M1 General Physics **2024-2025**

Particles

Final exam

January 8th 2025

Documents allowed. No laptops. No cell phones. No tablets.

Notes:
- **The subject is deliberately long**. It is not requested to reach the end to get a good mark!
- We use the system of unit in which $c = 1, \hbar = 1$, $\epsilon_0 = 1$, $\mu_0 = 1$.
- Space coordinates may be freely denoted as (x, y, z) or (x^1, x^2, x^3).
- Drawings are welcome!

The exercise and the problem are independent.

1 Exercise: Energy-momentum tensor of a scalar field theory

We consider the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \,. \tag{1}$$

where m and λ are constants.

1. Dimension analysis.

i) What is the mass dimension of $\mathcal{L}$?

ii) What is the mass dimension of ∂_μ and ∂^μ?

iii) What is the mass dimension of the constant m?

iv) What is the mass dimension of ϕ?

v) What is the mass dimension of λ?

2. Write the Euler-Lagrange equations for the field ϕ.

3. The energy-momentum tensor is defined as

$$T^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\partial^\nu \phi - g^{\mu\nu}\mathcal{L} \,. \tag{2}$$

i) Compute the energy-momentum tensor corresponding to the Lagrangian density (1).

ii) Explain why it is conserved, and write the equation of conservation.

iii) Check directly that $T^{\mu\nu}$ is conserved.

iv) What are the symmetry properties of $T^{\mu\nu}$?

2 Problem: Electromagnetic fields in two frames

2.1 Four-velocity and four-vector force

Consider a frame R' traveling with speed $\vec{\beta} = \vec{v}$ ($c = 1$) with respect to the frame R. For convenience, $\vec{v}$ can be taken along the x-axis.

We denote by $\vec{u} = (u_x, u_y, u_z)$ the velocity of a particle in frame R and $\vec{u}\,' = (u'_x, u'_y, u'_z)$ the corresponding acceleration of this particle in frame R'.

1. Briefly show that

$$u'_x = \frac{u_x - \beta}{1 - \beta u_x} \tag{3}$$

$$u'_y = \frac{1}{\gamma}\frac{u_y}{1 - \beta u_x} \tag{4}$$

$$u'_z = \frac{1}{\gamma}\frac{u_z}{1 - \beta u_x}\,. \tag{5}$$

where $\gamma = 1/\sqrt{1-\beta^2}$.

2. Show that

$$1 - u'^2_x = \frac{(1-\beta^2)(1-u_x^2)}{(1-\beta u_x)^2}\,. \tag{6}$$

3. Compute $1 - \vec{u}\,'^2$.

4. Show that the two ratios $\dfrac{1-\vec{u}\,'^2}{1-\vec{u}\,^2}$ and $\dfrac{1-u'^2_x}{1-u_x^2}$ are related in a very simple way.

The above particle has a mass m. Its four-momentum is $p^\mu = (E, \vec{p})$ in frame R, and $p'^\mu = (E', \vec{p}\,')$ in frame R'. In frame R, we define

$$\mathcal{F}^\mu = \frac{dp^\mu}{d\tau}\,. \tag{7}$$

5. Why $\mathcal{F}^\mu$ is a four-vector? Why can we call it the four-vector force in frame R?

6. Denoting as $\vec{F}$ the force experienced by the particle in frame R, show that

$$\mathcal{F}^\mu = \Gamma(u)(\vec{F}\cdot\vec{u}, \vec{F}) \tag{8}$$

with $u = \|\vec{u}\|$ and $\Gamma(u) = 1/\sqrt{1-u^2}$.

We similarly define $\mathcal{F}'^{\mu}$ in frame R', and $\vec{F}\,'$ the force experienced by the particle in this frame.
7. Relate the components of $\mathcal{F}'^{\mu}$ in terms of $\mathcal{F}^{\mu}$.

8. Prove finally that

$$\vec{F}'_{\parallel} = \frac{1}{1-\vec{\beta}\cdot\vec{u}}\left[\vec{F}_{\parallel} - \vec{\beta}(\vec{F}\cdot\vec{u})\right], \tag{9}$$

$$\vec{F}'_{\perp} = \frac{1}{\gamma(1-\vec{\beta}\cdot\vec{u})}\vec{F}_{\perp} \tag{10}$$

where $\parallel$ and $\perp$ correspond respectively to the component collinear and transverse to $\vec{\beta}$.

9. Show that the time derivative of the energy in frames R and R' are related by the equation

$$\frac{dE'}{dt'} = \frac{1}{1-\vec{\beta}\cdot\vec{u}}\left[\frac{dE}{dt} - \vec{F}\cdot\vec{\beta}\right]. \tag{11}$$

10. Show that the previous relation, written as

$$\vec{F}'\cdot\vec{u}\,' = \frac{1}{1-\vec{\beta}\cdot\vec{u}}\left[\vec{F}\cdot\vec{u} - \vec{F}\cdot\vec{\beta}\right], \tag{12}$$

can be directly obtained from the transformation laws for $\vec{u}_{\parallel}$ and $\vec{u}_{\perp}$ obtained in question 1. and for $\vec{F}_{\parallel}$ and $\vec{F}_{\perp}$ obtained in question 8.

2.2 Electromagnetic field created by an infinite plane

In an inertial reference frame R, consider an infinite plane of electric charges, of constant surface density σ. We choose an origin and a system of axes such it that coincides with the Oxy plane. These charges move at constant velocity $\vec{v}$ in the direction of the Ox axis. We denote $\vec{e}_x, \vec{e}_y, \vec{e}_z$ the unit vectors on each of the axes Ox, Oy, Oz.

Preliminary (bonus).
11. Show that the electromagnetic field, in the reference frame R, is given, for $z > 0$, by

$$\vec{E} = \frac{\sigma}{2}\vec{e}_z \quad \text{and} \quad \vec{B} = -\frac{\sigma v}{2}\vec{e}_y \tag{13}$$

What are the expressions of $\vec{E}$ and $\vec{B}$ for $z < 0$?

Let R' be another inertial frame, with moves at a constant velocity $\vec{V}$ in the direction of $x > 0$ with respect to the frame R.

12. Give the expression of $\vec{v}\,'$ in frame R', as a function of V and v.

13. Compute $\Gamma' = 1/\sqrt{1-v'^2}$ in terms of $\Gamma = 1/\sqrt{1-v^2}$, $\gamma = 1/\sqrt{1-V^2}$, V and v.

14. Show that the surface density σ' in frame R' reads

$$\sigma' = \sigma \frac{1 - Vv}{\sqrt{1-V^2}}\,. \tag{14}$$

15. We now consider the frame R'.
i) By analogy with the expression given above for the electromagnetic field in the frame R, give the expression for the electromagnetic field $(\vec{E}\,', \vec{B}\,')$ in frame R'.

ii) Check that it is consistent with Lorentz's transformation of fields.

iii) What happens if $\vec{V} = \vec{v}$?

16. A particle, of charge q and velocity $\vec{v}$ in frame R, is subjected to the action of this electromagnetic field.
i) What force does it experience in frame R?

ii) And in its rest frame?

iii) Compare these two forces.

iv) Does the result agree with that obtained using the quadrivector force?