M1 General Physics **2021-2022**
Major PNU

Particles

Mid-term exam

October 26th 2021

Documents allowed

Relativistic Doppler effect and aberration of light

Notes:
- **The subject is deliberately long**. It is not requested to reach the end to get a good mark!
- Space coordinates may be freely denoted as (x, y, z) or (x^1, x^2, x^3).
- Any drawing, at any stage, is welcome, and will be rewarded!

1 Particle of arbitrary mass

We consider two frames K and K'. The frame F' moves with respect to the frame F with a velocity $\vec{v}_{F'/F} = \vec{\beta} c$ along the z axis.

1. Assuming that a particle has a velocity $\vec{v}$ in the frame F, show that the velocity $\vec{v}^{\,\prime}$ in the frame F' is given by

$$v^{3\prime} = \frac{v^3 - \beta\, c}{1 - \beta \frac{v^3}{c}}\,, \tag{1}$$

$$v^{1\prime} = \frac{1}{\gamma} \frac{v^1}{1 - \beta \frac{v^3}{c}}\,, \tag{2}$$

and similarly for $v^{2\prime}$.
Hint: consider the differential of a boost, and use the fact that

$$\forall i \in \{1, 2, 3\}, \quad v^i = c\frac{dx^i}{dx^0} \quad \text{and} \quad v^{i\prime} = c\frac{dx^{i\prime}}{dx^{0\prime}}\,.$$

2. Consider a particle of velocity $\vec{v}$ in a frame K, with spherical coordinates (v, θ, φ) with respect to the above z-axis used to defined the boost from F to F'. We now observe the particle in the frame K'. Its velocity $\vec{v}^{\,\prime}$ is described using the spherical coordinates (v', θ', φ') defined in the same (x, y, z) reference system.

(a) Why does it make sense to use the same system of coordinates for both F and F' frames?

(b) It is convenient to separate the transverse and the longitudinal velocity with respect to the direction z of the boost, as $\vec{v} = \vec{v}_\parallel + \vec{v}_\perp$ where $\vec{v}_\parallel$ is along the $z-$axis and $\vec{v}_\perp$ is in the xy plane. Express $\vec{v}^{\,\prime}_\parallel$ and $\vec{v}^{\,\prime}_\perp$ in terms of $\vec{v}_\parallel$, $\vec{v}_\perp$, β, γ and c.

(c) Change of the magnitude of the velocity.

(i) Show that

$$\left(1-\frac{v'^2}{c^2}\right)\left(1-\frac{v_\parallel}{c}\beta\right)=\left(1-\frac{v^2}{c^2}\right)\left(1-\beta^2\right). \tag{3}$$

(ii) Show that if $v \leqslant c$, this remains true in any frame boosted by a velocity $\beta c \leqslant c$. What about the special case of $v = c$?

(d) Transformation of the direction of the particle velocity.

(i) Write the expressions of $\vec{v}_\perp$ and $v_\parallel$ as functions of v, φ, θ in the frame F, and provide the corresponding expressions in the frame F'.

(ii) What is the relation between φ and φ'?

(iii) Give the expression of $\sin\theta'$, $\cos\theta'$ as functions of v, v', β and γ. Check that

$$\tan\theta' = \frac{1}{\gamma}\frac{v\,\sin\theta}{v\,\cos\theta-\beta c}\,. \tag{4}$$

2 Photon

We now consider the case of a photon. We again consider the two frames F and F' related by the same above boost along the $z-$axis.

1. Write the expression of $\sin\theta'$, $\cos\theta'$ and $\tan\theta'$. One should in particular check that

$$\tan\theta' = \frac{1}{\gamma}\frac{\sin\theta}{\cos\theta-\beta}\,. \tag{5}$$

The change of θ to θ' is known under the name of aberration of light.

2. Boost of the photon momentum

(a) Write the way a photon of 4-momentum $k = (k^0, \vec{k})$ is transformed into $k' = (k^{0\prime}, \vec{k}')$ under the boost from F to F', separating $\perp$ and $\parallel$ components. Introduce the angles θ and θ'.

(b) Relate θ' and θ and check the consistency with the results obtained in the question 2.1.

(c) Show that the change of frequency of the photon under the above boost is given by

$$\nu' = \nu\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}}\,. \tag{6}$$

3. A few particular cases

(a) Discuss the cases $\theta = 0$, $\theta = \frac{\pi}{2}$ and $\theta = \pi$, both for the aberration of light and for the frequency. Comment in each case whether there is a blue shift or a red shift.

(b) Show that there is a particular angle θ for which the photon frequency is unchanged. What is the aberration in this case?

4. Forward cone.

(a) Show that any light propagating outside a forward cone in the frame F will appear to propagate backward in the frame F'. Express the half-opening angle of this cone in term of β.

(b) Show that this half-opening angle in the ultra-relativistic limit $\beta \to 1$ is given by

$$\theta_{\text{cone}} \sim \frac{1}{\gamma}\,. \tag{7}$$

(c) What would be the consequence for an hypothetical interstellar traveler moving at a speed close to the light speed?

5. We consider the non-relativistic limit $\beta \ll 1$.

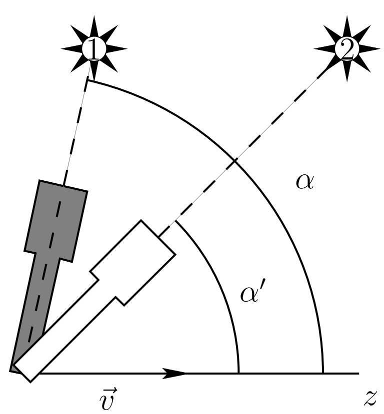
(a) Show that the aberration angle $\Delta\theta = \theta' - \theta$ is then given by

$$\Delta\theta = \beta \sin\theta\,. \tag{8}$$

(b) Show that this result can be obtained when performing a classical galilean change of frame.

3 Aberration of light in astronomy (Bonus)

1. We assume that a star is at rest with respect to the Sun. A telescope points in the direction of this star. We denote the direction of the instant motion of Earth with respect to Sun as z.



In the rest frame of the Sun, the star is in the position (1), and α is the angle with respect to the axis z toward which the telescope should be pointed if fixed with respect to the Sun. Due to the motion of Earth with respect to the Sun, the star seems to be in the position (2) and α' is the real angle (i.e. in the rest frame of the Earth at instant time) toward which the telescope should be pointed. Note: the average distance between Sun and Earth is $150 \cdot 10^6$ km (one astronomical unit).

(i) Discuss the importance of relativistic effect with respect to the classical one.

(ii) Show that the exact expression of $\tan\alpha'$ as observed by the telescope, as a function of α and v_{Earth}, is given by

$$\tan\alpha' = \frac{1}{\gamma}\frac{\sin\alpha}{\cos\alpha + \frac{v_{\text{Earth}}}{c}}\,. \tag{9}$$

(iii) Give a suitable approximation for $\alpha' - \alpha$.

2. We assume that the star is asymptotically far from the Sun. We recall that the *ecliptic* is the plane of Earth's orbit around the Sun.

(a) For which angle with respect to the direction of motion of Earth with respect to Sun is the aberration angle maximal? Compute numerically (as a precise fraction of degrees) this maximum aberration angle $\Delta\theta$.

(b) What is the numerical difference of the two observed angles for a star of maximal aberration when performing two measurements separated by half a year?

(c) Describe the phenomena seen over a year in the case of a star:

(i) in the ecliptic, therefore with 0° ecliptic latitude.

(ii) at the pole of the ecliptic, therefore with 90° (north ecliptic pole) or −90° ecliptic latitude (south ecliptic pole).

(iii) for an arbitrary star.

(c) The phenomenon of parallax is well known: a rather close object seems to move when you observe it either from your right eye or from you left eye. More generally, it is the difference in the apparent position of an object viewed along two different lines of sight. Compare the aberration effect discussed previously with the parallax effect due to the fact that the observation angle of a nearby star (typically of the order of a few light-year (ly) from us; Proxima Centaurus is at 4.24 ly) varies when observed at two times separated by half a year.

(d) Explain why this allowed to perform one of the very first measurement of the speed of light (James Bradley, 1729), although limited by the precision on the Sun-Earth distance, by an astronomical measurement of γ-Draconis ($\sim$ 148 light-year from us), which is almost at the north ecliptic pole.