M1 General Physics **2020-2021**
Major PNU

Particles

Mid-term exam

Documents allowed

Notes:
- **The subject is deliberately long**. It is not requested to reach the end to get a good mark!
- For convenience, one may freely put $c = 1$ everywhere.
- Space coordinates maybe freely denoted as (x, y, z) or (x^1, x^2, x^3).
- The standard notation according to which a quantity with superscript * is measured in the center-of-mass frame will be used.

Any process is characterized by the scattering amplitude which gives the amplitude of probability to pass from a given initial state to another given final state. Knowing the phase space for the initial and final states, the modulus square of this amplitude, and the flux of initial particles, one can compute the cross-section of the process, which is an experimental observable. Our purpose here is obtain several generic properties of the flux and of the amplitude.

1 Flux

Considering the scattering of a beam on a target, the flux term accounts for the fact that a target has a given number density, and that the beam has a given number density, made of particles of type B with a velocity v_B.
More generally, for a head-on scattering (let us say along the z axis), the masses, velocities, energies of particles A and B being denoted as m_A, $\vec{p}_A$, E_A and m_B, $\vec{p}_B$, E_B, it can be shown that this flux factor reads

$$2K = |\vec{v}_A - \vec{v}_B|\, 2E_A\, 2E_B\,. \tag{1}$$

1. Consider a boost along the z axis, the new frame F' moving with respect to initial one F with a velocity $\vec{\beta}c$. As usual, the rapidity ϕ of this boost is defined through the relation $\beta = \tanh\phi$.

(a) Give the expression of γ and $\gamma\beta$ as hyperbolic trigonometric functions of the rapidity ϕ.
Hint: $\cosh^2\phi - \sinh^2\phi = 1$.

Solution

One has

$$\begin{aligned}
\beta &= \tanh\phi\,,\\
\gamma &= \frac{1}{\sqrt{1-\tanh^2\phi}} = \cosh\phi\,,\\
\gamma\beta &= \sinh\phi\,.
\end{aligned}$$

(b) Write this boost using ϕ and then using β and γ.

Solution

A pure boost of rapidity ϕ along z can be written as

$$\left\{\begin{array}{rcrcr} x^{0\prime} & = & \cosh\phi\, x^0 & - & \sinh\phi\, x^3 \\ x^{3\prime} & = & -\sinh\phi\, x^0 & + & \cosh\phi\, x^3\,, \end{array}\right.$$

or equivalently, β being algebraic,

$$\left\{\begin{array}{rcrcr} x^{0\prime} & = & \gamma\, x^0 & - & \gamma\beta\, x^3 \\ x^{3\prime} & = & -\gamma\beta\, x^0 & + & \gamma\, x^3 \end{array}\right.$$

2. Prove that under such a boost, the velocity of a particle transforms as

$$v^{1\prime} = \frac{1}{\gamma}\frac{v^1}{1-\beta\frac{v^3}{c}} \qquad \text{and} \qquad v^{2\prime} = \frac{1}{\gamma}\frac{v^2}{1-\beta\frac{v^3}{c}}\,, \tag{2}$$

$$v^{3\prime} = \frac{v^3-\beta\, c}{1-\beta\frac{v^3}{c}}\,. \tag{3}$$

Hint: consider the differential of a boost.
Discuss the non-relativistic limit $v \ll c$, $\beta \ll 1$ and comment.

Solution

We have, through differentiation,

$$\left\{\begin{array}{rcrcr} dx^{0\prime} & = & \gamma\, dx^0 & - & \gamma\beta\, dx^1 \\ dx^{3\prime} & = & -\gamma\beta\, dx^0 & + & \gamma\, dx^3 \end{array}\right.$$

which gives, since $dx^0 = cdt$ and $dx^{0\prime} = cdt'$,

$$v^{3\prime} = c\frac{dx^{3\prime}}{dx^{0\prime}} = c\frac{-\gamma\beta dx^0 + \gamma dx^3}{\gamma dx^0 - \gamma\beta dx^3} = \frac{v^3-\beta\, c}{1-\beta\frac{v^3}{c}}\,.$$

Besides,

$$v^{1\prime} = c\frac{dx^{1\prime}}{dx^{0\prime}} = c\frac{dx^1}{\gamma dx^0 - \gamma\beta dx^1} = \frac{1}{\gamma}\frac{v^2}{1-\beta\frac{v^3}{c}}\,,$$

and similarly for $v^{2\prime}$.
At lowest order, we get $v^{1\prime} \sim v^1$, $v^{2\prime} \sim v^2$ and $v^{3\prime} \sim v^3 - \beta c$ as expected in the change of inertial frame in non-relativistic mechanics.

3. Show that the flux factor $2K$ given by Eq. (1) is invariant under boosts along the z axis.

Solution

Under a boost along the z axis, one has, according to Eq. (2),

$$v'_B - v'_A = (v_B - v_A)\frac{1-\beta^2}{(1-\beta v_A)(1-\beta v_B)}\,.$$

Besides,

$$\begin{aligned} E'_A &= \gamma\, E_A - \gamma\beta p^3_A\,, \\ E'_B &= \gamma\, E_B - \gamma\beta p^3_B\,, \end{aligned}$$

so that, since $v = p/E$,

$$E'_A\, E'_B = E_A\, E_B\, \gamma^2 \left(1 - \beta\frac{p^3_A}{E_A}\right)\left(1 - \beta\frac{p^3_B}{E_B}\right) = E_A\, E_B\, \gamma^2 (1-\beta v_A)(1-\beta v_B)\,,$$

and finally

$$E'_A\, E'_B\,(v'_B - v'_A) = E_A\, E_B\,(v_B - v_A)$$

where we have used the fact that $\gamma^2(1-\beta^2) = 1$, which implies that $2K$ is invariant under boost along the z axis.

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4. Prove that the flux factor can be expressed as

$$2K = 4(E_B|\vec{p}_A| + E_A|\vec{p}_B|)\,. \qquad (4)$$

Solution

Using $\vec{v} = \vec{p}/E$ one gets from Eq. (1)

$$2K = 4E_A\, E_B \left|\frac{\vec{p}_A}{E_A} - \frac{\vec{p}_B}{E_B}\right| = 4\,|E_B\vec{p}_A - E_A\vec{p}_B| = 4(E_B|\vec{p}_A| + E_A|\vec{p}_B|)$$

since $\vec{p}_A$ and $\vec{p}_B$ point in opposite directions.

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5. Demonstrate that

$$2K = 4\sqrt{(p_A\cdot p_B)^2 - m^2_A m^2_B}\,. \qquad (5)$$

Solution

For convenience, without loss of generality, let us assume that A moves in the z direction, and B in the $-z$ direction. Then, since $p_A = (E_A, 0, 0, |\vec{p}_A|)$ and $p_B = (E_B, 0, 0, -|\vec{p}_B|)$ we have on the one hand

$$(p_A \cdot p_B)^2 = (E_A E_B + |\vec{p}_A||\vec{p}_B|)^2 = E_A^2 E_B^2 + 2|\vec{p}_A||\vec{p}_B| E_A E_B + \vec{p}_A^{\,2}\, \vec{p}_B^{\,2}$$

and on the other hand

$$m_A^2 m_B^2 = (E_A^2 - \vec{p}_A^{\,2})(E_B^2 - \vec{p}_B^{\,2}) = E_A^2 E_B^2 - E_A^2 \vec{p}_B^{\,2} - E_B^2 \vec{p}_A^{\,2} + \vec{p}_A^{\,2} \vec{p}_B^{\,2}$$

and thus

$$(p_A \cdot p_B)^2 - m_A^2 m_B^2 = E_B^2 \vec{p}_A^{\,2} + E_A^2 \vec{p}_B^{\,2} + 2|\vec{p}_A||\vec{p}_B| E_A E_B = (E_B |\vec{p}_A| + E_A |\vec{p}_B|)^2$$

which ends the proof.

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6. In the center-of-mass frame, show that

$$2K = 4\, p_i^*\, W^*\,, \tag{6}$$

where $W^* = E_A^* + E_B^*$ is the total center of mass energy.

Solution

Starting from

$$\begin{cases} E_A^{*2} - p_i^{*2} = m_A^2 \\ E_B^{*2} - p_i^{*2} = m_B^2 \end{cases}$$

with $p_i^* \equiv |\vec{p}_A^{\,*}| = |\vec{p}_B^{\,*}|$, since $\vec{p}_A^{\,*} + \vec{p}_B^{\,*} = \vec{0}$, one deduces, using the expressions

$$p_A = (E_A^*, 0, 0, p_i^*) \quad \text{and} \quad p_B = (E_B^*, 0, 0, -p_i^*)\,,$$

that the flux factor (5) can be expressed as

$$\begin{aligned} 2\,K &= 4\left[(E_A^*\, E_B^* + p_i^{*2})^2 - (E_A^{*2} - p_i^{*2})(E_B^{*2} - p_i^{*2})\right]^{1/2} = 4\left[(E_A^* + E_B^*)^2 p_i^{*2})\right]^{1/2} \\ &= 4\,(E_A^* + E_B^*)\, p_i^* = 4W^* p_i^*\,. \end{aligned}$$

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2 Mandelstam variables

Any 2 body $\to$ 2 body scattering between particles P_1, P_2, producing particles P_3 and P_4,

$$P_1(p_1)\, P_2(p_2) \to P_3(p_3)\, P_3(p_4) \tag{7}$$

is completely characterized, if one does not take into account spin effects, by the Mandelstam variables defined by

$$\boxed{\begin{aligned} s &\equiv (p_1 + p_2)^2 = (p_3 + p_4)^2\,, \\ t &\equiv (p_1 - p_3)^2 = (p_2 - p_4)^2\,, \\ u &\equiv (p_1 - p_4)^2 = (p_2 - p_3)^2\,, \end{aligned}} \tag{8}$$

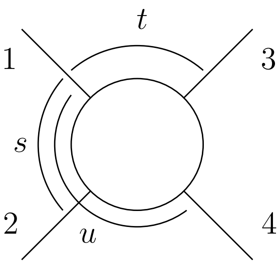

Figure 1: Mandelstam variables for a 2 body → 2 body process.

the various equivalent expressions coming from energy-momentum conservation. These variables are illustrated on Fig. 1. Each of these variables can be considered as a "s"-variable for a crossed-channel process:

		"s"-variable	"t"-variable	"u"-variable
s-channel:	$1+2 \to 3+4$	s	t	u
t-channel:	$1+\bar{3} \to \bar{2}+4$	t	s	u
$u-$channel:	$1+\bar{4} \to 3+\bar{2}$	u	t	s

(9)

Indeed, a particle i of momentum p_i with $p^0 > 0$ should be considered as its antiparticle $\bar{i}$, of momentum $-p_i$ when $p^0 < 0$. The *same amplitude* $\mathcal{M}(s,\, t,\, u)$ thus describes these 3 reactions, as well as every desintegration process 1 body → 3 body (for example $1 \to \bar{2}+3+4$) and every back reaction (for example $3+4 \to 1+2$), by analytic continuation on variables s, t, u.

1. For further use, we denote, using the fact that $2p_1 \cdot p_2 = s - m_1^2 - m_2^2$,

$$\lambda(s, m_1^2, m_2^2) = 4[(p_1 \cdot p_2)^2 - m_1^2 m_2^2]\,. \tag{10}$$

We thus have

$$K(s) = \sqrt{\lambda(s, m_1^2, m_2^2)}\,. \tag{11}$$

(a) Show that

$$\lambda(s,\, m_1^2,\, m_2^2) = [s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]\,. \tag{12}$$

Solution

We have

$$\begin{aligned}
\lambda(s,\, m_1^2,\, m_2^2) &= 4[(p_1 \cdot p_2)^2 - m_1^2\, m_2^2] = [2(p_1 \cdot p_2) - 2m_1 m_2][2(p_1 \cdot p_2) + 2m_1 m_2] \\
&= [s - m_1^2 - m_2^2 - 2m_1 m_2][s - m_1^2 - m_2^2 + 2m_1 m_2] \\
&= [s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]\,.
\end{aligned}$$

(b) Give the expression of $\lambda(s, 0, M)$ when one mass vanishes, and of $\lambda(s, 0, 0)$ when both vanish.

Solution

$$\begin{aligned}
\lambda(s,\,0,\,M) &= (s-M^2)^2 &&: \text{ one of the masses vanishes}\\
\lambda(s,\,0,\,0) &= s^2 &&: \text{ the two masses vanish.}
\end{aligned}$$

(c) Show that in the center-of-mass frame, $p_i^* = |\vec{p}_1| = |\vec{p}_2|$ and $p_f^* = |\vec{p}_3| = |\vec{p}_4|$ have very simple relativistic invariant forms:

$$p_i^* = \frac{\sqrt{\lambda(s,\,m_1^2,\,m_2^2)}}{2\sqrt{s}}\,, \tag{13}$$

and

$$p_f^* = \frac{\sqrt{\lambda(s,\,m_3^2,\,m_4^2)}}{2\sqrt{s}}\,. \tag{14}$$

Solution

This expression of p_i^* is obvious from the fact that $W^* = \sqrt{s}$ and from the relations (6) and (11). The expression of p_f^* is obtained in a similar way by considering the particles 3 and 4: the proof which led to I.6 can be reproduced identically for the particles 3 and 4, leading to

$$2\sqrt{\lambda(s,\,m_3^2,\,m_4^2)} = 4W^* p_f^* = 4\sqrt{s}\,p_f^*\,,$$

therefore ending the proof.

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2. Prove that

$$s+t+u = \sum_i m_i^2\,. \tag{15}$$

Hint: compute $2(s+t+u)$ is a “democratic way“, using Eq. (8).

Solution

This is proven by computing $2(s+t+u)$, summing the 6 terms of (8) and then using the energy-momentum conservation:

$$\begin{aligned}
2\,(s+t+u) &= 3\sum_i m_i^2 + 2\,p_1\cdot p_2 + 2\,p_3\cdot p_4 - 2\,p_1\cdot p_3 - 2\,p_2\cdot p_4 - 2\,p_1\cdot p_4 - 2\,p_2\cdot p_3\\
&= 3\sum_i m_i^2 + 2\,p_1\cdot p_2 + 2\,p_3\cdot p_4 - 2\,(p_1+p_2)\cdot(p_3+p_4)\\
&= 3\sum_i m_i^2 + (s-m_1^2-m_2^2) + (s-m_3^2-m_4^2) - 2s = 2\sum_i m_i^2\,.
\end{aligned}$$

Consequently, the scattering amplitude only depends on two independent variables. One conventionally writes

$$\mathcal{M} = \mathcal{M}(s,\, t)\,.$$

3. (a) In the center-of-mass frame, show that

$$\begin{aligned} E_1^* &= \frac{s+m_1^2-m_2^2}{2\sqrt{s}}\,, \qquad & E_3^* &= \frac{s+m_3^2-m_4^2}{2\sqrt{s}}\,, \\ E_2^* &= \frac{s+m_2^2-m_1^2}{2\sqrt{s}}\,, \qquad & E_4^* &= \frac{s+m_4^2-m_3^2}{2\sqrt{s}}\,. \end{aligned} \tag{16}$$

Hint: use Eqs. (13) and (14).

Solution

For example, in order to express E_1^*, it is enough to combine (13) and $E_1^{*2} = \vec{p}_1^{\,*2} + m_1^2$. This gives

$$\begin{aligned} E_1^* &= \sqrt{p_i^{*2}+m_1^2} = \sqrt{\frac{\lambda(s,m_1^2,m_2^2)+4m_1^2 s}{4s}} \\ &= \frac{\sqrt{[s-(m_1+m_2)^2][s-(m_1-m_2)^2]+4m_1^2 s}}{2\sqrt{s}} \\ &= \frac{\sqrt{s^2+(m_1^2-m_2^2)^2-2s(m_1^2+m_2^2)+4m_1^2 s}}{2\sqrt{s}} \\ &= \frac{\sqrt{s^2+(m_1^2-m_2^2)^2+2s(m_1^2-m_2^2)}}{2\sqrt{s}} \\ &= \frac{s+m_1^2-m_2^2}{2\sqrt{s}}\,. \end{aligned}$$

The proofs are similar for the other E_i^*: for E_2^*, starting from E_1^* one should just permute the roles of particles 1 and 2, for E_3^*, use p_f^* instead of p_i^* and then for E_4^* permute the roles of particles 3 and 4.

(b) Prove that the following threshold conditions should be satisfied, in each indicated channel:

$$\begin{aligned} &s\text{-channel:} \quad 1+2 \to 3+4 \;: \quad s \geqslant (m_1+m_2)^2 \;\text{ and }\; (m_3+m_4)^2 \\ &t\text{-channel:} \quad 1+\bar{3} \to \bar{2}+4 \;: \quad t \geqslant (m_1+m_3)^2 \;\text{ and }\; (m_2+m_4)^2 \\ &u\text{-channel:} \quad 1+\bar{4} \to 3+\bar{2} \;: \quad u \geqslant (m_1+m_4)^2 \;\text{ and }\; (m_2+m_3)^2 \end{aligned} \tag{17}$$

Solution

In the center-of-mass frame, we have, on the one hand

$$\sqrt{s} = W^* = \sum_{k=1,2} E_k^* = \sum_{k=1,2} \sqrt{p_k^{*2} + m_k^2} \geqslant m_1 + m_2\,,$$

since for each k, $p_k^* \geqslant 0$, and on the other hand

$$\sqrt{s} = W^* = \sum_{k=3,4} E_k^* = \sum_{k=3,4} \sqrt{p_k^{*2} + m_k^2} \geqslant m_3 + m_4\,,$$

since for each k, $p_k^* \geqslant 0$. This leads to the constraint $s \geqslant (m_1+m_2)^2$ and $s \geqslant (m_3+m_4)^2$. Note that this can be equivalently obtained from the relations (13) and (14): for p_i^* and p_f^* to be defined, i.e. p_i^{*2} and p_f^{*2} to be positive, a necessary and sufficient condition is that $\lambda(t, m_1^2, m_2^2) \geqslant 0$ and $\lambda(t, m_3^2, m_4^2) \geqslant 0$ respectively, i.e. $s \geqslant (m_1+m_2)^2$ and $s \geqslant (m_3+m_4)^2$ respectively.
There is still a less direct but equivalent way to get the same result: require that $E_1^* \geqslant m_1$ and $E_2^* \geqslant m_2$. Using the relations (16) leads after a careful analysis to the same constraint $s \geqslant (m_1+m_2)^2$, and a similar analysis for $E_3^* \geqslant m_3$ and $E_4^* \geqslant m_4$ leads to the constraint $s \geqslant (m_3+m_4)^2$.
Analogous discussions and results hold for crossed-channels:
- in the $t-$channel one should consider $\lambda(t, m_1^2, m_3^2)$ and $\lambda(t, m_2^2, m_4^2)$. This leads to the conditions $t \geqslant (m_1+m_3)^2$ and $t \geqslant (m_2+m_4)^2$.
- in the $u-$channel one should consider $\lambda(u, m_1^2, m_4^2)$ and $\lambda(u, m_2^2, m_3^2)$. This leads to the conditions $u \geqslant (m_1+m_4)^2$ and $u \geqslant (m_2+m_3)^2$.

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4. The diffusion angle is by definition the scattering angle between particles 1 and 3, i.e. the scattering angle in the $s-$channel.

(a) Prove that

$$\cos\theta = \frac{t - m_1^2 - m_3^2 + 2\,E_1 E_3}{2\,|\vec{p}_1||\vec{p}_3|}\,. \tag{18}$$

Solution

For that purpose, let us rewrite the diffusion angle θ for the $s-$channel process:

$$t = (p_1 - p_3)^2 = m_1^2 + m_3^2 - 2\,p_1 \cdot p_3 = m_1^2 + m_3^2 - 2\,E_1\,E_3 + 2\,|\vec{p}_1||\vec{p}_3|\cos\theta\,. \tag{19}$$

that is

$$\cos\theta = \frac{t - m_1^2 - m_3^2 + 2\,E_1 E_3}{2\,|\vec{p}_1||\vec{p}_3|}\,.$$

(b) In an arbitrary reference frame, for fixed s and E_1, explain why the discussion on the maximal/minimal values of $\cos\theta$ as a function of t is in general complicate.

Solution

In an arbitrary frame, the variables E_3 and $|\vec{p}_3|$ are $\cos\theta$-dependent, therefore the expression of $\cos\theta$ as a function of t is very complicate, making the discussion somehow tricky.

(c) In the center-of-mass frame, one may write

$$\cos\theta^* \quad = \quad \frac{t - m_1^2 - m_3^2 + 2\, E_1^* E_3^*}{2\, p_1^*\, p_3^*}\,. \tag{20}$$

(i) At fixed values of s and E_1, to which limit in t corresponds the forward reaction $\theta^* = 0$?

Solution

The forward reaction $\theta^* = 0$ corresponds to the maximal algebraic value for t.

(ii) At fixed values of s and E_1, to which limit in u corresponds the backward reaction $\theta^* = \pi$?

Solution

We have

$$\cos\theta^* \quad = \quad \frac{m_2^2 + m_4^2 - s - u + 2\, E_1^* E_3^*}{2\, p_1^*\, p_3^*}\,,$$

and thus the backward reaction $\theta^* = \pi$ corresponds to the maximal algebraic value for u.

(d) Show that

$$\cos\theta^* = \frac{s^2 + s(2t - m_1^2 - m_2^2 - m_3^2 - m_4^2) + (m_1^2 - m_2^2)(m_3^2 - m_4^2)}{\sqrt{\lambda(s, m_1^2, m_2^2)\,\lambda(s, m_3^2, m_4^2)}}\,. \tag{21}$$

Solution

In this center-of-mass frame, the angle θ^* can be expressed as:

$$\begin{aligned}
\cos\theta^* &= \frac{t - m_1^2 - m_3^2 + 2\, E_1^* E_3^*}{2\, p_1^*\, p_2^*} \\
&= \left[t - m_1^2 - m_3^2 + \frac{(s + m_1^2 - m_2^2)(s + m_3^2 - m_4^2)}{2s}\right] \frac{2s}{\sqrt{\lambda(s, m_1^2, m_2^2)\,\lambda(s, m_3^2, m_4^2)}} \\
&= \frac{s^2 + s(2t - m_1^2 - m_2^2 - m_3^2 - m_4^2) + (m_1^2 - m_2^2)(m_3^2 - m_4^2)}{\sqrt{\lambda(s, m_1^2, m_2^2)\,\lambda(s, m_3^2, m_4^2)}}\,.
\end{aligned}$$

(e) In the large energy limit where $s \sim -u \gg -t$, m_i^2, called *Regge limit*, show that $\theta^* \to 0$.

Solution

In this limit, since $\lambda(s, m_i^2) \sim s^2$ we get $\cos\theta^* \sim 1$ *i.e.* $\theta^* \to 0$.

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5. Physical region for t.
(a) For a given s, show that the physical region for t looks like $t^- \leqslant t \leqslant t^+$ and give the values of t^- and t^+.

Solution

The constraint $-1 \leqslant \cos\theta^* \leqslant 1$ provides the range in t:

$$t^- \leqslant t \leqslant t^+ \quad \text{with}$$
$$t^\pm = m_1^2 + m_3^2 - \frac{1}{2s}\{(s + m_1^2 - m_2^2)(s + m_3^2 - m_4^2) \mp \sqrt{\lambda(s, m_1^2, m_2^2)\,\lambda(s, m_3^2, m_4^2)}\}\,.$$

(b) In the case of equal masses ($m_i^2 = m^2$), give the physical region in t.

Solution

One gets $\lambda(s,\, m^2,\, m^2) = s(s - 4m^2)$ and thus

$$\cos\theta^* = 1 + \frac{2t}{s - 4m^2}\,.$$

The physical region for the reaction is then given by the conditions

$$s \geq 4m^2 \qquad \text{and} \qquad t^- = 4m^2 - s \leqslant t \leqslant t^+ = 0\,.$$

(c) Still in the case of equal masses ($m_i^2 = m^2$), express t and u as functions of s, m^2 and $\cos\theta^*$. Comment.

Solution

One can check that

$$\begin{aligned} t &= -\frac{s - 4m^2}{2}(1 - \cos\theta^*)\,, \\ u &= -\frac{s - 4m^2}{2}(1 + \cos\theta^*) \end{aligned}$$

where we have used the fact that $s + t + u = 4m^2$ to pass from the first to the second expression.
One then recovers the fact that
- t negative, small in absolute value, corresponding to θ^* small (forward)
- u negative, small in absolute value, corresponding to $\pi - \theta^*$ small (forward).
