M1 General Physics **2020-2021**
Major PNU

Particles

Final exam

January 4th 2021

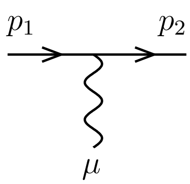
Documents allowed

Amplitudes

In the whole problem, “electron” and “positron“ should be understood as scalar particle of charge $e = -|e|$ and $|e|$ respectively.

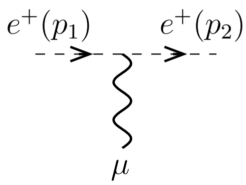
0. Preliminary

(a) The feynman rule for the spinless electron-photon vertex is known to be



$$= ie(p_1 + p_2)^\mu \,.$$

Explain why the feynman rule for the spinless positron-photon vertex is



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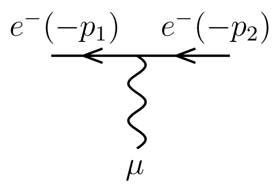

$$= ie(-p_1 - p_2)^\mu \,.$$

Solution

Relying on the antiparticle prescription, we know that a positron of momentum p propagating forward in time is equivalent to an electron of momentum $-p$ propagating backward in time. The Feynman rule is then obvious.

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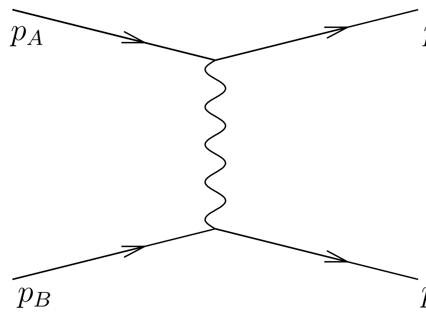
1. We consider the process

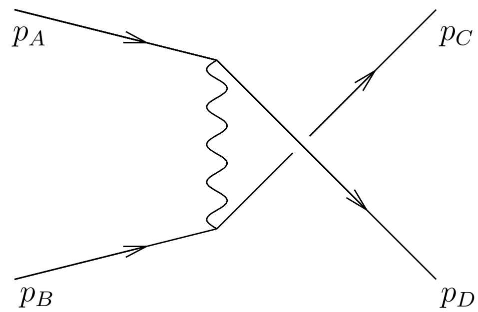
$$e^-(p_A)\, e^-(p_B) \to e^-(p_C)\, e^-(p_D)\,. \tag{1}$$

(a) At lowest order in perturbation theory, how many Feynman diagrams can be drawn? Draw them.

Solution

There are two diagrams:





(b) Write the scattering amplitude $\mathcal{M}^{e^-e^-\to e^-e^-}$ of this process.

Solution

We have

$$\begin{aligned} & i\mathcal{M}^{e^-e^-\to e^-e^-}(p_A,p_B,p_C,p_D) \\ = \; & (ie)(-i)(ie)\left[(p_A+p_C)^\mu \frac{g_{\mu\nu}}{(p_D-p_B)^2}(p_B+p_D)^\nu + (p_A+p_D)^\mu \frac{g_{\mu\nu}}{(p_B+p_C)^2}(p_B+p_C)^\nu\right] \\ = \; & ie^2\left[\frac{(p_A+p_C)\cdot(p_B+p_D)}{(p_D-p_B)^2} + \frac{(p_A+p_D)\cdot(p_B+p_C)}{(p_C-p_B)^2}\right]. \end{aligned}$$

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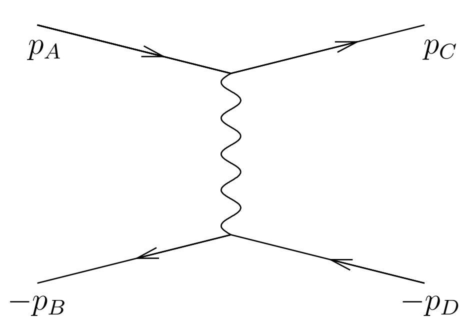
2. We consider the process

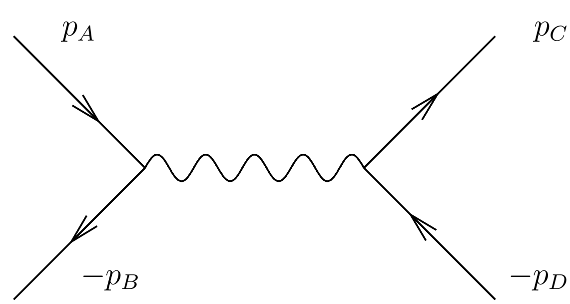
$$e^-(p_A)\, e^+(p_B) \to e^-(p_C)\, e^+(p_D)\,. \tag{2}$$

(b) At lowest order in perturbation theory, how many Feynman diagrams can be drawn? Draw them, using only electron lines, relying on the antiparticle prescription.

Solution

There are two diagrams. Using the antiparticle prescription, which says that an antiparticle of momentum p propagating forward in time is equivalent to a particle of momentum $-p$ propagating backward in time, they should be drawn as





(c) Write the scattering amplitude $\mathcal{M}_{e^-e^+\to e^-e^+}$ of this process.

Solution

We have

$$
\begin{aligned}
& i\mathcal{M}^{e^-e^+\to e^-e^+}(p_A, p_B, p_C, p_D) \\
= \; & (ie)(-i)(ie)\left[(p_A+p_C)^\mu \frac{g_{\mu\nu}}{(p_D-p_B)^2}(-p_B-p_D)^\nu + (p_A-p_B)^\mu \frac{g_{\mu\nu}}{(p_C+p_D)^2}(p_C-p_D)^\nu\right] \\
= \; & ie^2\left[\frac{(p_A+p_C)\cdot(-p_B-p_D)}{(p_D-p_B)^2} + \frac{(p_A-p_B)\cdot(p_C-p_D)}{(p_C+p_D)^2}\right].
\end{aligned}
$$

3. Due to the antiparticle prescription, we know that for an arbitrary particle P, the set $P(p)\bar{P}(-p)$ is the same as the vacuum. Starting from an arbitrary $2\to 2$ process generically written as

$$A(p_A)\,B(p_B) \to C(p_C)\,D(p_D) \tag{3}$$

and adding on the left hand side a particle-antiparticle pair of a suitable type, and on the right hand side another particle-antiparticle pair of different type, show that this process is equivalent to the process

$$A(p_A)\,\bar{D}(-p_D) \to C(p_C)\,\bar{B}(-p_B) \tag{4}$$

and thus that

$$\mathcal{M}^{AB\to CD}(p_A, p_B, p_C, p_D) = \mathcal{M}^{A\bar{D}\to C\bar{B}}(p_A, -p_D, p_C, -p_B)\,, \tag{5}$$

a property known under the name of crossing symmetry.

Solution

One should simply add $D(p_D)\bar{D}(-p_D)$ in the left hand side, and $B(p_B)\bar{B}(-p_B)$ in the right hand side, so that the process (3) is identical to

$$A(p_A)\,D(p_D)\bar{D}(-p_D)\,B(p_B) \to C(p_C)\,B(p_B)\bar{B}(-p_B)\,D(p_D)$$

so that removing the state $B(p_B)D(p_D)$ from both left and right hand sides, we obtain the process

$$A(p_A)\,\bar{D}(-p_D) \to C(p_C)\,\bar{B}(-p_B)\,,$$

which proves that the processes (3) and (4) are identical, so that they are described by the same scattering amplitudes.

4. Compare the two amplitudes

$$\mathcal{M}^{e^-e^+\to e^-e^+}(p_A, p_B, p_C, p_D)$$

and

$$\mathcal{M}^{e^-e^-\to e^-e^-}(p_A, -p_D, p_C, -p_B)\,.$$

Comment and explain why this should be expected.

Solution

From the two results of questions 1. and 2., we readily have

$$\begin{aligned}\mathcal{M}^{e^-e^-\to e^-e^-}(p_A, -p_D, p_C, -p_B) &= e^2\left[\frac{(p_A+p_C)\cdot(-p_D-p_B)}{(-p_B+p_D)^2} + \frac{(p_A-p_B)\cdot(p_C-p_D)}{(p_C+p_D)^2}\right] \\ &= \mathcal{M}^{e^-e^+\to e^-e^+}(p_A, p_B, p_C, p_D)\,,\end{aligned}$$

so that these two amplitudes are equal. This is simply due to the above crossing symmetry.

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5. We introduce the three Mandelstam variables

$$\begin{aligned} s &= (p_A+p_B)^2 & (6)\\ t &= (p_A-p_C)^2 & (7)\\ u &= (p_A-p_D)^2. & (8)\end{aligned}$$

(a) Show that we also have

$$\begin{aligned} s &= (p_C+p_D)^2 & (9)\\ t &= (p_B-p_D)^2 & (10)\\ u &= (p_C-p_B)^2. & (11)\end{aligned}$$

Solution

This is obvious using energy-momentum conservation $p_A+p_B=p_C+p_D$.

(b) Translate the crossing discussed in question 4. in terms of the exchange of two variables among s, t, u. Deduce a relation between the amplitudes of the two processes when expressed as functions of s, t, u.

Solution

Passing from

$$A(p_A)\,B(p_B)\to C(p_C)\,D(p_D)$$

to

$$A(p_A)\,\bar{D}(-p_D) \to C(p_C)\,\bar{B}(-p_B)$$

corresponds to the exchange $s \leftrightarrow u$. Therefore, the equality

$$\mathcal{M}^{AB\to CD}(p_A, p_B, p_C, p_D) = \mathcal{M}^{A\bar{D}\to C\bar{B}}(p_A, -p_D, p_C, -p_B)\,,$$

can also be written as

$$\mathcal{M}^{AB\to CD}(s, t, u) = \mathcal{M}^{A\bar{D}\to C\bar{B}}(u, t, s)\,.$$

(c) Find another crossed reaction involving the exchange of another subset of two variables among s, t, u, and provide the relation between the two amplitudes, expressed as functions of momenta, and then expressed as functions of Mandelstam variables.

Solution

Passing from

$$A(p_A)\,B(p_B) \to C(p_C)\,D(p_D)$$

to

$$A(p_A)\,\bar{C}(-p_C) \to \bar{B}(-p_B)\,D(p_D)$$

corresponds to the exchange $s \leftrightarrow t$. It means that

$$\mathcal{M}^{AB\to CD}(p_A, p_B, p_C, p_D) = \mathcal{M}^{A\bar{C}\to \bar{B}D}(p_A, -p_C, -p_B, p_D)\,,$$

can also be written as

$$\mathcal{M}^{AB\to CD}(s, t, u) = \mathcal{M}^{A\bar{C}\to \bar{B}D}(t, s, u)\,.$$

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6. Explicit expressions of the amplitudes and crossing properties

(a) Compute the scattering amplitude of the process (1) as a function of e^2, s, t, u.

Solution

We have

$$\begin{aligned}(p_A + p_C)\cdot(p_B + p_D) &= p_A\cdot p_B + p_A\cdot p_D + p_C\cdot p_B + p_C\cdot p_D \\ &= \frac{1}{2}\left[s - m_A^2 - m_B^2 + m_A^2 + m_D^2 - u + m_C^2 + m_B^2 - u + s - m_C^2 - m_D^2\right] = s - u\end{aligned}$$

and

$$\begin{aligned}(p_A + p_D)\cdot(p_B + p_C) &= p_A\cdot p_B + p_A\cdot p_C + p_D\cdot p_B + p_D\cdot p_C \\ &= \frac{1}{2}\left[s - m_A^2 - m_B^2 + m_A^2 + m_C^2 - t + m_B^2 + m_D^2 - t + s - m_C^2 - m_D^2\right] = s - t\end{aligned}$$

so that

$$\mathcal{M}^{e^-e^-\to e^-e^-}(p_A, p_B, p_C, p_D) = e^2\left[\frac{s-u}{t} + \frac{s-t}{u}\right]$$

(b) Compute the scattering amplitude of the process (2) as a function of e^2, s, t, u.

Solution

We have, from the previous question,

$$(p_A + p_C) \cdot (-p_B - p_D) = u - s$$

and

$$\begin{aligned}(p_A - p_B) \cdot (-p_D + p_C) &= -p_A \cdot p_D + p_A \cdot p_C + p_B \cdot p_D - p_B \cdot p_C \\ &= \frac{1}{2}\left[u - m_A^2 - m_D^2 + m_A^2 + m_C^2 - t + m_B^2 + m_D^2 - t + u - m_C^2 - m_B^2\right] = u - t\end{aligned}$$

so that

$$\mathcal{M}^{e^-e^+ \to e^-e^+}(p_A, p_B, p_C, p_D) = e^2 \left[\frac{u-s}{t} + \frac{u-t}{s}\right].$$

(c) Crossing properties:

(i) Comment on s, t, u crossing properties of the two diagrams involved in the process (1).

Solution

The two diagrams are conjugated through $t \leftrightarrow u$ crossing. This is indeed satisfied by the obtained amplitudes, see question 6. (a).

(ii) Comment on s, t, u crossing properties of the two diagrams involved in the process (2).

Solution

The two diagrams are conjugated through $s \leftrightarrow$ crossing. This is indeed satisfied by the obtained amplitudes, see question 6. (b).

(iii) Comment on s, t, u crossing properties between the two processes (1) and (2).

Solution

The two amplitudes are conjugated through $s \leftrightarrow u$ crossing. This is indeed satisfied by the obtained amplitudes, see questions 6. (a) and 6. (b), and it is also true diagram per diagram.

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7. Kinematics in the center-of-mass frame.

We now consider the center-of-mass frame, and we denote $\vec{p}_i = \vec{p}_A = -\vec{p}_B$ and $\vec{p}_f = \vec{p}_C = -\vec{p}_D$, and $p_i^* = |\vec{p}_i|$ and $p_f^* = |\vec{p}_f|$.

(a) Explain why $p_A^0 = p_B^0 = \sqrt{s}/2$ and $p_C^0 = p_D^0 = \sqrt{s}/2$.

Solution

Since in the c.m.f, $\vec{p}_A + \vec{p}_B = 0$, one denotes $\vec{p}_i = \vec{p}_A = -\vec{p}_B$, and similarly, due to $\vec{p}_C + \vec{p}_D = 0$, one denotes $\vec{p}_f = \vec{p}_C = -\vec{p}_D$, which implies that

$$p_A^2 = (p_A^0)^2 - p_i^{*\,2} = m_e^2 \qquad \text{and} \qquad p_B^2 = (p_B^0)^2 - p_i^{*\,2} = m_e^2\,,$$

as well as

$$p_C^2 = (p_C^0)^2 - p_f^{*\,2} = m_e^2 \qquad \text{and} \qquad p_D^2 = (p_D^0)^2 - p_f^{*\,2} = m_e^2\,,$$

so that $p_A^0 = p_B^0$ and thus, since $s = (p_A + p_B)^2 = (p_C + p_D)^2 = (p_A^0 + p_B^0)^0 = (p_C^0 + p_D^0)^0$, we get $p_A^0 = p_B^0 = \sqrt{s}/2$ and $p_C^0 = p_D^0 = \sqrt{s}/2$.

(b) Show that $p_i^* = p_f^*$.

Solution

This is obvious from

$$(p_A^0)^2 - p_i^{*\,2} = m_e^2 = (p_C^0)^2 - p_f^{*\,2}$$

with $p_A^0 = p_C^0$.

(c) We denote $k = p_i^* = p_f^*$ and introduce the scattering angle θ, i.e. the angle between $\vec{p}_i$ and $\vec{p}_f$.

(i) Show that

$$s = 4m^2 + 4k^2\,. \tag{12}$$

Solution

We have

$$s = 4(p_A^0)^2 = 4m^2 + 4k^2\,.$$

(ii) Show that

$$t = -2k^2(1 - \cos\theta)\,. \tag{13}$$

Solution

One can choose the unit vector $\vec{u}_x$ so that $(\vec{u}_x, \vec{u}_z)$ is a basis of the reaction plane. Then

$$\begin{aligned} p_A &= (p_A^0, 0, 0, k) \\ p_B &= (p_A^0, 0, 0, -k) \\ p_C &= (p_A^0, k\sin\theta, 0, k\cos\theta) \\ p_D &= (p_A^0, -k\sin\theta, 0, -k\cos\theta) \end{aligned}$$

thus

$$p_A \cdot p_C = (p_A^0)^2 - k^2\cos\theta$$

so that

$$t = (p_A - p_C)^2 = p_A^2 + p_C^2 - 2p_A \cdot p_C = 2m^2 - 2(p_A^0)^2 + 2k^2\cos\theta = -2k^2(1 - \cos\theta)\,.$$

(iii) Show that

$$u = -2k^2(1 + \cos\theta)\,. \tag{14}$$

Solution

We have

$$p_A \cdot p_D = (p_A^0)^2 + k^2\cos\theta$$

so that

$$u = (p_A - p_D)^2 = p_A^2 + p_D^2 - 2p_A \cdot p_D = 2m^2 - 2(p_A^0)^2 - 2k^2\cos\theta = -2k^2(1 + \cos\theta)\,.$$

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8. Cross-sections

(a) Write the differential cross-section $d\sigma/d\Omega$ in the center-of-mass frame for the process (1) as a function of s, t, u, introducing the fine structure constant

$$\alpha_{em} = \frac{e^2}{4\pi}.$$

Solution

Since $e^4 = 16\pi^2\alpha_{em}^2$, we thus get

$$\left.\frac{d\sigma^{e^-e^-\to e^-e^-}}{d\Omega}\right|_{c.m.f} = \frac{\alpha_{em}^2}{4s}\left[\frac{s-u}{t}+\frac{s-t}{u}\right]^2.$$

(b) Write the differential cross-section $d\sigma/d\Omega$ in the center-of-mass frame for the process (2) as a function of s, t, u.

Solution

Similarly, we get

$$\left.\frac{d\sigma^{e^-e^+\to e^-e^+}}{d\Omega}\right|_{c.m.f} = \frac{\alpha_{em}^2}{4s}\left[\frac{u-s}{t}+\frac{u-t}{s}\right]^2.$$

(c) Write the two above cross-sections as functions of k^2 and $\cos\theta$.

Solution

One gets

$$\left.\frac{d\sigma^{e^-e^-\to e^-e^-}}{d\Omega}\right|_{c.m.f} = \frac{\alpha_{em}^2}{4(m^2+k^2)}\left[\frac{m^2+2k^2}{k^2(1-\cos\theta)}+\frac{m^2+2k^2}{k^2(1+\cos\theta)}-1\right]^2$$

and

$$\left.\frac{d\sigma^{e^-e^+\to e^-e^+}}{d\Omega}\right|_{c.m.f} = \frac{\alpha_{em}^2}{16(m^2+k^2)}\left[2\frac{m^2+2k^2}{k^2(1-\cos\theta)}+\frac{k^2\cos\theta}{m^2+k^2}-1\right]^2.$$

(d) Comment on the behavior of the cross-section for the process $e^-e^- \to e^-e^-$ when $\theta\to 0$ or $\theta\to\pi$. What is the technical origin of this? Can one find a physical explanation?

Solution

The cross-section has a (divergent) peak from the term $1/(1-\cos\theta)$ when $\theta\to 0$ (forward peak) and a (divergent) peak from the term $1/(1+\cos\theta)$ when $\theta\to\pi$ (backward peak). They respectively come from the pole contribution in $1/t$ in the diagram with a t-channel photon exchange (for $\theta\to 0$) and from the pole contribution in $1/u$ in the diagram with a u-channel photon exchange (for $\theta\to\pi$). Indeed, the virtual photon then goes to its mass shell. In the c.m.f, since the t-channel photon and the $u-$channel photon are purely space-like, it means from the Heisenberg uncertainty principle that since its 3-momentum goes to zero, its interaction range goes to infinity (just like a real photon), which leads to large cross-sections.
