{(b^2 + v^2\gamma^2 t^2)^{3/2}} \end{cases}$$

Field of a highly boosted charge
In the boosted frame K of the observer

Electromagnetic fields in K, at P



Boosted fields: $K' \longrightarrow K$

$$\begin{aligned}
\vec{E} &= (\vec{E}'\cdot\vec{n})\vec{n} + \gamma\left[\vec{E}' - (\vec{E}'\cdot\vec{n})\vec{n}\right] - \gamma\,\vec{v}\wedge\vec{B}'\,,\\
\vec{B} &= (\vec{B}'\cdot\vec{n})\vec{n} + \gamma\left[\vec{B}' - (\vec{B}'\cdot\vec{n})\vec{n}\right] + \gamma\,\vec{v}\wedge\vec{E}'\,.
\end{aligned}$$

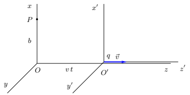
Thus, since $\vec{v} = v\vec{u}_z = \beta\vec{u}_z$,

$$\begin{aligned}
\vec{E} &= E'_z\vec{u}_z + \gamma\,E'_x\vec{u}_x\,,\\
\vec{B} &= \gamma\vec{\beta}\wedge\vec{E}' = \gamma v\vec{u}_1\wedge\vec{u}_x E'_x = \gamma v E'_x\vec{u}_y\,,
\end{aligned}$$

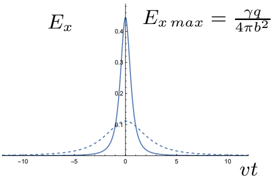
and one gets

$$\left\{\begin{aligned}
E'_x &= \frac{q}{4\pi}\frac{b}{(b^2+v^2\gamma^2t^2)^{3/2}}\,,\\
E'_y &= 0\,,\\
E'_z &= -\frac{q}{4\pi}\frac{v\gamma t}{(b^2+v^2\gamma^2t^2)^{3/2}}
\end{aligned}\right.
\longrightarrow
\left\{\begin{aligned}
E_x &= \gamma E'_x = \frac{q}{4\pi}\frac{\gamma b}{(b^2+v^2\gamma^2t^2)^{3/2}}\,,\\
B_y &= \gamma\beta E'_x = \beta E_x\,,\\
E_z &= E'_z = -\frac{q}{4\pi}\frac{v\gamma t}{(b^2+v^2\gamma^2t^2)^{3/2}}\,.
\end{aligned}\right.$$

Field of a highly boosted charge
In the boosted frame K of the observer

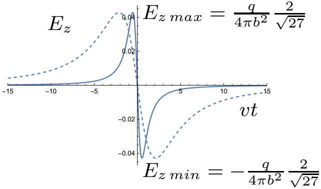


Space-time structure of the fields



Continuous: $\gamma = 4$, dashed: $\gamma = 1$.

arbitrary vertical scale $q = 4\pi$



Continuous: $\gamma = 4$, dashed: $\gamma = 1$.

arbitrary vertical scale $q = 4\pi$

Sizable for

$$b^2 \gtrsim \gamma^2 v^2 t^2 \quad \text{i.e.} \quad |t| \lesssim \frac{b}{\gamma v} = \Delta t\,.$$

$E_{x\,max} = \frac{\gamma q}{4\pi b^2}$ gets amplified by a factor γ wrt non-relativistic case.

Pulse significant over a time of the order of Δt, which vanishes on average.

For t averaged on scales larger than Δt, $(\vec{E}, \vec{B})$ looks like a plane wave: $E_x \sim B_y$ $(\beta \to 1)$ with $(\vec{E}, \vec{B}, \vec{u}_z)$ = direct basis.

Field of a highly boosted charge

Effect on a test charge

Let the moving charge be a nucleus, i.e. $q = Ze$, scattering an atomic e^- (charge $-e$) at P, with $b > R_{atom}$.
The transfered momentum is thus

$$\Delta p \sim -eE_x \Delta t \sim -ze^2 \frac{b}{\gamma v} \frac{1}{4\pi} \frac{\gamma}{b^2} \sim -\frac{ze^2}{4\pi b v} \quad \text{independent of } \gamma\,.$$

It increases when b gets smaller.

An exact calculation would give:

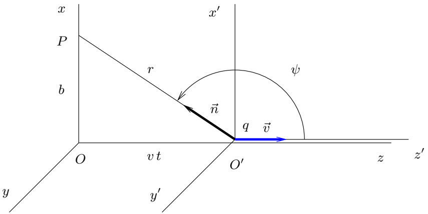
$$\begin{aligned}\int_{-\infty}^{+\infty} -eE_x(t)\, dt &= -\frac{ze^2}{4\pi v b} \int_{-\infty}^{+\infty} \frac{\gamma v t / b}{[1 + (\gamma v t / b)^2]^{3/2}} = -\frac{ze^2}{4\pi v b} \int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^{3/2}} \\ &= -\frac{ze^2}{4\pi v b} \left[\frac{x}{\sqrt{1+x^2}} \right]_{-\infty}^{+\infty} = -\frac{ze^2}{2\pi v b}\,.\end{aligned}$$

Field of a highly boosted charge
Space structure

At time t, the field $\vec{E}$ points from the position O' of the charge q at time t towards the observer P (for $q > 0$). Indeed:

$$\begin{cases} E_x = \dfrac{q}{4\pi} \dfrac{\gamma b}{(b^2 + v^2\gamma^2 t^2)^{3/2}} \,, \\ E_z = -\dfrac{q}{4\pi} \dfrac{v\gamma t}{(b^2 + v^2\gamma^2 t^2)^{3/2}} \,, \end{cases} \quad \text{and thus} \quad E_z / E_x = -\frac{vt}{b} \,,$$

i.e. $\vec{E} = \mathrm{sign}(q) \|\vec{E}\| \, \vec{n}$ with $\vec{n} = \dfrac{b\,\vec{u}_x - vt\,\vec{u}_z}{\sqrt{b^2 + v^2 t^2}}$ = unit vector pointing from O' to P.



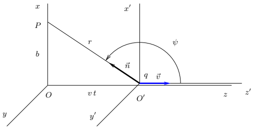
Actually **[back-up]**, $\vec{E} = \dfrac{q\vec{r}}{4\pi r^3 \gamma^2 (1 - \beta^2 \sin^2 \psi)^{3/2}}$, with $\vec{r} = r\,\vec{n}$ and $\psi = (\vec{v}, \vec{n})$.

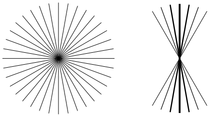
Here r = radial distance between q at time t and P at the same time t.

Field of a highly boosted charge

Space structure

Angular distribution





$\gamma = 1$ $\qquad$ $\gamma \gg 1$

thickness = magnitude of $\|\vec{E}\|$

$$\vec{E} = \frac{q\vec{r}}{4\pi r^3 \gamma^2 (1-\beta^2 \sin^2\psi)^{3/2}}$$: the field is radial, but anisotropic.

It is isotropic only for $\beta = 0$ (static Coulomb case):

$$\|\vec{E}\|_{\text{Coul.}} = \frac{q}{4\pi r^2} .$$

Perpendicularly to the movement axis, i.e. $\psi = \pm\pi/2$ one gets

$$\|\vec{E}\|_{\perp} = \frac{q}{4\pi r^2}\frac{1}{\gamma^2}\left(\frac{1}{\sqrt{1-\beta^2}}\right)^3 = \frac{\gamma q}{4\pi r^2} = \gamma \|\vec{E}\|_{\text{Coul.}} .$$

Along the movement axis, i.e. $\psi = 0$ or π, one gets

$$\|\vec{E}\|_{\parallel} = \frac{q}{4\pi r^2}\frac{1}{\gamma^2} = \frac{\|\vec{E}\|_{\text{Coul.}}}{\gamma^2} .$$

Energy distribution

Introducing the Poynting vector

$$\vec{S}(t,\vec{r}) = \vec{E} \times \vec{B}$$

and using the Fourier transform:

$$\vec{E}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \vec{E}(t)\, e^{i\omega t} d\omega\,,$$

the total energy radiated per solid angle reads (sphere of radius R)

$$\frac{d^2 W}{d^2\Omega} = \int_{-\infty}^{+\infty} dt\, \vec{S}(t,\vec{r}) \cdot d^2\vec{\mathcal{S}} = 2\pi R^2 \int_{-\infty}^{+\infty} d\omega |\vec{E}_(\omega)|^2\,.$$

The frequency spectrum i.e. energy per unit area per unit frequency interval reads ($\omega > 0$ thus a factor 2)

$$\frac{dI}{d\omega} = 4\pi |\vec{E}(\omega)|^2\,,$$

and, separately for transverse and longitudinal components,

$$\begin{aligned} \frac{dI_x}{d\omega} &= 4\pi |E_x(\omega)|^2\,, \\ \frac{dI_z}{d\omega} &= 4\pi |E_z(\omega)|^2\,. \end{aligned}$$

Energy distribution

The field components

$$\begin{cases} E_x = \dfrac{q}{4\pi}\dfrac{\gamma b}{(b^2+v^2\gamma^2t^2)^{3/2}}\,, \\ E_z = -\dfrac{q}{4\pi}\dfrac{v\gamma t}{(b^2+v^2\gamma^2t^2)^{3/2}}\,, \end{cases}$$

read, in ω space (with $x = \gamma v t/b$)

$$\begin{aligned} E_x &= \frac{qb\gamma}{4\pi 2\pi}\int_{-\infty}^{+\infty}\frac{e^{i\omega t}}{(b^2+v^2\gamma^2t^2)^{3/2}}dt = \frac{q}{4\pi 2\pi b v}\int_{-\infty}^{+\infty}\frac{e^{i\omega\frac{bx}{\gamma v}}}{(1+x^2)^{3/2}}dx \\ &= \frac{q}{4\pi^2 b v}\frac{\omega b}{\gamma v}K_1\left(\frac{\omega b}{\gamma v}\right)\,, \\ E_z &= \frac{-q\gamma}{4\pi 2\pi}\int_{-\infty}^{+\infty}\frac{e^{i\omega t}vt}{(b^2+v^2\gamma^2t^2)^{3/2}}dt = \frac{-q}{4\pi 2\pi b v\gamma}\int_{-\infty}^{+\infty}\frac{x\,e^{i\omega\frac{bx}{\gamma v}}}{(1+x^2)^{3/2}}dx \\ &= -i\frac{q}{4\pi^2\gamma b v}\frac{\omega b}{\gamma v}K_0\left(\frac{\omega b}{\gamma v}\right)\,. \end{aligned}$$

Energy distribution and Number of virtual quanta

We thus get

$$\begin{aligned}\frac{dI_x}{d\omega}(\omega,b) &= 4\pi|E_x(\omega)|^2 = \frac{q^2}{4\pi}\frac{1}{\pi^2 v^2 b^2}\left(\frac{\omega b}{\gamma v}\right)^2 K_1^2\left(\frac{\omega b}{\gamma v}\right),\\ \frac{dI_z}{d\omega}(\omega,b) &= 4\pi|E_z(\omega)|^2 = \frac{1}{\gamma^2}\frac{q^2}{4\pi}\frac{1}{\pi^2 v^2 b^2}\left(\frac{\omega b}{\gamma v}\right)^2 K_0^2\left(\frac{\omega b}{\gamma v}\right).\end{aligned}$$

Density of modes $N(\hbar\omega)$:

From

$$\frac{dI}{d\omega}d\omega = \hbar\omega N(\hbar\omega)\, d(\hbar\omega)\,,$$

i.e.

$$N(\hbar\omega) = \frac{1}{\hbar^2\omega}\frac{dI}{d\omega}$$

one gets (still with $\hbar = 1, c = 1$)

$$N(\hbar\omega) = \frac{q^2}{4\pi}\frac{1}{\omega}\frac{1}{\pi^2 v^2 b^2}\left(\frac{\omega b}{\gamma v}\right)^2\left[K_1^2\left(\frac{\omega b}{\gamma v}\right) + \frac{1}{\gamma^2}K_0^2\left(\frac{\omega b}{\gamma v}\right)\right].$$

Integration over the impact parameter

Integration over the impact parameter:

$$\frac{dI_x}{d\omega} = 2\pi \int_{b_{min}}^{\infty} \left[\frac{dI_x}{d\omega}(\omega, b) + \frac{dI_z}{d\omega}(\omega, b)\right] d\, db\,.$$

Final result **[back-up]**:

$$\frac{dI}{d\omega} = \frac{dI_x}{d\omega} + \frac{dI_z}{d\omega} = \frac{2}{\pi}\frac{q^2}{4\pi}\frac{1}{v^2}\left[xK_0(x)\,K_1(x) - \frac{v^2}{2}x^2\left(K_1^2(x) - K_0^2(x)\right)\right],$$

with $x = \frac{\omega b_{min}}{\gamma v}$.
Note: neglecting the longitudinal photons means $v^2/2 \to 1/2$.

Energy distribution
Shape

Low frequency region $\omega \ll \gamma v/b_{min}$: dominates

$$K_0(x) \underset{x\to 0}{\sim} -\ln(x) - \gamma_E + \ln(2) + O\left(x^2\right) \quad \text{and} \quad K_1(x) \underset{x\to 0}{\sim} \frac{1}{x} + O\left(x^1\right).$$

Leads to

$$x K_0(x)\, K_1(x) - \frac{v^2}{2} x^2 (K_1^2(x) - K_0^2(x))$$

$$\underset{x\to 0}{\sim} -\frac{v^2}{2} - \ln(x) - \gamma + \ln(2) = \ln \frac{2e^{-\gamma_E}}{x} - \frac{v^2}{2} \simeq \ln \frac{1.123}{x} - \frac{v^2}{2}$$

so that

$$\frac{dI}{d\omega} \sim \frac{2}{\pi} \frac{q^2}{4\pi} \frac{1}{v^2} \left[\ln \frac{1.123\,\gamma v}{\omega b_{min}} - \frac{v^2}{2} \right] \quad \text{for } \omega \ll \gamma v/b_{min}\,.$$

High frequencies $\omega \gg \gamma v/b_{min}$: exponential damping

$$K_0(x) \underset{x\to\infty}{\sim} e^{-x}\sqrt{\frac{\pi}{2x}} \quad \text{and} \quad K_1(x) \underset{x\to\infty}{\sim} e^{-x}\sqrt{\frac{\pi}{2x}}$$

Leads to

$$x K_0(x)\, K_1(x) - \frac{v^2}{2} x^2 (K_1^2(x) - K_0^2(x)) \underset{x\to\infty}{\sim} \frac{\pi}{2} \left(1 - \frac{v^2}{2}\right) e^{-2x}$$

so that

$$\frac{dI}{d\omega} \sim \frac{q^2}{4\pi} \frac{1}{v^2} \left(1 - \frac{v^2}{2}\right) e^{-2\frac{\omega b_{min}}{\gamma v}} \quad \text{for } \omega \gg \gamma v/b_{min}\,.$$

Number of virtual quanta in the low energy limit

Thus, the spectrum is dominated by low-energy photons with $\omega \lesssim 2\gamma v/b_{min}$.

Density of modes $N(\hbar\omega)$, in the low frequency limit (with $q = Ze$ and $\alpha = \frac{e^2}{4\pi}$):

$$N(\hbar\omega) \sim \frac{2}{\pi} Z\alpha \frac{1}{v^2}\left[\ln\frac{1.123\,\gamma v}{\omega b_{min}} - \frac{v^2}{2}\right] \quad \text{for } \omega \ll \gamma v/b_{min}\,.$$

This dominance is expected:

the pulse has a time width $\Delta t \sim \frac{b}{\gamma v}$.

$\Longrightarrow$ the ω spectrum is such that $\omega \lesssim 1/\Delta t$.

So highest frequencies correspond to $b = b_{min}$, which is a key parameter:

at the smallest impact parameter, energy transfer is greatest.

no surprise, think about classical Coulomb scattering...

Example: AA scattering: $b_{min} = 2R_A$

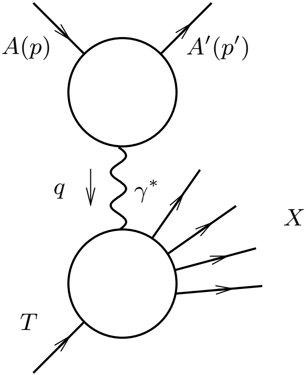
$\Longrightarrow$ maximum photon energy:

$$E_{max} \sim \hbar\gamma v/2R_A\,.$$

One photon exchange process

Generic process $AT \to AX$

Consider a process involving the exchange of a single virtual photon between a particle A (projectile) scattering off a particle T (target).

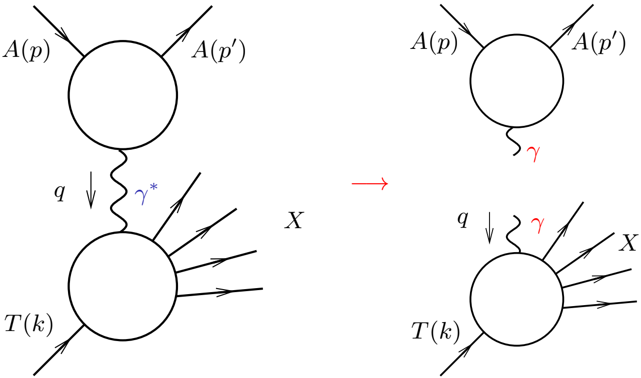


A is assumed to remain almost intact.

T may or may not break up during the process.

One photon exchange process
Equivalent photon approximation

Suppose that in the rest frame of T, the projectile A is highly boosted and almost undeflected by the collision



The process can then be described as a flux of real photons, ultimately emitted by a classical source (eikonal coupling, spin independent) which scatter off the target (photonuclear process). Let us go beyond eikonal, and see how this settle in.

Photon polarization tensor
Transverse polarizations

In the cms of $T\gamma^*$,

$$q^\mu = (E_q, 0, 0, q_z)\,, \quad k^\mu = (E, 0, 0, -q_z)\,,$$

the transverse polarization vectors $\epsilon_\pm$ ($\epsilon_\pm^2 = -1$) satisfy both conditions:

$$q \cdot \epsilon_\pm = 0\,, \quad k \cdot \epsilon_\pm = 0\,.$$

The transverse projector (actually, $-P$ is a projector)

$$P^{\mu\nu} = \sum_{\lambda=\pm} \epsilon_\lambda^\mu (\epsilon_\lambda^\nu)^*, \quad \text{with} \quad P_{\mu\nu}\epsilon_\pm^\nu = -\epsilon_{\pm\mu}\,,$$

satisfies the transverse conditions

$$k_\mu P^{\mu\nu} = q_\mu P^{\mu\nu} = 0$$

as well, as for any projector,

$$(-P_{\mu\sigma})(-P^{\sigma\nu}) = -P_\mu{}^\nu.$$

It is an instructive exercise to show that $P^{\mu\nu}$ can be expanded as

$$P_{\mu\nu} = -g_{\mu\nu} + \frac{(q\cdot k)(k^\mu q^\nu + q^\mu k^\nu) - q^2 k^\mu k^\nu - k^2 q^\mu q^\nu}{(q\cdot k)^2 - q^2 k^2}\,.$$

Photon polarization tensor

Longitudinal polarization

The longitudinal polarization ϵ_0 ($\epsilon_0^2 = +1$) can be written as

$$e_0^\mu = \sqrt{\frac{-q^2}{(q\cdot k)^2 - q^2 k^2}} \left(k^\mu - q^\mu \frac{q\cdot k}{q^2} \right)$$

The longitudinal projector

$$L^{\mu\nu} = e_0^\mu e_0^\nu$$

satisfies

$$q_\mu L^{\mu\nu} = 0\,, \quad L_{\mu\sigma} P^{\sigma\nu} = 0\,, \quad L_{\mu\sigma} L^{\sigma\nu} = L_\mu{}^\nu\,.$$

In Feynman gauge, in the numerator of the photon propagator

$$g^{\mu\nu} = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) + \frac{q^\mu q^\nu}{q^2}$$

only the first part is relevant, when contracted to a conserved current.
This part is actually the total polarization sum

$$g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} = \sum_{\lambda=0,\pm} (-1)^\lambda \epsilon_\lambda^\mu (\epsilon_\lambda^\nu)^* = -P^{\mu\nu} + L^{\mu\nu}\,.$$

The process $AT \to AX$

Cross-section for the process $AT \to AX$

Amplitude: $$i\mathcal{M}(AT \to AX) = (iZ_A e \ell^\mu)\left(-i\frac{g_{\mu\nu}}{q^2}\right)(iZ_T e H^\mu)$$

$\ell^\mu \sim$ current associated to the source

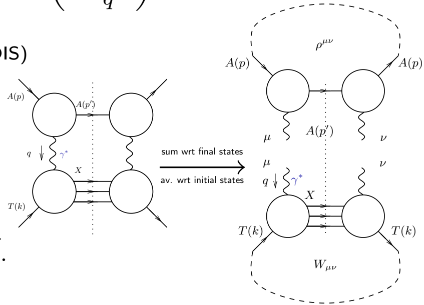
(denoted as ℓ^μ, in the spirit of the leptonic current for DIS)

$H^\mu \sim$ current associated to the target.

In the case of unpolarized cross-sections,

$$\overline{|\mathcal{M}(AT \to AX)|^2} = \frac{Z_A^2 Z_T^2 e^4}{(q^2)^2}\rho^{\mu\nu}W_{\mu\nu}$$

with $\rho^{\mu\nu} = \overline{\ell^\mu(\ell^\nu)^*}$ and $W^{\mu\nu} = \overline{H^\mu(H^\nu)^*}$.



$\overline{(\cdots)}$ = sum over the spin of produced particles, and average over initial-state spins

Differential cross-section: $$d\sigma(AT \to TX) = \frac{Z_A^2 Z_T^2 e^4}{(q^2)^2}\frac{\rho^{\mu\nu}W_{\mu\nu}}{4\sqrt{(p\cdot k)^2 - p^2k^2}}dPS\,,$$

with the phase space $$dPS = (2\pi)^4\delta^{(4)}(p + k - p' - p_X)\frac{d^3p'}{(2\pi)^3 2E'}d\Gamma_X$$

$d\Gamma_X$ = phase space of the system of particles X .

The process $\gamma^* T \to X$

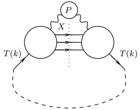
Cross-section for the process $\gamma^{(*)} T \to X$

Amplitude: $$i\mathcal{M}_\lambda(\gamma^* T \to X) = (iZ_A e \ell_\mu)\, \epsilon^\mu_\lambda(q)$$

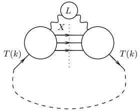
$\ell^\mu \sim$ current associated to the source

Spin averaged squared amplitudes:

$$\gamma^*_T : \overline{|\mathcal{M}_T(\gamma^{(*)} T \to X)|^2} = \frac{1}{2} Z_A^2 e^2 \overline{H_\mu (H_\nu)^*} \sum_{\lambda=\pm} \epsilon^\mu_\lambda (\epsilon^\nu_\lambda)^* = \frac{1}{2} Z_A^2 e^2 W_{\mu\nu} P^{\mu\nu}$$



$$\gamma^*_L : \overline{|\mathcal{M}_L(\gamma^{(*)} T \to X)|^2} = Z_A^2 e^2 \overline{H_\mu (H_\nu)^*} \epsilon^\mu_0 (\epsilon^\nu_0)^* = Z_A^2 e^2 W_{\mu\nu} L^{\mu\nu}$$



Differential cross-sections:

$$d\hat{\sigma}_T(\gamma^* T \to X) = \frac{1}{2} Z_A^2 e^2 \frac{W_{\mu\nu} P^{\mu\nu}}{4\sqrt{(q\cdot k)^2 - q^2 k^2}} (2\pi)^4 \delta^{(4)}(p + k - p' - p_X)\, d\Gamma_X \,,$$

$$d\hat{\sigma}_L(\gamma^* T \to X) = Z_A^2 e^2 \frac{W_{\mu\nu} L^{\mu\nu}}{4\sqrt{(q\cdot k)^2 - q^2 k^2}} (2\pi)^4 \delta^{(4)}(p + k - p' - p_X)\, d\Gamma_X \,,$$

$d\Gamma_X$ = phase space of the system of particles X .

The process $AT \to AX$
Factorization of the cross-section

Differential cross-section:

$$d\sigma(AT \to TX) = \frac{Z_A^2 Z_T^2 e^4}{(q^2)^2} \frac{\rho^{\mu\nu} W_{\mu\nu}}{4\sqrt{(p\cdot k)^2 - p^2 k^2}} \, dPS \,,$$

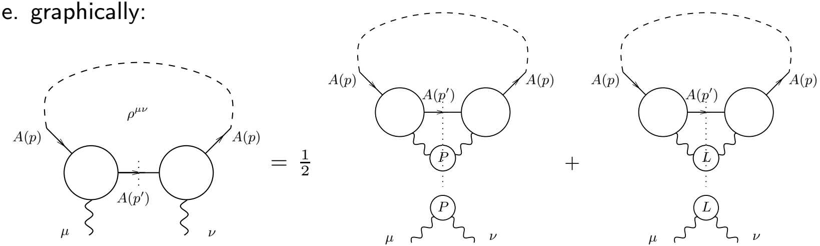
Remember that the density matrix for the virtual photon is built from the conserved current ℓ^μ $\quad \rho^{\mu\nu} = \overline{\ell^\mu (\ell^\nu)^*}$

$\rho^{\mu\nu}$ would be the leptonic tensor in the case of DIS

ℓ^μ being conserved implies that $\rho^{\mu\nu}$ can expanded in terms of $P^{\mu\nu}$ and $L^{\mu\nu}$:

$$\rho^{\mu\nu} = \frac{1}{2}[P_{\tau\sigma}\rho^{\tau\sigma}]P^{\mu\nu} + [L_{\tau\sigma}\rho^{\tau\sigma}]L^{\mu\nu} = \frac{1}{2} T\, P^{\mu\nu} + L\, L^{\mu\nu},$$

i.e. graphically:



The process $AT \to AX$

Factorization of the cross-section

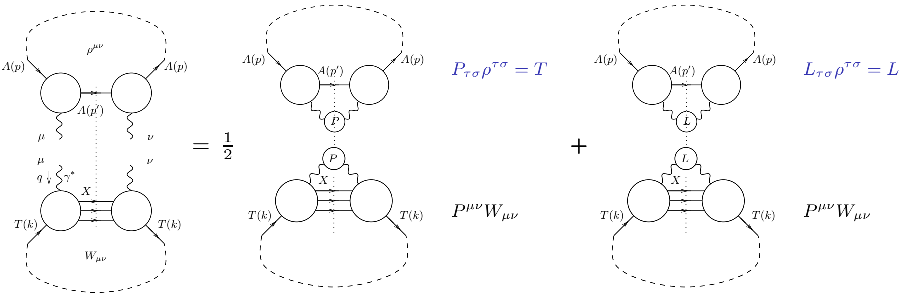
Differential cross-section:

$$d\sigma(AT \to TX) = \frac{Z_A^2 Z_T^2 e^4}{(q^2)^2} \frac{\rho^{\mu\nu} W_{\mu\nu}}{4\sqrt{(p\cdot k)^2 - p^2 k^2}} \, dPS \,,$$

with thus

$$\rho^{\mu\nu} W_{\mu\nu} = \frac{1}{2} T \left[P^{\mu\nu} W_{\mu\nu} \right] + L \left[L^{\mu\nu} W_{\mu\nu} \right]$$

which reads graphically:



The process $AT \to AX$

Factorization of the cross-section

Differential cross-section for the process $AT \to AX$:

$$d\sigma(AT \to TX) = \frac{Z_A^2 Z_T^2 e^4}{(q^2)^2} \frac{\frac{1}{2} T \left[P^{\mu\nu} W_{\mu\nu}\right] + L \left[L^{\mu\nu} W_{\mu\nu}\right]}{4\sqrt{(p \cdot k)^2 - p^2 k^2}} \times (2\pi)^4 \delta^{(4)}(p + k - p' - p_X) \frac{d^3 p'}{(2\pi)^3 2E'} d\Gamma_X \,,$$

Differential cross-section for the process $\gamma^* T \to X$:

$$d\hat{\sigma}_T(\gamma^* T \to X) = \frac{1}{2} Z_A^2 e^2 \frac{W_{\mu\nu} P^{\mu\nu}}{4\sqrt{(q \cdot k)^2 - q^2 k^2}} (2\pi)^4 \delta^{(4)}(p + k - p' - p_X) \, d\Gamma_X \,,$$

$$d\hat{\sigma}_L(\gamma^* T \to X) = Z_A^2 e^2 \frac{W_{\mu\nu} L^{\mu\nu}}{4\sqrt{(q \cdot k)^2 - q^2 k^2}} (2\pi)^4 \delta^{(4)}(p + k - p' - p_X) \, d\Gamma_X \,.$$

thus:

$$\begin{aligned} & T \, d\hat{\sigma}_T(\gamma^* T \to X) + L d\hat{\sigma}_L(\gamma^* T \to X) \\ = \quad & Z_A^2 e^2 \frac{\frac{1}{2} T \left[P^{\mu\nu} W_{\mu\nu}\right] + L \left[L^{\mu\nu} W_{\mu\nu}\right]}{4\sqrt{(q \cdot k)^2 - q^2 k^2}} (2\pi)^4 \delta^{(4)}(p + k - p' - p_X) \, d\Gamma_X \end{aligned}$$

The process $AT \to AX$

Factorization of the cross-section

Differential cross-section for the process $AT \to AX$:

$$d\sigma(AT \to TX) = \frac{Z_A^2 Z_T^2 e^4}{(q^2)^2} \frac{\frac{1}{2}T\,[P^{\mu\nu}W_{\mu\nu}] + L\,[L^{\mu\nu}W_{\mu\nu}]}{4\sqrt{(p\cdot k)^2 - p^2k^2}}$$
$$\times (2\pi)^4 \delta^{(4)}(p + k - p' - p_X) \frac{d^3p'}{(2\pi)^3 2E'} d\Gamma_X \,,$$

Differential cross-section for the weighted process $\gamma^* T \to X$:

$$T\, d\hat{\sigma}_T(\gamma^* T \to X) + L\, d\hat{\sigma}_L(\gamma^* L \to X)$$
$$= \quad Z_A^2 \frac{\frac{1}{2}T\,[P^{\mu\nu}W_{\mu\nu}] + L\,[L^{\mu\nu}W_{\mu\nu}]}{4\sqrt{(q\cdot k)^2 - q^2k^2}} (2\pi)^4 \delta^{(4)}(p + k - p' - p_X)\, d\Gamma_X$$

Conclusion:

$$d\sigma(AT \to TX) \quad = \quad \frac{Z_T^2 e^2}{(q^2)^2} \sqrt{\frac{(q\cdot k)^2 - q^2k^2}{(p\cdot k)^2 - p^2k^2}} \frac{d^3p'}{(2\pi)^3 2E'}$$
$$\times \left[T\, d\hat{\sigma}_T(\gamma^* T \to X) + L\, d\hat{\sigma}_L(\gamma^* T \to X)\right].$$

Kinematics
Reorganizing a bit...

Since $q = p - p'$, $\quad q^2 = (p - p')^2 = 2\,p^2 - 2\,p \cdot p' = 2\,p \cdot q = -2\,p' \cdot q = -Q^2\,.$

Mandelstam invariants:

$$\begin{array}{lrcl} A\,T \ \text{channel:} & s & = & (p+k)^2. \\ \gamma\,T \ \text{channel:} & \hat{s} & = & (q+k)^2. \end{array}$$

It is natural to introduce $\quad y = \dfrac{q \cdot k}{p \cdot k}\,.$

Indeed, in the target rest frame (i.e. $\vec{k} = 0$)

$y = \dfrac{q^0}{E} = \dfrac{E - E'}{E} =$ fractional energy loss of the projectile, $0 \leq y \leq 1$.

Neglecting masses in the high energy limit, $\quad \hat{s} \simeq y\,s\,.$

Kinematics

Here comes the truth...

Now, the key point:

if we do not measure Q^2, it will turn to be almost zero when integrating over the phase space...

$Q^2 = -(p-p')^2 = -2M^2 + 2(E\,E' - |\vec{p}||\vec{p}\,'|\cos\theta)$ $\theta =$ angle between $\vec{p}$ and $\vec{p}\,'$.

In the high energy limit (M^2 negligible), the γ^* propagator exhibit a collinear pole:

$Q^2 \to 0$ when $\theta \to 0$: this is the dominance we are looking for.

The virtual γ^* is practically a real photon.

Note: the very same idea makes sense in QCD ! Collinear partons (quarks, gluons) $\Longrightarrow$ DGLAP equation...

This survives in QCD when dealing with loop corrections:

when masses are negligible, integration over 4-momenta leads to a trapping of propagator at their collinear pole ("collinear pinch", at the heart of collinear factorization)

Kinematics

Some more insight

The minimum value of Q^2 is obtained for $\theta = 0$, and reads (easy exercise):

$$Q^2_{min} = M^2 \frac{y^2}{1-y} \,.$$

Simple picture:

After a suitable choice of frame, let the scattering $A\,T$ be along the z axis.

At high energy, neglecting masses

$$\begin{aligned} p &= (E, 0_\perp, E) \\ p' &\simeq (E', -q_\perp, E') \\ q = p - p' &\simeq (E - E', q_\perp, E - E') = y\,p + q_\perp \,, \end{aligned}$$

with $q_T \ll E, E'$ and $Q^2 = -q_\perp^2$.

$\Longrightarrow y =$ longitudinal fraction of momentum carried by the virtual photon.

It looks like the familiar x of the parton model, which enters the convolution:

$\mathcal{M} =$ hard part $\otimes$ non-perturbative matrix elements (e.g PDF)

The photon density matrix $\rho^{\mu\nu}$

Generically, $\rho^{\mu\nu}$ being symmetric,

$$\rho^{\mu\nu} = A\, p^\mu p^\nu + B\, (p^\mu p^\nu + q^\mu p^\nu) + C\, q^\mu q^\nu + D\, g^{\mu\nu}\,.$$

We know from current conservation that only A and D will contribute.

We just need $T = P_{\mu\nu}\rho^{\mu\nu}$.

One gets (exercise!):

$$T = -A\, q^2 \left(\frac{M^2}{q^2} + \frac{1-y}{y} \right) - 2D$$

where the leading and next-to-leading terms in the small q^2 expansion are kept.

The photon density matrix $\rho^{\mu\nu}$
A few specific cases

In the case of DIS, $\rho^{\mu\nu}$ reduces to the leptonic tensor $\ell^{\mu\nu}$, and can be computed perturbatively.

For non point-like sources (pion, proton, heavy ion), some non-perturbative structure will appear $\Longrightarrow$ $\rho^{\mu\nu}$ cannot be computed from first principles.

One needs to introduce elastic form factors, the number of which depends on the spin of the source

Examples:

pion (spin 0): one form factor

Think about scalar QED: the $\gamma-$pion vertex ℓ^{μ} is proportional to $(p+p')^{\mu}$.

It comes from the lagrangian, but it cannot be anything else, due to current conservation:

ℓ^{μ} is a combination of $(p+p')^{\mu}=(2p-q)^{\mu}$ and $(p-p')^{\mu}=q^{\mu}$.

But $q_{\mu}(p-p')^{\mu}=q^{2}\neq 0$ in general.

$$\Longrightarrow \qquad \rho^{\mu\nu}=F^{2}(Q^{2})(2p-q)^{\mu}(2p-q)^{\nu}$$

Thus the two relevant coefficients A and D read $A=4\,F^{2}(Q^{2})$ and $D=0\,.$

The photon density matrix $\rho^{\mu\nu}$

A few specific cases

Proton (spin 1/2): two form factors $F_1(Q^2)$ and $F_2(Q^2)$

(or equivalently $G_E(Q^2)$ and $G_M(Q^2)$)

starting from: e^- (of charge $e = -|e|$) vertex $-ie\gamma^\mu$

one gets for the proton the parametrization: $+ie\Gamma^\mu$

with (combining current conservation + parity invariance of QED)

$$\begin{aligned} \Gamma^\mu &= F_1(Q^2)\gamma^\mu - \frac{i\kappa}{2M} F_2(Q^2)\sigma^{\mu\nu} q_\nu \\ &= (4M^2 G_E - q^2 G_M)\frac{\gamma^\mu}{P^2} - i\frac{2M}{P^2}(G_E - G_M)\sigma^{\mu\nu} q_\nu \,, \end{aligned}$$

leading to the current $\ell^\mu = \bar{u}(s', p')\Gamma^\mu u(s,p)$

$$\begin{aligned} \Longrightarrow \rho^{\mu\nu} &= \overline{\ell^\mu (\ell^\nu)^*} = \frac{1}{2}\mathrm{Tr}\left[(\not{p}' + M)\Gamma^\mu(\not{p} + M)\gamma^\nu\right] \\ &= \frac{G_E^2(Q^2) + \tau G_M^2(Q^2)}{1+\tau}(2p-q)_\mu(2p-q)_\nu + G_M^2(Q^2)(q^2 g_{\mu\nu} - q_\mu q_\nu) \end{aligned}$$

with $\tau = Q^2/(4M^2)$. Thus the two relevant coefficients A and B read

$$\begin{aligned} A &= 4\frac{G_E^2(Q^2) + \tau G_M^2(Q^2)}{1+\tau} \,, \\ D &= -Q^2 G_M^2(Q^2) \,. \end{aligned}$$

The photon density matrix $\rho^{\mu\nu}$

The classical limit

In the limit of soft photon exchange, the coupling becomes eikonal and somehow universal:
it does not depend on the spin of the emitter, and is basically the classical current corresponding to the Fourier transform of an untilded line (i.e. unaffected by the emission of the soft photon): $\ell^\mu \sim p^\mu$.

We are then back to the classical treatment: high energy i.e. high frequency of the source wrt the photon.
This is the link with the first (classical) point of view.

The equivalent photon approximation

The $q^2 \to 0$ limit

Starting from

$$d\sigma(AT \to TX) = \frac{Z_T^2 e^2}{(q^2)^2}\sqrt{\frac{(q\cdot k)^2 - q^2k^2}{(p\cdot k)^2 - p^2k^2}}\frac{d^3p'}{(2\pi)^3 2E'} \times \left[T\, d\hat{\sigma}_T(\gamma^* T \to X) + L\, d\hat{\sigma}_L(\gamma^* T \to X)\right]$$

$$\underset{q^2\to 0}{\sim} \frac{Z_T^2 e^2}{(q^2)^2}\sqrt{\frac{(q\cdot k)^2}{(p\cdot k)^2 - p^2k^2}}\frac{d^3p'}{(2\pi)^3 2E'} T\, d\hat{\sigma}(\gamma T \to X)\,.$$

Although e_0^μ is singular in the limit $q^2 \to 0$, the contraction $e_0^\mu e_0^\nu W_{\mu\nu} = L^{\mu\nu}W_{\mu\nu} \propto d\hat{\sigma}_L$ vanishes and L is finite when $q^2 \to 0$.

Indeed, just as we did for the density matrix $\rho^{\mu\nu}$, we can expand $W^{\mu\nu}$ as

$$W^{\mu\nu} = \frac{1}{2}[P_{\tau\sigma}W^{\tau\sigma}]P^{\mu\nu} + [L_{\tau\sigma}W^{\tau\sigma}]L^{\mu\nu}$$

In order that $W^{\mu\nu}$ remains finite when $q^2 \to 0$, $[L_{\tau\sigma}W^{\tau\sigma}]$ should go to zero fast enough to compensate the divergence of $L^{\mu\nu}$. Same argument for L.

$\Longrightarrow$ we can safely:

- remove the contribution $L\, d\hat{\sigma}_L(\gamma^* T \to X)$
- put $q^2 = 0$ in $d\hat{\sigma}_T(\gamma^* T \to X) \to d\hat{\sigma}(\gamma T \to X)$: photoproduction cross-section

The equivalent photon approximation
Phase space integration

Let us reorganize $\frac{d^3p'}{(2\pi)^3 2E'}$:

In the projectile's rest frame, $E = M$, $\vec{p} = 0$ and $|\vec{p}\,'| = |\vec{q}\,|$. Thus

$$Q^2 = -q^2 = -2M^2 + 2M\,E' = 2M(\sqrt{|\vec{q}\,|^2 + M^2} - M)\,.$$

Introducing the angle θ between $\vec{p}\,'$ and $\vec{k}$ we have

$$y = \frac{q \cdot k}{p \cdot k} = \frac{q^0}{M} + \frac{|\vec{q}\,||\vec{k}|\cos\theta}{Mk^0}\,.$$

An easy Jacobian calculation gives

$$dy\,dQ^2 = \frac{2|\vec{k}||\vec{q}\,|^2}{E'k^0} d(\cos\theta) d|\vec{p}\,'|$$

and integrating out the azimuthal angle gives

$$\frac{d^3p'}{(2\pi)^3 2E'} = \frac{1}{4(2\pi)^2} \frac{k^0}{|\vec{k}|} dy\,dQ^2.$$

Since $k^2 = m^2 \ll (k^0)^2 \sim |\vec{k}|^2$ is negligible at high energies,

$$\frac{d^3p'}{(2\pi)^3 2E'} \sim \frac{1}{4(2\pi)^2} dy\,dQ^2.$$

Again, from $k^2 = m^2 \ll (k^0)^2 \sim |\vec{k}|^2$, we have, using $q \cdot k = y\,(p \cdot k)$:

The equivalent photon approximation

Almost done!

Let us perform a final high-energy simplification, starting from

$$
\begin{aligned}
d\sigma(AT \to TX) &= \frac{Z_T^2 e^2}{(q^2)^2}\sqrt{\frac{(q\cdot k)^2 - q^2k^2}{(p\cdot k)^2 - p^2k^2}}\frac{d^3p'}{(2\pi)^3 2E'} \\
&\quad \times [T\, d\hat{\sigma}_T(\gamma T \to X) + L\, d\hat{\sigma}_L(\gamma T \to X)] \\
&\sim \frac{Z_T^2 e^2}{(q^2)^2}\sqrt{\frac{(q\cdot k)^2}{(p\cdot k)^2 - p^2k^2}}\frac{1}{4(2\pi)^2} dy\, dQ^2 T\, d\hat{\sigma}(\gamma T \to X)\,.
\end{aligned}
$$

Again, from $k^2 = m^2 \ll (k^0)^2 \sim |\vec{k}|^2$, we have, using $q\cdot k = y\,(p\cdot k)$:

$$
\sqrt{\frac{(q\cdot k)^2}{(p\cdot k)^2 - p^2k^2}} \sim y
$$

and finally

$$
\frac{d\sigma(AT \to TX)}{dy\, dQ^2} \sim \frac{Z_T^2 e^2}{Q^4}\frac{y}{4(2\pi)^2} T\, d\hat{\sigma}(\gamma T \to X) = \frac{\alpha}{4\pi}\frac{y}{Q^4} Z_T^2\, T\, d\hat{\sigma}(\hat{s}, \gamma T \to X)\,.
$$

The equivalent photon approximation
Photon distribution

From

$$\frac{d\sigma(AT \to TX)}{dy\, dQ^2} = \frac{\alpha}{4\pi}\frac{y}{Q^4} Z_T^2\, T\, d\hat{\sigma}(\hat{s}, \gamma T \to X)\,.$$

we define the photon distribution

$$\begin{aligned} f_{\gamma/A}(y, Q^2) &= \frac{\alpha}{4\pi}\frac{y}{Q^4} Z_T^2\, T(y, Q^2, M^2) \\ &= \frac{\alpha}{2\pi}\frac{y}{Q^2} Z_T^2 \left[\frac{A}{2}\left(\frac{1-y}{y^2} - \frac{M^2}{Q^2}\right) - \frac{D}{Q^2}\right] \end{aligned}$$

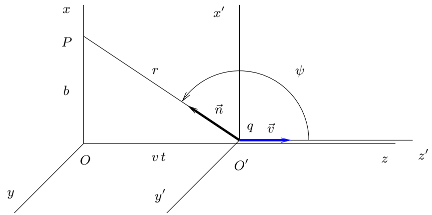
so that

$$\frac{d\sigma(AT \to TX)}{dy\, dQ^2\, d\Gamma_X} = f_{\gamma/A}(y, Q^2)\, \frac{d\hat{\sigma}(ys, \gamma T \to X)}{d\Gamma_X}\,.$$

Since $d\hat{\sigma}$ does not depend anymore on Q^2, integration over y and Q^2 reads

$$\begin{aligned} \frac{d\sigma(AT \to TX)}{d\Gamma_X} &= \int dy \left[\int_{Q^2_{min}}^{Q^2_{max}} dQ^2\, f_{\gamma/A}(y, Q^2)\right] \frac{d\hat{\sigma}(ys, \gamma T \to X)}{\Gamma_X} \\ &= \int dy\, f_{\gamma/A}(y)\, \frac{d\hat{\sigma}(ys, \gamma T \to X)}{d\Gamma_X}\,, \end{aligned}$$

with typically, for a source of size R_A, $\quad Q^2_{max} \sim 1/R_A^2$, and $Q^2_{min} = M^2 \frac{y}{1-y}$.



We have $\quad \vec{r} = b\,\vec{u}_x - vt\,\vec{u}_z$

and
$$\vec{E} = \frac{q\gamma\vec{r}}{4\pi(b^2+v^2\gamma^2t^2)^{3/2}} = \frac{q\gamma\vec{r}}{4\pi\gamma^3(\frac{b^2}{\gamma^2}+v^2t^2)^{3/2}} = \frac{q\vec{r}}{4\pi\gamma^2(\frac{b^2}{\gamma^2}+v^2t^2)^{3/2}}$$
$$= \frac{q\vec{r}}{4\pi\gamma^2(b^2+v^2t^2-\beta^2b^2)^{3/2}}$$

since
$$\frac{b^2}{\gamma^2} = b^2 + b^2\frac{1-\gamma^2}{\gamma^2} = b^2 - \beta^2 b^2\,.$$

Finally, due to $\sin\psi = b/r$, one gets
$$\vec{E} = \frac{q\vec{r}}{4\pi\gamma^2r^3(1-\beta^2b^2/r^2)^{3/2}} = \frac{q\vec{r}}{4\pi r^3\gamma^2(1-\beta^2\sin^2\psi)^{3/2}}\,.$$