M1 General Physics **2024-2025**

Particles

Final exam

January 8th 2025

Documents allowed. No laptops. No cell phones. No tablets.

Notes:
- **The subject is deliberately long**. It is not requested to reach the end to get a good mark!
- We use the system of unit in which $c = 1, \hbar = 1$, $\epsilon_0 = 1$, $\mu_0 = 1$.
- Space coordinates may be freely denoted as (x, y, z) or (x^1, x^2, x^3).
- Drawings are welcome!

The exercise and the problem are independent.

1 Exercise: Energy-momentum tensor of a scalar field theory

We consider the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!}\phi^4 \,. \tag{1}$$

where m and λ are constants.

1. Dimension analysis.

i) What is the mass dimension of $\mathcal{L}$?

Solution

The action is dimensionless, therefore $\mathcal{L}$ has mass dimension M^4 since $[d^4x] = M^{-4}$.

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ii) What is the mass dimension of ∂_μ and ∂^μ?

Solution

There are both of the dimension of a mass.

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iii) What is the mass dimension of the constant m?

Solution

The first and second terms in (1) should have dimension M^4 and both contains ϕ^2. Thus m has the dimension of ∂_μ, i.e. it is a mass.

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iv) What is the mass dimension of ϕ?

Solution

Either starting from term 1 or term 2 in (1), one sees that $[\phi] = M$.

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v) What is the mass dimension of λ?

Solution

From the fact that $[\phi] = M$ and $\mathcal{L}$ has mass dimension M^4, one gets that λ is dimensionless.

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2. Write the Euler-Lagrange equations for the field ϕ.

Solution

The Euler-Lagrange equation reads

$$\frac{\delta \mathcal{L}}{\delta \phi} = \partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \,.$$

We have

$$\frac{\delta \mathcal{L}}{\delta \phi} = -m^2 \phi - \frac{\lambda}{3!}\phi^3$$

and

$$\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} = \partial^\mu \phi$$

so that

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} = \Box \phi \,.$$

Thus,

$$\Box \phi + m^2 \phi + \frac{\lambda}{3!}\phi^3 = 0 \,.$$

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3. The energy-momentum tensor is defined as

$$T^{\mu\nu} = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial^\nu\phi - g^{\mu\nu}\mathcal{L}\,. \tag{2}$$

i) Compute the energy-momentum tensor corresponding to the Lagrangian density (1).

Solution

The energy-momentum tensor reads

$$T^{\mu\nu} = (\partial^\mu\phi)(\partial^\nu\phi) - \frac{1}{2}g^{\mu\nu}(\partial_\sigma\phi)(\partial^\sigma\phi) + \frac{1}{2}g^{\mu\nu}m^2\phi^2 + g^{\mu\nu}\frac{\lambda}{4!}\phi^4\,.$$

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ii) Explain why it is conserved, and write the equation of conservation.

Solution

The Lagrangian density (1) does not depend explicitly on coordinates x^μ, therefore the action is invariant under any global space-time translation. From Noether theorem, there is an associated conserved current, which is $T^{\mu\nu}$. The fact that $T^{\mu\nu}$ is conserved reads

$$\partial_\mu T^{\mu\nu} = 0\,.$$

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iii) Check directly that $T^{\mu\nu}$ is conserved.

Solution

We compute directly

$$\begin{aligned}\partial_\mu T^{\mu\nu} &= \partial_\mu\left[(\partial^\mu\phi)(\partial^\nu\phi) - \frac{1}{2}g^{\mu\nu}(\partial_\sigma\phi)(\partial^\sigma\phi) + \frac{1}{2}g^{\mu\nu}m^2\phi^2 + g^{\mu\nu}\frac{\lambda}{4!}\phi^4\right]\\ &= (\Box\phi)(\partial^\nu\phi) + (\partial_\mu\phi)(\partial^\mu\partial^\nu\phi) - (\partial^\nu\partial_\sigma\phi)(\partial^\sigma\phi) + m^2\phi\,\partial^\nu\phi + \frac{\lambda}{3!}\phi^3\partial^\nu\phi\,.\end{aligned}$$

In this equation, the second and third terms cancel since μ and σ are dummy indices, while the first, fourth and fifth terms compensate after factorizing $\partial^\nu\phi$ and using Euler-Lagrange equation.

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iv) What are the symmetry properties of $T^{\mu\nu}$?

Solution

The tensor $T^{\mu\nu}$ is symmetric, as a direct consequence of the conservation of the angular-momentum tensor, which implies the symmetry of $T^{\mu\nu}$ for a scalar theory.

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2 Problem: Electromagnetic fields in two frames

2.1 Four-velocity and four-vector force

Consider a frame R' traveling with speed $\vec{\beta} = \vec{v}$ ($c = 1$) with respect to the frame R. For convenience, $\vec{v}$ can be taken along the x-axis.

We denote by $\vec{u} = (u_x, u_y, u_z)$ the velocity of a particle in frame R and $\vec{u}\,' = (u'_x, u'_y, u'_z)$ the corresponding acceleration of this particle in frame R'.

1. Briefly show that

$$u'_x = \frac{u_x - \beta}{1 - \beta u_x} \tag{3}$$

$$u'_y = \frac{1}{\gamma}\frac{u_y}{1 - \beta u_x} \tag{4}$$

$$u'_z = \frac{1}{\gamma}\frac{u_z}{1 - \beta u_x}\,. \tag{5}$$

where $\gamma = 1/\sqrt{1-\beta^2}$.

Solution

We have, through differentiation,

$$\begin{cases} dt' = \gamma\, dt - \gamma\beta\, dx \\ dx' = -\gamma\beta\, dt + \gamma\, dx \end{cases}$$

which gives

$$u'_x = \frac{dx'}{dt'} = \frac{-\gamma\beta dt + \gamma dx}{\gamma dt - \gamma\beta dx} = \frac{u_x - \beta}{1 - \beta u_x}\,.$$

Besides,

$$u'_y = \frac{dy'}{dt'} = \frac{dy}{\gamma dt - \gamma\beta dy} = \frac{1}{\gamma}\frac{u_y}{1 - \beta u_x}\,,$$

and similarly

$$u'_z = \frac{dz'}{dt'} = \frac{dz}{\gamma dt - \gamma\beta dz} = \frac{1}{\gamma}\frac{u_z}{1 - \beta u_x}\,.$$

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2. Show that

$$1 - u'^2_x = \frac{(1-\beta^2)(1-u_x^2)}{(1-\beta u_x)^2}\,. \tag{6}$$

Solution

We have

$$1 - u_x'^2 = 1 - \left(\frac{u_x - \beta}{1 - \beta u_x}\right)^2 = \frac{1 + \beta^2 u_x^2 - 2\beta u_x - u_x^2 + 2\beta u_x - \beta^2}{(1 - \beta u_x)^2} = \frac{(1 - \beta^2)(1 - u_x^2)}{(1 - \beta u_x)^2} \,.$$

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3. Compute $1 - \vec{u}\,'^2$.

Solution

Using the fact that $1/\gamma^2 = 1 - \beta^2$, let us first compute

$$\begin{aligned}
\vec{u}\,'^2 &= u_x'^2 + u_y'^2 + u_z'^2 = \frac{1}{(1 - \beta u_x)^2}\left[(u_x - \beta)^2 + \frac{u_y^2}{\gamma^2} + \frac{u_z^2}{\gamma^2}\right] \\
&= \frac{1}{(1 - \beta u_x)^2}\left[u_x^2 + \beta^2 - 2\beta u_x + u_y^2 - \beta^2 u_y^2 + u_z^2 - \beta^2 u_z^2\right] \\
&= \frac{1}{(1 - \beta u_x)^2}\left[\vec{u}^{\,2} + \beta^2 - 2\beta u_x + \beta^2 u_x^2 - \beta^2 \vec{u}^{\,2}\right] \\
&= \frac{1}{(1 - \beta u_x)^2}\left[(1 - \beta^2)\vec{u}^{\,2} + \beta(\beta - 2u_x + \beta u_x^2)\right] .
\end{aligned}$$

Thus,

$$\begin{aligned}
1 - \vec{u}\,'^2 &= \frac{(1 - \beta u_x)^2 - (1 - \beta^2)\vec{u}^{\,2} - \beta(\beta - 2u_x + \beta u_x^2)}{(1 - \beta u_x)^2} \\
&= \frac{1 - 2\beta u_x + \beta^2 u_x^2 - (1 - \beta^2)\vec{u}^{\,2} - \beta^2 + 2\beta u_x - \beta^2 u_x^2}{(1 - \beta u_x)^2} \\
&= \frac{(1 - \beta^2)(1 - \vec{u}^{\,2})}{(1 - \beta u_x)^2} \,.
\end{aligned}$$

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4. Show that the two ratios $\dfrac{1 - \vec{u}\,'^2}{1 - \vec{u}^{\,2}}$ and $\dfrac{1 - u_x'^2}{1 - u_x^2}$ are related in a very simple way.

Solution

From questions 2 and 4, we have

$$\frac{1 - \vec{u}\,'^2}{1 - \vec{u}^{\,2}} = \frac{1 - u_x'^2}{1 - u_x^2} \,.$$

The above particle has a mass m. Its four-momentum is $p^\mu = (E, \vec{p})$ in frame R, and $p'^\mu = (E', \vec{p}\,')$ in frame R'. In frame R, we define

$$\mathcal{F}^\mu = \frac{dp^\mu}{d\tau} \,. \tag{7}$$

5. Why $\mathcal{F}^\mu$ is a four-vector? Why can we call it the four-vector force in frame R?

Solution

It is the derivative of the four-momentum with respect to the proper time. The proper time being a Lorentz invariant, it is thus a four-vector. It is a natural relativistic extension of the usual relation

$$\vec{F} = \frac{d\vec{p}}{dt},$$

thus the name.

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6. Denoting as $\vec{F}$ the force experienced by the particle in frame R, show that

$$\mathcal{F}^\mu = \Gamma(u)(\vec{F}\cdot\vec{u}, \vec{F}) \tag{8}$$

with $u = \|\vec{u}\|$ and $\Gamma(u) = 1/\sqrt{1-u^2}$.

Solution

First, one has

$$\frac{dE}{d\tau} = \frac{dt}{d\tau}\frac{dE}{dt} = \Gamma(u)\vec{F}\cdot\vec{u}\,.$$

Second,

$$\frac{d\vec{p}}{d\tau} = \frac{dt}{d\tau}\frac{d\vec{p}}{dt} = \Gamma(u)\vec{F}\,.$$

We similarly define $\mathcal{F}'^\mu$ in frame R', and $\vec{F}\,'$ the force experienced by the particle in this frame.

7. Relate the components of $\mathcal{F}'^\mu$ in terms of $\mathcal{F}^\mu$.

Solution

We have

$$\begin{aligned}
\mathcal{F}'^0 &= \gamma\mathcal{F}^0 - \gamma\beta\mathcal{F}^x,\\
\mathcal{F}'^x &= -\gamma\beta\mathcal{F}^0 + \gamma\mathcal{F}^x,\\
\mathcal{F}'^y &= \mathcal{F}^y,\\
\mathcal{F}'^z &= \mathcal{F}^z.
\end{aligned}$$

or in a more compact way

$$\begin{aligned}
\mathcal{F}'^0 &= \gamma\mathcal{F}^0 - \gamma\beta\mathcal{F}_\parallel,\\
\mathcal{F}'_\parallel &= -\gamma\beta\mathcal{F}^0 + \gamma\mathcal{F}_\parallel,\\
\mathcal{F}'_\perp &= \mathcal{F}_\perp\,.
\end{aligned}$$

8. Prove finally that

$$\vec{F}'_{\parallel} = \frac{1}{1-\vec{\beta}\cdot\vec{u}}\left[\vec{F}_{\parallel} - \vec{\beta}(\vec{F}\cdot\vec{u})\right] \tag{9}$$

$$\vec{F}'_{\perp} = \frac{1}{\gamma(1-\vec{\beta}\cdot\vec{u})}\vec{F}_{\perp} \tag{10}$$

where $\parallel$ and $\perp$ correspond respectively to the component collinear and transverse to $\vec{\beta}$.

Solution

Let us denote $\Gamma = \Gamma(u)$ and $\Gamma' = \Gamma(u')$. From the previous question, one gets

$$\Gamma'\vec{F}'_{\parallel} = \gamma\Gamma[\vec{F}_{\parallel} - \beta\vec{F}\cdot\vec{u}]$$
$$\Gamma'\vec{F}'_{\perp} = \Gamma\vec{F}_{\perp}$$

Besides, question 2. implies the relationship

$$\Gamma' = \Gamma\gamma(1-\vec{\beta}\cdot\vec{u})\,,$$

so that

$$\vec{F}'_{\parallel} = \frac{1}{1-\vec{\beta}\cdot\vec{u}}\left[\vec{F}_{\parallel} - \vec{\beta}(\vec{F}\cdot\vec{u})\right]\,,$$
$$\vec{F}'_{\perp} = \frac{1}{\gamma(1-\vec{\beta}\cdot\vec{u})}\vec{F}_{\perp}\,.$$

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9. Show that the time derivative of the energy in frames R and R' are related by the equation

$$\frac{dE'}{dt'} = \frac{1}{1-\vec{\beta}\cdot\vec{u}}\left[\frac{dE}{dt} - \vec{F}\cdot\vec{\beta}\right]. \tag{11}$$

Solution

From question 7. one gets

$$\Gamma'\frac{dE'}{dt'} = \gamma\Gamma\frac{dE}{dt} - \gamma\beta\Gamma F_{\parallel}$$

and thus

$$\frac{dE'}{dt'} = \frac{\gamma\Gamma}{\Gamma'}\frac{dE}{dt} - \frac{\gamma\beta\Gamma}{\Gamma'}F_{\parallel} = \frac{1}{1-\vec{\beta}\cdot\vec{u}}\frac{dE}{dt} - \frac{\vec{F}\cdot\vec{\beta}}{1-\vec{\beta}\cdot\vec{u}}\,.$$

10. Show that the previous relation, written as

$$\vec{F}' \cdot \vec{u}\,' = \frac{1}{1 - \vec{\beta} \cdot \vec{u}} \left[\vec{F} \cdot \vec{u} - \vec{F} \cdot \vec{\beta} \right], \tag{12}$$

can be directly obtained from the transformation laws for $\vec{u}_\parallel$ and $\vec{u}_\perp$ obtained in question 1. and for $\vec{F}_\parallel$ and $\vec{F}_\perp$ obtained in question 8.

Solution

From question 1. we have

$$\begin{aligned} \vec{u}\,'_\parallel &= \frac{\vec{u}_\parallel - \vec{\beta}}{1 - \vec{\beta} \cdot \vec{u}}, \\ \vec{u}\,'_\perp &= \frac{1}{\gamma} \frac{\vec{u}_\perp}{1 - \vec{\beta} \cdot \vec{u}}. \end{aligned}$$

Thus,

$$\vec{F}' \cdot \vec{u}\,' = \vec{F}'_\parallel u'_\parallel + \vec{F}'_\perp \cdot \vec{u}\,'_\perp = \frac{1}{(1 - \vec{\beta} \cdot \vec{u})^2} \left[\left(F_\parallel - \beta (\vec{F} \cdot \vec{u}) \right) (u_\parallel - \beta) + \frac{1}{\gamma^2} \vec{F}_\perp \cdot \vec{u}_\perp \right].$$

Using $1/\gamma^2 = 1 - \beta^2$ and performing the replacement $\vec{F}_\perp \cdot \vec{u}_\perp = \vec{F} \cdot \vec{u} - F_\parallel u_\parallel$, we get

$$\begin{aligned} \vec{F}' \cdot \vec{u}\,' &= \frac{1}{(1 - \vec{\beta} \cdot \vec{u})^2} \left[F_\parallel u_\parallel - \beta F_\parallel - \beta u_\parallel \vec{F} \cdot \vec{u} + \beta^2 \vec{F} \cdot \vec{u} + \vec{F} \cdot \vec{u} - \beta^2 \vec{F} \cdot \vec{u} - F_\parallel u_\parallel + \beta^2 F_\parallel u_\parallel \right] \\ &= \frac{1}{(1 - \vec{\beta} \cdot \vec{u})^2} \left[\vec{F} \cdot \vec{u} (1 - \beta u_\parallel) - \beta F_\parallel (1 - \beta u_\parallel) \right] = \frac{1}{1 - \vec{\beta} \cdot \vec{u}} \left[\vec{F} \cdot \vec{u} - \vec{F} \cdot \vec{\beta} \right] \end{aligned}$$

2.2 Electromagnetic field created by an infinite plane

In an inertial reference frame R, consider an infinite plane of electric charges, of constant surface density σ. We choose an origin and a system of axes such that that coincides with the Oxy plane. These charges move at constant velocity $\vec{v}$ in the direction of the Ox axis. We denote $\vec{e}_x, \vec{e}_y, \vec{e}_z$ the unit vectors on each of the axes axes Ox, Oy, Oz.

Preliminary (bonus).
11. Show that the electromagnetic field, in the reference frame R, is given, for $z > 0$, by

$$\vec{E} = \frac{\sigma}{2} \vec{e}_z \quad \text{and} \quad \vec{B} = -\frac{\sigma v}{2} \vec{e}_y \tag{13}$$

What are the expressions of $\vec{E}$ and $\vec{B}$ for $z < 0$?

Solution

Consider a point M in which we want to evaluate the electromagnetic fields. The plane $P_1 = (Myz)$ is a symmetry plane for the charge distribution, and an antisymmetry plane for the current distribution. Under the symmetry S_{P_1} with respect to this plane, the vector field $\vec{E}$ and the pseudo-vector $\vec{B}$ transform as

$$\begin{aligned}
\vec{B} = B_x\vec{e}_x + B_y\vec{e}_y + B_z\vec{e}_z &\xrightarrow{S_{P_1}} B_x\vec{e}_x - B_y\vec{e}_y - B_z\vec{e}_z \\
\vec{E} = E_x\vec{e}_x + E_y\vec{e}_y + E_z\vec{e}_z &\xrightarrow{S_{P_1}} -E_x\vec{e}_x + E_y\vec{e}_y + E_z\vec{e}_z
\end{aligned}$$

so that $E_x = 0$ and $B_x = 0$. Similarly, consider the plane $P_2 = (Mxz)$. It is a symmetry plane for both the charge and the current distributions. Under the symmetry S_{P_2} with respect to this plane, one has

$$\begin{aligned}
\vec{B} = B_y\vec{e}_y + B_z\vec{e}_z &\xrightarrow{S_{P_2}} B_y\vec{e}_y - B_z\vec{e}_z \\
\vec{E} = E_y\vec{e}_y + E_z\vec{e}_z &\xrightarrow{S_{P_2}} -E_y\vec{e}_y + E_z\vec{e}_z
\end{aligned}$$

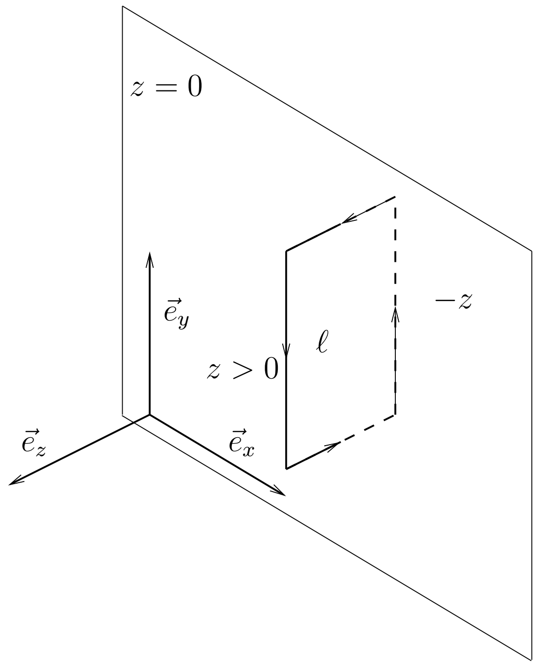
so that $E_y = 0$ and $B_z = 0$. Finally, by translation invariance along $\vec{e}_x$ and $\vec{e}_y$, we conclude that $\vec{E} = E_z(z)\vec{e}_z$ and $\vec{B} = B_y(z)\vec{e}_y$.
The plane $P_3 = (Oxy)$ is a symmetry plane for both the charge and the current distributions. Under the symmetry S_{P_3} with respect to this plane,

$$\begin{aligned}
\vec{B} = B_y(z)\vec{e}_y &\xrightarrow{S_{P_3}} -B_y(-z)\vec{e}_y \\
\vec{E} = E_z(z)\vec{e}_z &\xrightarrow{S_{P_3}} -E_z(-z)\vec{e}_z
\end{aligned}$$

Let $z > 0$. Consider a cylinder symmetric with respect to $P_3 = (Oxy)$, with its faces of surface s, located at z and $-z$. Applying Gauss theorem on this cylinder gives $2sE_z(z) = s\sigma$ so that

$$\vec{E}(z > 0) = \frac{\sigma}{2}\vec{e}_z \quad \text{and} \quad \vec{E}(z < 0) = -\frac{\sigma}{2}\vec{e}_z$$

Consider now a closed path made of a two segments of length ℓ, the first one being at fixed x and z, pointing in the direction $-\vec{e}_y$, the second one at fixed x and $-z$, pointing in the direction $\vec{e}_y$, these two segments being connected by two segments pointing along $\vec{e}_z$ and $-\vec{e}_z$, of length $2z$, see figure below.



Applying Ampère theorem

$$\iint \vec{B} \cdot d\vec{\ell} = I$$

on this contour gives

$$-B_y(z)\ell + B_y(-z)\ell = v\sigma\ell$$

so that

$$B_y(z > 0) = -\frac{v\sigma}{2} \quad \text{and} \quad B_y(z < 0) = \frac{v\sigma}{2} \,.$$

Let R' be another inertial frame, with moves at a constant velocity $\vec{V}$ in the direction of $x > 0$ with respect to the frame R.

12. Give the expression of $\vec{v}\,'$ in frame R', as a function of V and v.

Solution

From question 1., we have $\vec{v}\,' = v'\vec{e}_x$ with

$$v' \;=\; \frac{v - V}{1 - Vv} \,.$$

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13. Compute $\Gamma' = 1/\sqrt{1 - v'^2}$ in terms of $\Gamma = 1/\sqrt{1 - v^2}$, $\gamma = 1/\sqrt{1 - V^2}$, V and v.

Solution

From question 3., we have

$$\Gamma' = \Gamma\gamma(1 - Vv) \,.$$

14. Show that the surface density σ' in frame R' reads

$$\sigma' = \sigma \frac{1 - Vv}{\sqrt{1 - V^2}} \,. \tag{14}$$

Solution

Passing from the charges rest frame R_0 to the frame R, one experience length contraction along x axis, namely $dx = dx_{R_0}/\Gamma$, while $dy = dy_{R_0}$. Since the charge in a given surface remains identical in frames R_0 and R, one can write

$$\sigma_{R_0} dx_{R_0} dy_{R_0} = \sigma dx dy$$

i.e.

$$\sigma = \sigma_{R_0} \frac{dx_{R_0}}{dx} = \Gamma \sigma_{R_0} \,.$$

Similarly,

$$\sigma' = \sigma_{R_0} \frac{dx_{R_0}}{dx'} = \Gamma' \sigma_{R_0} \,,$$

so that

$$\sigma' = \frac{\Gamma'}{\Gamma} \sigma = \gamma(1 - \beta V)\sigma \,.$$

15. We now consider the frame R'.
i) By analogy with the expression given above for the electromagnetic field in the frame R, give the expression for the electromagnetic field $(\vec{E}\,', \vec{B}\,')$ in frame R'.

Solution

In frame R', we have

$$\begin{aligned} \vec{E}\,' &= \frac{\sigma'}{2} \vec{e}_z = \gamma(1 - Vv) \frac{\sigma}{2} \vec{e}_z \,, \\ \vec{B}\,' &= -\frac{\sigma' v'}{2} \vec{e}_y = -\frac{\sigma}{2} \gamma(1 - Vv) \frac{v - V}{1 - Vv} \vec{e}_y = -\frac{\sigma}{2} \gamma(v - V) \vec{e}_y \end{aligned}$$

ii) Check that it is consistent with Lorentz's transformation of fields.

Solution

We know that the transformation law for $(\vec{E}, \vec{B})$ reads

$$\begin{aligned} \vec{E}' &= (\vec{E}\cdot\vec{n})\vec{n} + \gamma\left[\vec{E} - (\vec{E}\cdot\vec{n})\vec{n}\right] + \gamma\,\vec{V}\wedge\vec{B}\,, \\ \vec{B}' &= (\vec{B}\cdot\vec{n})\vec{n} + \gamma\left[\vec{B} - (\vec{B}\cdot\vec{n})\vec{n}\right] - \gamma\,\vec{V}\wedge\vec{E}\,. \end{aligned}$$

Here $\vec{n} = \vec{e}_x$, $\vec{V} = V\vec{e}_x$, $\vec{E} = E_z\vec{e}_z$ and $\vec{B} = B_y\vec{e}_y$ so that

$$\begin{aligned} \vec{E}' &= \gamma E_z\vec{e}_z + \gamma\,V\vec{e}_x\wedge B_y\vec{e}_y\,, \\ \vec{B}' &= \gamma B_y\vec{e}_y - \gamma\,V\vec{e}_x\wedge E_z\vec{e}_z\,, \end{aligned}$$

which leads to

$$\begin{aligned} \vec{E}' &= \gamma\frac{\sigma}{2}\vec{e}_z + \gamma\,V\left(-\frac{v\sigma}{2}\right)\vec{e}_z = \gamma(1-Vv)\frac{\sigma}{2}\vec{e}_z\,, \\ \vec{B}' &= \gamma\left(-\frac{v\sigma}{2}\right)\vec{e}_y + \gamma\,V\frac{\sigma}{2}\vec{e}_y = -\frac{\sigma}{2}\gamma\,(v-\ V)\,\vec{e}_y\,, \end{aligned}$$

in accordance with question 15. i).

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iii) What happens if $\vec{V} = \vec{v}$?

Solution

When $\vec{V} = \vec{v}$, $\gamma(1-Vv) = 1/\gamma = 1/\Gamma$ so that

$$\begin{aligned} \vec{E}' &= \frac{\sigma}{\Gamma 2}\vec{e}_z = \frac{\sigma_{R_0}}{2}\vec{e}_z\,, \\ \vec{B}' &= 0\,. \end{aligned}$$

As expected, this is the field produced by an infinite plane of electric charges at rest, of constant surface density σ_{R_0}.

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16. A particle, of charge q and velocity $\vec{v}$ in frame R, is subjected to the action of this electromagnetic field.
i) What force does it experience in frame R?

Solution

The Lorentz force reads

$$\vec{F} = q\vec{E} = +q\vec{v}\wedge\vec{B}$$

In frame R, we thus have

$$\vec{F} = q\sigma\left[\frac{1}{2}\vec{e}_z - \frac{v^2}{2}\vec{e}_x\wedge\vec{e}_y\right] = q\frac{\sigma}{2}(1-v^2)\vec{e}_z = q\frac{\sigma}{2\,\Gamma^2}\vec{e}_z\,.$$

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ii) And in its rest frame?

Solution

The rest frame of the charge is also the frame in which the charge of the plane are at rest, since both have the same velocity in frame R. We thus have

$$\vec{F}_{R_0} = q\frac{\sigma_{R_0}}{2}\vec{e}_z = \frac{\sigma}{2\,\Gamma}\vec{e}_z\,.$$

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iii) Compare these two forces.

Solution

We observe that

$$\vec{F}_{R_0} = \Gamma\vec{F}\,.$$

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iv) Does the result agree with that obtained using the quadrivector force?

Solution

From question 8., since $\vec{F}_{\parallel} = \vec{0}$ and $\vec{F}\cdot\vec{v} = 0$, the force after Lorentz's boost from frame R to the rest frame R_0 reads

$$\begin{aligned} F_{R_0\,x} &= F_x = 0\,,\\ F_{R_0\,y} &= F_y = 0\,,\\ F_{R_0\,z} &= \frac{1}{\Gamma(1-v^2)}F_z = \Gamma F_z\,, \end{aligned}$$

i.e.

$$\vec{F}_{propre} = \Gamma\vec{F}\,.$$

in accordance to the result of question 16. iii).
