M1 General Physics **2023-2024**

Particles

Final exam

January 10th 2024

Documents allowed

Notes:
- **The subject is deliberately long**. It is not requested to reach the end to get a good mark!
- The system of unit is such that $c = 1, \hbar = 1$, $\epsilon_0 = 1$, $\mu_0 = 1$.
- Space coordinates may be freely denoted as (x, y, z) or (x^1, x^2, x^3).
- One may always assume that fields are rapidly decreasing at infinity.
- Drawings are welcome!

1 Belinfante tensor

We consider a Lagrangian describing particles with spin. The Lagrangian density is assumed to have no explicit dependence with respect to space-time position.

1. Recall the origin of the conservation of the canonical energy-momentum tensor, denoted as $T^{\mu\nu}$, and the expression of the conserved charge in terms of $T^{\mu\nu}$. How is it named?

Solution

This is due to the invariance of the Lagrangian with respect to space-time translations, which implies the invariance of the action, and thus the existence of conserved currents by Noether theorem. The conserved charge is the 4-momentum, which reads

$$P^\nu = \int T^{0\nu}\, d^3x\,.$$

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2. Recall the origin of the conservation of angular momentum, denoted as $J^{\mu,\nu\lambda}$, and the expression of the conserved charge $J^{\nu\lambda}$. What are the symmetry properties of $J^{\mu,\nu\lambda}$ and $J^{\nu\lambda}$?

Solution

This is due to the invariance of the Lagrangian with respect to Lorentz transformations, which implies the invariance of the action, and thus the existence of conserved currents by Noether theorem. The conserved charge is

$$J^{\nu\lambda} = \int J^{0,\nu\lambda}\, d^3x\,.$$

Both $J^{\mu,\nu\lambda}$ and $J^{\nu\lambda}$ are antisymmetric with respect to $\nu \leftrightarrow \lambda$.

3. In the case of a particle of arbitrary spin, the angular momentum tensor takes the form

$$J^{\mu,\nu\lambda} = x^\nu \, T^{\mu\lambda} \, - x^\lambda \, T^{\mu\nu} \, + \, \Delta^{\mu\nu\lambda} \, , \tag{1}$$

where $\Delta^{\mu\nu\lambda}$ is a function of the fields, antisymmetric with respect to $\nu \leftrightarrow \lambda$.

i) What would be the value of $T^{\mu\nu} - T^{\nu\mu}$ for a particle of spin 0?

Solution

In this case, $\Delta^{\mu\nu\lambda} = 0$ and thus the conservation of $J^{\mu,\nu\lambda}$ implies that, using the conservation of $T^{\mu\nu}$

$$\partial_\mu J^{\mu,\nu\lambda} = g^\nu_\mu T^{\mu\lambda} - g^\lambda_\mu T^{\mu\nu} = T^{\nu\lambda} - T^{\lambda\nu} = 0 \, .$$

Thus $T^{\mu\nu}$ is symmetric.

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ii) In the general case, compute $T^{\nu\lambda} - T^{\lambda\nu}$ in terms of $\Delta^{\mu\nu\lambda}$.

Solution

We now get

$$\partial_\mu J^{\mu,\nu\lambda} = g^\nu_\mu T^{\mu\lambda} - g^\lambda_\mu T^{\mu\nu} + \partial_\mu \Delta^{\mu\nu\lambda} = T^{\nu\lambda} - T^{\lambda\nu} + \partial_\mu \Delta^{\mu\nu\lambda} = 0$$

and thus

$$T^{\lambda\nu} - T^{\nu\lambda} = \partial_\mu \Delta^{\mu\nu\lambda}.$$

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4. We introduce the Belinfante energy-momentum tensor

$$T^{\mu\nu}_B = T^{\mu\nu} + \frac{1}{2} \partial_\lambda \left[\Delta^{\mu\nu\lambda} + \Delta^{\nu\mu\lambda} - \Delta^{\lambda\nu\mu} \right] . \tag{2}$$

i) Show that $T^{\mu\nu}_B$ is conserved.

Solution

We have

$$\partial_\mu T^{\mu\nu}_B = \partial_\mu T^{\mu\nu} + \frac{1}{2} \partial_\mu \partial_\lambda \left[\Delta^{\mu\nu\lambda} + \Delta^{\nu\mu\lambda} - \Delta^{\lambda\nu\mu} \right] = 0 \, .$$

Indeed:
- the second term gives 0 since $\partial_\mu \partial_\lambda$ is symmetric while $\Delta^{\nu\mu\lambda}$ is antisymmetric.
- λ and μ are summation indexes, thus the first and the third term compensate.

ii) What can be said on the symmetry properties of $T_B^{\mu\nu}$?

Solution

$$
\begin{aligned}
T_B^{\mu\nu} - T_B^{\nu\mu} &= T^{\mu\nu} - T^{\nu\mu} + \frac{1}{2}\partial_\lambda \left[\Delta^{\mu\nu\lambda} + \Delta^{\nu\mu\lambda} - \Delta^{\lambda\nu\mu} - \Delta^{\nu\mu\lambda} - \Delta^{\mu\nu\lambda} + \Delta^{\lambda\mu\nu}\right] \\
&= \partial_\lambda[\Delta^{\lambda\nu\mu} + \Delta^{\lambda\mu\nu}] = 0
\end{aligned}
$$

where we have used the antisymmetry of $\Delta^{\lambda\mu\nu}$ with respect to $\mu \leftrightarrow \nu$.

iii) Compare the charge associated to $T^{\mu\nu}$ to the one associated to $T_B^{\mu\nu}$. Conclusion?

Solution

The charge associated to $T^{\mu\nu}$ is the 4-momentum of the field. It reads

$$
P^\nu = \int T^{0\nu} d^3x \,.
$$

The one associated to $T_B^{\mu\nu}$ is

$$
P_B^\nu = \int T_B^{0\nu} d^3x = \int T^{0\nu} d^3x + \frac{1}{2}\int \partial_\lambda \left[\Delta^{\mu\nu\lambda} + \Delta^{\nu\mu\lambda} - \Delta^{\lambda\nu\mu}\right] d^3x \,.
$$

The second term in the RHS is the integral over the whole space of a total derivative, thus it vanishes thanks to the fast decreasing fields at infinity. In conclusion, the total 4-momentum of the field is the same, although its local density differs. This is the reason why $T_B^{\mu\nu}$ can be named energy-momentum tensor.

iv) Show that the total angular momentum of the field can be defined using a local density built from the local density of the 4-momentum, just like in the scalar case, using $T_B^{\mu\nu}$ instead of $T^{\mu\nu}$, i.e. show that

$$
J^{\nu\lambda} = \int (x^\nu T_B^{0\lambda} - x^\lambda T_B^{0\nu})\, d^3x \tag{3}
$$

Solution

We have

$$
\begin{aligned}
J_B^{\nu\lambda} &= \int (x^\nu T_B^{0\lambda} - x^\lambda T_B^{0\nu})\, d^3x \\
&= \int (x^\nu T^{0\lambda} - x^\lambda T^{0\nu})\, d^3x \\
&+ \frac{1}{2}\int \left[x^\nu \partial_\alpha \left(S^{0\lambda\alpha} + S^{\lambda 0\alpha} - S^{\alpha\lambda 0}\right) - x^\lambda \partial_\alpha \left(S^{0\nu\alpha} + S^{\nu 0\alpha} - S^{\alpha\nu 0}\right)\right].
\end{aligned}
$$

In this equation, the second term in the RHS can be rewritten after performing an integration by part, so that

$$J_B^{\nu\lambda} = \int (x^\nu T^{0\lambda} - x^\lambda T^{0\nu})\, d^3x - \frac{1}{2}\int \left[g_\alpha^\nu \left(S^{0\lambda\alpha} + S^{\lambda 0\alpha} - S^{\alpha\lambda 0}\right) - g_\alpha^\lambda \left(S^{0\nu\alpha} + S^{\nu 0\alpha} - S^{\alpha\nu 0}\right)\right] d^3x$$

$$= \int (x^\nu T^{0\lambda} - x^\lambda T^{0\nu})\, d^3x - \frac{1}{2}\int \left[S^{0\lambda\nu} + S^{\lambda 0\nu} - S^{\nu\lambda 0} - S^{0\nu\lambda} - S^{\nu 0\lambda} + S^{\lambda\nu 0}\right] d^3x$$

$$= \int (x^\nu T^{0\lambda} - x^\lambda T^{0\nu})\, d^3x - \frac{1}{2}\int \left[S^{0\lambda\nu} - S^{0\nu\lambda}\right] d^3x$$

where to get the last line, we have canceled the second with the last term, as well as the third term with the fifth one, using the antisymmetry of $S^{\mu\nu\rho}$ with respect to $\nu \leftrightarrow \rho$. Using again this antisymmetry, we finally get

$$J_B^{\nu\lambda} = \int (x^\nu T^{0\lambda} - x^\lambda T^{0\nu} + S^{0\nu\lambda})\, d^3x = J^{\nu\lambda}$$

as expected.

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2 Lorentz Transformation of Electric and Magnetic Fields

Einstein's first postulate of the Special Theory of Relativity tells that the laws of physics have the same mathematical form in inertial frames moving with constant velocity with respect to each other. In this problem, we will rely on this postulate to get the law of transformation of electric and magnetic fields.

Consider, in frame S, a particle of rest mass m and charge q moves with velocity $\vec{u}$ in an electric field $\vec{E}$ and a magnetic field $\vec{B}$ and experiences a force

$$\vec{F} = q(\vec{E} + \vec{u} \wedge \vec{B}) \tag{4}$$

so that, denoting as $\vec{p}$ the momentum of the particle in S,

$$\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{u} \wedge \vec{B})\,. \tag{5}$$

In frame S', which moves along the $x-$axis of S with speed $\beta = v$, the velocity of the particle is $\vec{u}\,'$ and the particle experiences a force

$$\vec{F}\,' = q(\vec{E}\,' + \vec{u}\,' \wedge \vec{B}\,') \tag{6}$$

where $\vec{E}\,'$ and $\vec{B}\,'$ are the electric and magnetic field, respectively, in S', so that

$$\frac{d\vec{p}\,'}{dt'} = q(\vec{E}\,' + \vec{u}\,' \wedge \vec{B}\,')\,, \tag{7}$$

where $\vec{p}\,'$ is the momentum of the particle in S'.

1. Show that

$$\frac{dt'}{dt} = \gamma(1 - u_x\beta)\,. \tag{8}$$

Solution

This is obvious from the expression of a Lorentz transformation, using the fact that $dx = u_x dt$ and

$$dt' = \gamma dt - \gamma\beta dx\,. \tag{9}$$

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2. Express $(p_0', \vec{p}')$ in terms of $(p_0, \vec{p})$.

Solution

We have

$$\begin{aligned} p_0' &= \gamma(p_0 - \beta p_x) && (10)\\ p_x' &= \gamma(p_x - \beta p_0) && (11)\\ p_y' &= p_y && (12)\\ p_z' &= p_z\,. && (13) \end{aligned}$$

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3. Justify that

$$\frac{dp_0}{dt} = q\vec{u}\cdot\vec{E} \tag{14}$$

and

$$\frac{dp_0'}{dt'} = q\vec{u}'\cdot\vec{E}' \tag{15}$$

Solution

This just comes from the fact that

$$\frac{dp_0}{dt} = \vec{u}\cdot\vec{F} \quad \text{and} \quad \frac{dp_0'}{dt'} = \vec{u}'\cdot\vec{F}'.$$

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4. Show that

$$\begin{aligned} u_x' &= \frac{u_x - \beta}{1 - \beta u_x} && (16)\\ u_y' &= \frac{1}{\gamma}\frac{u_y}{1 - \beta u_x} && (17)\\ u_z' &= \frac{1}{\gamma}\frac{u_z}{1 - \beta u_x}\,. && (18) \end{aligned}$$

Comment on the non-relativistic limit $u_x \sim \beta \ll 1$.

Solution

We have, through differentiation,

$$\left\{ \begin{array}{rcrcr} dt' & = & \gamma\, dt & - & \gamma\beta\, dx \\ dx' & = & -\gamma\beta\, dt & + & \gamma\, dx \end{array} \right.$$

which gives

$$u'_x = \frac{dx'}{dt'} = \frac{-\gamma\beta dt + \gamma dx}{\gamma dt - \gamma\beta dx} = \frac{u_x - \beta}{1 - \beta u_x}\,.$$

Besides,

$$u'_y = \frac{dy'}{dt'} = \frac{dy}{\gamma dt - \gamma\beta dy} = \frac{1}{\gamma}\frac{u_y}{1 - \beta u_x}\,,$$

and similarly

$$u'_z = \frac{dz'}{dt'} = \frac{dz}{\gamma dt - \gamma\beta dz} = \frac{1}{\gamma}\frac{u_z}{1 - \beta u_x}\,.$$

At lowest order, we get $u'_x \sim u_x - \beta$, $u'_y \sim u_y$ and $u'_z \sim u_z$ as expected in the change of inertial frame in non-relativistic mechanics.

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5. From the known result based on the fact that $F^{\mu\nu}$ transforms as a 2-contravariant tensor under Lorentz transformations, write the components of $\vec{E}\,'$ and $\vec{B}\,'$ in terms of the components of $\vec{E}$ and $\vec{B}$.

Solution

Using

$$\begin{aligned} \vec{E}' &= (\vec{E}\cdot\vec{n})\vec{n} + \gamma\left[\vec{E} - (\vec{E}\cdot\vec{n})\vec{n}\right] + \gamma\,\vec{v}\wedge\vec{B}\,, \\ \vec{B}' &= (\vec{B}\cdot\vec{n})\vec{n} + \gamma\left[\vec{B} - (\vec{B}\cdot\vec{n})\vec{n}\right] - \gamma\,\vec{v}\wedge\vec{E}\,, \end{aligned}$$

we obtain

$$\begin{aligned} E'_x &= E_x\,, \\ E'_y &= \gamma(E_y - \beta B_z)\,, \\ E'_z &= \gamma(E_z + \beta B_y)\,, \\ B'_x &= B_x\,, \\ B'_y &= \gamma(B_y + \beta E_z)\,, \\ B'_z &= \gamma(B_z - \beta E_y)\,. \end{aligned}$$

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6. Determine directly the relationship between the electric and magnetic fields in S and S', obtained in the previous question.

Hint: using the various expressions obtained above in questions 1,2,3,4, express the LHS of Eqs. (14) and of (7) in terms of quantities in frame S, and in each case, compare then LHS with RHS.

Solution

We get from Eq. (15), using Eqs. (16), (17) and (18),

$$\begin{aligned}\frac{dp'_0}{dt'} &= q\vec{u}'\cdot\vec{E}' = q\left[\frac{u_x-\beta}{1-\beta u_x}E'_x + \frac{1}{\gamma}\frac{u_y}{1-\beta u_x}E'_y + \frac{1}{\gamma}\frac{u_z}{1-\beta u_x}E'_z\right] \\ &= \frac{1}{1-\beta u_x}\left(\frac{dp_0}{dt} - \beta\frac{dp_x}{dt}\right)\end{aligned}$$

where we have used Eqs. (10) and (11). Using now (5) and (14), we have

$$\begin{aligned}\frac{dp_x}{dt} &= q[E_x + u_yB_z - u_zB_y]\,, \\ \frac{dp_y}{dt} &= q[E_y + u_zB_x - u_xB_z]\,, \\ \frac{dp_z}{dt} &= q[E_z + u_xB_y - u_yB_x]\end{aligned}$$

and

$$\frac{dp_0}{dt} = = q[u_xE_x + u_yE_y + u_zE_z]\,,$$

so that

$$\begin{aligned}& q\left[\frac{u_x-\beta}{1-\beta u_x}E'_x + \frac{1}{\gamma}\frac{u_y}{1-\beta u_x}E'_y + \frac{1}{\gamma}\frac{u_z}{1-\beta u_x}E'_z\right] \\ = & \frac{q}{1-\beta u_x}\left[u_xE_x + u_yE_y + u_zE_z - \beta(E_x + u_yB_z - u_zB_y)\right]\end{aligned}$$

and thus

$$\gamma(u_x-\beta)E'_x + u_yE'_y + u_zE'_z = \gamma(u_x-\beta)E_x + u_y\gamma(E_y-\beta B_z) + u_z\gamma(E_z+\beta B_y)\,.$$

This should be valid for any $\vec{u}$, which implies that

$$\begin{aligned}E'_x &= E_x\,, \\ E'_y &= \gamma(E_y - \beta B_z)\,, \\ E'_z &= \gamma(E_z + \beta B_y)\,.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\frac{dp'_x}{dt'} &= q[E'_x + u'_yB'_z - u'_zB'_y] = q\left[E'_x + \frac{1}{\gamma}\frac{u_y}{1-\beta u_x}B'_z - \frac{1}{\gamma}\frac{u_z}{1-\beta u_x}B'_y\right] \\ &= \frac{1}{1-\beta u_x}\left(\frac{dp_x}{dt} - \beta\frac{dp_0}{dt}\right) = \frac{q}{1-\beta u_x}\left[E_x + u_yB_z - u_zB_y - \beta(u_xE_x + u_yE_y + u_zE_z)\right]\end{aligned}$$

and thus

$$E'_x + u_y B'_z - u_z B'_y = E_x + u_y \gamma(B_z - \beta E_y) - u_z \gamma(B_y + \beta E_z)\,,$$

which implies that

$$\begin{aligned} E'_x &= E_x\,, \\ B'_y &= \gamma(B_y + \beta E_z)\,, \\ B'_z &= \gamma(B_z - \beta E_y)\,. \end{aligned}$$

Next,

$$\begin{aligned} \frac{dp'_y}{dt'} &= q[E'_y + u'_z B'_x - u'_x B'_z] = q\left[E'_y + \frac{1}{\gamma}\frac{u_z}{1-\beta u_x}B'_x - \frac{u_x - \beta}{1-\beta u_x}B'_z\right] \\ &= \frac{1}{\gamma}\frac{1}{1-\beta u_x}\frac{dp_y}{dt} = \frac{q}{\gamma(1-\beta u_x)}\left[E_y + u_z B_x - u_x B_z\right] \end{aligned}$$

and thus

$$u_z B'_x + \gamma(1-\beta u_x)E'_y - (u_x - \beta)\gamma B'_z = E_y + u_z B_x - u_x B_z\,,$$

which implies that

$$\begin{aligned} B'_x &= B_x\,, \\ E_y &= \gamma(E'_y + \beta B'_z)\,, \\ B_z &= \gamma(B'_z + \beta E'_y)\,. \end{aligned}$$

Finally,

$$\begin{aligned} \frac{dp'_z}{dt'} &= q[E'_z + u'_x B'_y - u'_y B'_x] = q\left[E'_z + \frac{u_x - \beta}{1-\beta u_x}B'_y - \frac{1}{\gamma}\frac{u_y}{1-\beta u_x}B'_x\right] \\ &= \frac{1}{\gamma}\frac{1}{1-\beta u_x}\frac{dp_z}{dt} = \frac{q}{\gamma(1-\beta u_x)}\left[E_z + u_x B_y - u_y B_x\right] \end{aligned}$$

and thus

$$\gamma(1-\beta u_x)E'_z + \gamma(u_x - \beta)B'_y - u_y B'_x = E_z + u_x B_y - u_y B_x\,,$$

which implies that

$$\begin{aligned} B'_x &= B_x\,, \\ B_y &= \gamma(B'_y - \beta E'_z)\,, \\ E_z &= \gamma(E'_z - \beta B'_y)\,. \end{aligned}$$

This is in agreement with the result of question 5.

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3 Breit frame

Consider the elastic scattering of two particles A and B, of masses m_A and m_B respectively. In a given inertial frame $\mathcal{F}$, the momenta of particles A and B are $P_A = (E_A, \vec{p}_A)$ and $P_B = (E_B, \vec{p}_B)$ before the scattering, and $P'_A = (E'_A, \vec{p}'_A)$ and $P'_B = (E'_B, \vec{p}'_B)$ after the scattering.

1. Show that there is an inertial frame $\mathcal{B}$, named Breit frame, in which $\vec{p}_A + \vec{p}'_A = \vec{0}$. What is the velocity of this frame with respect to the frame $\mathcal{F}$?

Solution

This choice is always possible: one should just boost from $\mathcal{F}$, in which $\vec{p}_A + \vec{p}'_A$ is arbitrary, to the frame $\mathcal{B}$ in which $\vec{p}_A + \vec{p}'_A = 0$. The velocity of this frame is thus

$$\vec{v} = c^2 \frac{\vec{p}_A + \vec{p}'_A}{E_A + E'_A} \,.$$

---

2. Show that in Breit's frame, the modulus of the momentum of each particle as well as their energies are conserved during elastic scattering.

Solution

Working now in Breit's frame, we denote as p_A, p_B, p'_A, p'_B the modulus of $\vec{p}_A$, $\vec{p}_B$, $\vec{p}'_A$, $\vec{p}'_B$ respectively. The fact that $p_A = p'_A$ is obvious from the definition of Breit's frame. Thus, $E_A = E'_A$. Conservation of energy, which reads $E_A + E_B = E'_A + E'_B$ thus implies that $E_B = E'_B$. Finally, one deduces that $p_B = p'_B$.

---

3. We use the standard notation * for a given quantity in the center-of-mass frame. Let us introduce the Mandelstam variable $t = (P_A - P'_A)^2$.

i) What is the property of t with respect to Lorentz transformations?

Solution

t is a Lorentz invariant.

---

ii) Compare p^*_A, p'^*_A, p^*_B, p'^*_B.

Solution

By definition of the center-of-mass frame, $\vec{p}^*_A + \vec{p}^*_B = 0$ and thus $p^*_A = p^*_B$. Conservation of momentum implies that $\vec{p}'^*_A + \vec{p}'^*_B = 0$ and thus $p'^*_A = p'^*_B$. Finally, conservation of momentum implies that

$$\sqrt{m_A^2 + p^{*2}_A} + \sqrt{m_B^2 + p^{*2}_A} = \sqrt{m_A^2 + p'^{*2}_A} + \sqrt{m_B^2 + p'^{*2}_A}$$

and thus $p^*_A = p'^*_A$.
Conclusion: $p^*_A = p'^*_A = p^*_B = p'^*_B$.

iii) Compute t in the center-of-mass frame, and show that the scattering angle θ^* satisfies

$$\cos\theta^* = 1 + \frac{t}{2p_A^{*2}} \,. \tag{19}$$

Solution

We have

$$\begin{aligned} t = (P_A^* - P_A'^*)^2 &= 2m_A^2 - 2P_A^* \cdot P_A'^* \\ &= 2m_A^2 - 2E_A^{*2} + 2\vec{p}_A \cdot \vec{p}_A' \\ &= -2p_A^{*2} + 2p_A^{*2}\cos^2\theta^* = 2p_A^{*2}(\cos^2\theta^* - 1) \end{aligned}$$

and thus

$$\cos\theta^* = 1 + \frac{t}{2p_A^{*2}} \,. \tag{20}$$

iv) Working now in the Breit's frame, deduce that

$$p_A = p_A^* \sin\frac{\theta^*}{2} \,. \tag{21}$$

Solution

Let us compute the Mandelstam variable t in the Breit's frame. Since $E_A = E_A'$ and $p_A = p_A'$, we get

$$t = (E_A - E_A')^2 - (\vec{p}_A - \vec{p}_A')^2 = -4p_A^2$$

so that

$$p_A^2 = p_A^{*2}\frac{1-\cos\theta^*}{2} = p_A^{*2}\sin^2\frac{\theta^*}{2} \,,$$

and thus

$$p_A = p_A^* \sin\frac{\theta^*}{2} \,.$$

4. (*) Show that

$$p_B = p_B' = p_A^*\sqrt{\sin^2\frac{\theta^*}{2} + \gamma^2\cos^2\frac{\theta^*}{2}\left(1 + \frac{E_B^*}{E_A^*}\right)^2} \tag{22}$$

and

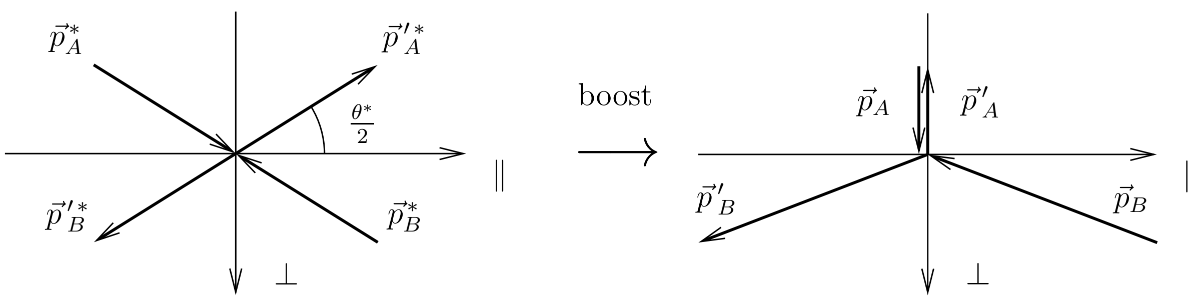
$$E_B = E'_B = \gamma \left(E_B^* + \frac{p_A^{*2}}{E_A^*} \cos^2 \frac{\theta^*}{2} \right) , \tag{23}$$

where γ is the Lorentz factor when boosting from the center-of-mass frame to the Breit's frame.

Hint: make a drawing and identify the direction of the boost. Then use the explicit form of this boost.

Solution

Let $\parallel$ be the direction of $\vec{p}_A + \vec{p}'_A$, and $\perp$ the direction of $\vec{p}_A - \vec{p}'_A$.
One should perform a boost along the direction $\parallel$, such that $\vec{p}_{A\parallel}$ and $\vec{p}'_{A\parallel}$ vanish, while their $\perp$ components remain identical to their value in the center-of-mass frame, and thus opposite. This will turn $\vec{p}_A$ and $\vec{p}'_{A\parallel}$ to be opposite, as it should be in Breit's frame.
This boost from the center-of-mass to the Breit's frame thus looks as follows:



It reads, for momentum p_A,

$$\begin{aligned} E_A &= \gamma E_A^* + \gamma\beta p_{A\parallel}^* , \\ p_{A\parallel} &= \gamma\beta E_A^* + \gamma p_{A\parallel}^* = 0 , \end{aligned}$$

and thus

$$\beta = -\frac{p_{A\parallel}^*}{E_A^*} .$$

For momentum p_B, the boost reads

$$\begin{aligned} E_B &= \gamma E_B^* + \gamma\beta p_{B\parallel}^* , \\ p_{B\parallel} &= \gamma\beta E_B^* + \gamma p_{B\parallel}^* . \end{aligned}$$

Thus, using the value of β and the fact that $p_{B\parallel}^* = -p_{A\parallel}^*$, we get

$$\begin{aligned} E_B &= \gamma \left(E_B^* + \frac{p_{A\parallel}^{*2}}{E_A^*} \right) , \\ p_{B\parallel} &= -\gamma p_{A\parallel}^* \left(\frac{E_B^*}{E_A^*} + 1 \right) . \end{aligned}$$

Besides, we have

$$
\begin{aligned}
p^*_{A\perp} &= p^*_A \sin\frac{\theta^*}{2}\,,\\
p^*_{A\parallel} &= p^*_A \cos\frac{\theta^*}{2}\,,
\end{aligned}
$$

and of course $p_A^2 = p_{A\perp}^2 + p_{A\parallel}^2$ and $p_B^2 = p_{B\perp}^2 + p_{B\parallel}^2$ with $p_{A\perp} = p^*_{A\perp}$ and $p_{B\perp} = p^*_{B\perp} = -p^*_{A\perp}$. Thus,

$$
E_B = \gamma\left(E^*_B + \frac{p^{*2}_A}{E^*_A}\cos^2\frac{\theta^*}{2}\right)
$$

and

$$
p_B = p'_B = p^*_A\sqrt{\sin^2\frac{\theta^*}{2} + \gamma^2\cos^2\frac{\theta^*}{2}\left(\frac{E^*_B}{E^*_A}+1\right)^2}
$$

Note that equivalently, using $E_B = \sqrt{m_B^2 + p_B^2}$, one can also write

$$
E_B = E'_B = \sqrt{E^{*2}_B + \left[\left(1+\frac{E^*_B}{E^*_A}\right)^2\gamma^2 - 1\right]p^{*2}_A\cos^2\frac{\theta^*}{2}}\,, \tag{24}
$$

The direct connexion between these two formula for E_B is not obvious and requires to use the explicit expression of γ, namely $\gamma = 1/\sqrt{1 - p^{*2}_A/E^{*2}_A\cos^2\theta^*/2}$ in order to relate them.

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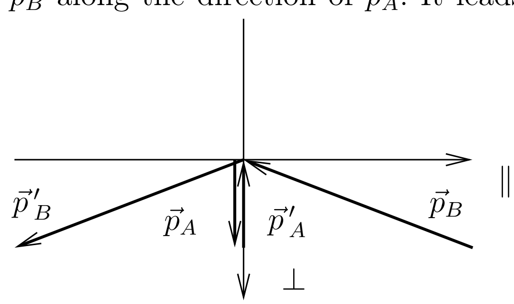
5. Study the relative position of momenta $\vec{p}_B$ and $\vec{p}_B^{\,\prime}$ with respect to the momenta $\vec{p}_A$ and $\vec{p}_A^{\,\prime}$, and justify the name "wall reference frame" given to the $\mathcal{B}$ reference frame.

Solution

The answer is obvious from the boost considered in the previous question, and the above figure.
Still, let us exhibit directly the geometry of the scattering in the Breit's frame, without using this boost from the center-of-mass frame. Just like in the previous question, for any momentum $\vec{p}$, denote $\vec{p}_\perp$ the component along $\vec{p}_A$ and $\vec{p}_\parallel$ the component orthogonal to $\vec{p}_A$. Since $\vec{p}_A + \vec{p}_B = \vec{p}^{\,\prime}_A + \vec{p}^{\,\prime}_B$, and $\vec{p}^{\,\prime}_A = -\vec{p}_A$, we have $\vec{p}^{\,\prime}_B = \vec{p}_B + 2\vec{p}_A$, which implies that $\vec{p}^{\,\prime}_{B\parallel} = \vec{p}_{B\parallel}$ and $\vec{p}^{\,\prime}_{B\perp} = \vec{p}_{B\perp} + 2\vec{p}_A$. From the fact that $p_B = p'_B$ one should thus have $\vec{p}^{\,\prime}_{B\perp} = \pm\vec{p}_{B\perp}$. Obviously, $\vec{p}^{\,\prime}_{B\perp} = \vec{p}_{B\perp}$ is impossible for a non vanishing $\vec{p}_A$, so that $\vec{p}^{\,\prime}_{B\perp} = -\vec{p}_{B\perp} = \vec{p}_{B\perp} + 2\vec{p}_A$, i.e. $\vec{p}_{B\perp} = -\vec{p}_A$. In the Breit frame the scattering thus looks like a scattering on a wall perpendicular to $\vec{p}_A$ direction: $\vec{p}_A$ get reversed, as well as the component of $\vec{p}_B$ along the direction of $\vec{p}_A$. It leads to the following geometry:

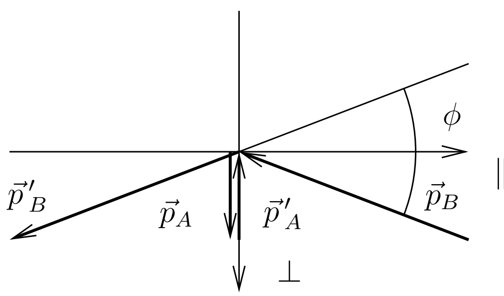


6. Calculate the deflection angle of each particle as a function of the modulus of momenta. One should in particular prove that the deflection angle ϕ of particle B is given by

$$\cos\phi = 1 - 2\frac{p_A^2}{p_B^2}\,. \tag{25}$$

Solution

The deflection angle of particle A is obviously $(\vec{p}_A, \vec{p}_A') = \pi$. Besides, from the following figure:



we see that $\sin\frac{\phi}{2} = \frac{p_A}{p_B}$ and thus $\cos\phi = 1 - 2\sin^2\frac{\phi}{2} = 1 - 2\frac{p_A^2}{p_B^2}$.