M1 General Physics **2023-2024**

Particles

Final exam

January 10th 2024

Documents allowed

Notes:
- **The subject is deliberately long**. It is not requested to reach the end to get a good mark!
- The system of unit is such that $c = 1, \hbar = 1$, $\epsilon_0 = 1$, $\mu_0 = 1$.
- Space coordinates may be freely denoted as (x, y, z) or (x^1, x^2, x^3).
- One may always assume that fields are rapidly decreasing at infinity.
- Drawings are welcome!

1 Belinfante tensor

We consider a Lagrangian describing particles with spin. The Lagrangian density is assumed to have no explicit dependence with respect to space-time position.

1. Recall the origin of the conservation of the canonical energy-momentum tensor, denoted as $T^{\mu\nu}$, and the expression of the conserved charge in terms of $T^{\mu\nu}$. How is it named?

2. Recall the origin of the conservation of angular momentum, denoted as $J^{\mu,\nu\lambda}$, and the expression of the conserved charge $J^{\nu\lambda}$. What are the symmetry properties of $J^{\mu,\nu\lambda}$ and $J^{\nu\lambda}$?

3. In the case of a particle of arbitrary spin, the angular momentum tensor takes the form

$$J^{\mu,\nu\lambda} = x^\nu \, T^{\mu\lambda} - x^\lambda \, T^{\mu\nu} + \Delta^{\mu\nu\lambda} \, , \tag{1}$$

where $\Delta^{\mu\nu\lambda}$ is a function of the fields, antisymmetric with respect to $\nu \leftrightarrow \lambda$.

i) What would be the value of $T^{\mu\nu} - T^{\nu\mu}$ for a particle of spin 0?

ii) In the general case, compute $T^{\nu\lambda} - T^{\lambda\nu}$ in terms of $\Delta^{\mu\nu\lambda}$.

4. We introduce the Belinfante energy-momentum tensor

$$T_B^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\lambda \left[\Delta^{\mu\nu\lambda} + \Delta^{\nu\mu\lambda} - \Delta^{\lambda\nu\mu}\right] . \tag{2}$$

i) Show that $T_B^{\mu\nu}$ is conserved.

ii) What can be said on the symmetry properties of $T_B^{\mu\nu}$?

iii) Compare the charge associated to $T^{\mu\nu}$ to the one associated to $T_B^{\mu\nu}$. Conclusion?

iv) Show that the total angular momentum of the field can be defined using a local density built from the local density of the 4-momentum, just like in the scalar case, using $T_B^{\mu\nu}$ instead of $T^{\mu\nu}$, i.e. show that

$$J^{\nu\lambda} = \int (x^\nu T_B^{0\lambda} - x^\lambda T_B^{0\nu})\, d^3x \tag{3}$$

2 Lorentz Transformation of Electric and Magnetic Fields

Einstein's first postulate of the Special Theory of Relativity tells that the laws of physics have the same mathematical form in inertial frames moving with constant velocity with respect to each other. In this problem, we will rely on this postulate to get the law of transformation of electric and magnetic fields.
Consider, in frame S, a particle of rest mass m and charge q moves with velocity $\vec{u}$ in an electric field $\vec{E}$ and a magnetic field $\vec{B}$ and experiences a force

$$\vec{F} = q(\vec{E} + \vec{u} \wedge \vec{B}) \tag{4}$$

so that, denoting as $\vec{p}$ the momentum of the particle in S,

$$\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{u} \wedge \vec{B})\,. \tag{5}$$

In frame S', which moves along the $x-$axis of S with speed $\beta = v$, the velocity of the particle is $\vec{u}\,'$ and the particle experiences a force

$$\vec{F}\,' = q(\vec{E}\,' + \vec{u}\,' \wedge \vec{B}\,') \tag{6}$$

where $\vec{E}\,'$ and $\vec{B}\,'$ are the electric and magnetic field, respectively, in S', so that

$$\frac{d\vec{p}\,'}{dt'} = q(\vec{E}\,' + \vec{u}\,' \wedge \vec{B}\,')\,, \tag{7}$$

where $\vec{p}\,'$ is the momentum of the particle in S'.

1. Show that

$$\frac{dt'}{dt} = \gamma(1 - u_x\beta)\,. \tag{8}$$

2. Express $(p_0', \vec{p}\,')$ in terms of $(p_0, \vec{p})$.

3. Justify that

$$\frac{dp_0}{dt} = q\vec{u} \cdot \vec{E} \tag{9}$$

and

$$\frac{dp'_0}{dt'} = q\vec{u}' \cdot \vec{E}' \tag{10}$$

4. Show that

$$\begin{aligned} u'_x &= \frac{u_x - \beta}{1 - \beta u_x} && (11) \\ u'_y &= \frac{1}{\gamma}\frac{u_y}{1 - \beta u_x} && (12) \\ u'_z &= \frac{1}{\gamma}\frac{u_z}{1 - \beta u_x}\,. && (13) \end{aligned}$$

Comment on the non-relativistic limit $u_x \sim \beta \ll 1$.

5. From the known result based on the fact that $F^{\mu\nu}$ transforms as a 2-contravariant tensor under Lorentz transformations, write the components of $\vec{E}'$ and $\vec{B}'$ in terms of the components of $\vec{E}$ and $\vec{B}$.

6. Determine directly the relationship between the electric and magnetic fields in S and S', obtained in the previous question.

Hint: using the various expressions obtained above in questions 1,2,3,4, express the LHS of Eqs. (9) and of (7) in terms of quantities in frame S, and in each case, compare then LHS with RHS.

3 Breit frame

Consider the elastic scattering of two particles A and B, of masses m_A and m_B respectively. In a given inertial frame $\mathcal{F}$, the momenta of particles A and B are $P_A = (E_A, \vec{p}_A)$ and $P_B = (E_B, \vec{p}_B)$ before the scattering, and $P'_A = (E'_A, \vec{p}'_A)$ and $P'_B = (E'_B, \vec{p}'_B)$ after the scattering.

1. Show that there is an inertial frame $\mathcal{B}$, named Breit frame, in which $\vec{p}_A + \vec{p}'_A = \vec{0}$. What is the velocity of this frame with respect to the frame $\mathcal{F}$?

2. Show that in Breit's frame, the modulus of the momentum of each particle as well as their energies are conserved during elastic scattering.

3. We use the standard notation * for a given quantity in the center-of-mass frame. Let us introduce the Mandelstam variable $t = (P_A - P'_A)^2$.

i) What is the property of t with respect to Lorentz transformations?

ii) Compare p^*_A, p'^*_A, p^*_B, p'^*_B.

iii) Compute t in the center-of-mass frame, and show that the scattering angle θ^* satisfies

$$\cos\theta^* = 1 + \frac{t}{2p_A^{*2}} \,. \tag{14}$$

iv) Working now in the Breit's frame, deduce that

$$p_A = p_A^* \sin\frac{\theta^*}{2} \,. \tag{15}$$

4. (*) Show that

$$p_B = p'_B = p_A^* \sqrt{\sin^2\frac{\theta^*}{2} + \gamma^2\cos^2\frac{\theta^*}{2}\left(1 + \frac{E_B^*}{E_A^*}\right)^2} \tag{16}$$

and

$$E_B = E'_B = \gamma\left(E_B^* + \frac{p_A^{*2}}{E_A^*}\cos^2\frac{\theta^*}{2}\right), \tag{17}$$

where γ is the Lorentz factor when boosting from the center-of-mass frame to the Breit's frame.

Hint: make a drawing and identify the direction of the boost. Then use the explicit form of this boost.

5. Study the relative position of momenta $\vec{p}_B$ and $\vec{p}_B^{\,\prime}$ with respect to the momenta $\vec{p}_A$ and $\vec{p}_A^{\,\prime}$, and justify the name "wall reference frame" given to the $\mathcal{B}$ reference frame.

6. Calculate the deflection angle of each particle as a function of the modulus of momenta. One should in particular prove that the deflection angle ϕ of particle B is given by

$$\cos\phi = 1 - 2\frac{p_A^2}{p_B^2} \,. \tag{18}$$