Particles

Exam

January 4th 2023

Documents allowed

Notes:
- One may use the usual system of units in which $c = 1$ and $\hbar = 1$.
- Space coordinates may be freely denoted as (x, y, z) or (x^1, x^2, x^3).
- Any drawing, at any stage, is welcome, and will be rewarded!

1 Study of the decay $\pi^0 \to \gamma\gamma$

1. Express the angle θ between the momenta of the two photons in the reaction $\pi^0 \to \gamma\gamma$ as a function of their energies and of the π^0 mass.

Hint: compute the scalar product of the 3-momenta of the two photons.

2. Compute separately $\cos\theta_1$ and $\cos\theta_2$, where θ_1 and θ_2 are the angle of the 3-momenta of each photon with respect to the direction of the incoming pion. One should obtain

$$\cos\theta_i = \frac{E - m^2/(2E_i)}{\sqrt{E^2 - m^2}} \, . \tag{1}$$

3. From the above expression, after computing $\sin\theta_i$, finally check your result for $\cos\theta$.

4. Detailed kinematics of the two photons

(i) Study in detail the variation of θ as a function of the fraction of the total energy carried by one of the photon. Give in particular the minimal value θ_{min} of this relative angle.

(ii) Discuss the range of energy covered by each photon.

5. Discuss the two extreme limits $E = m$ and $E \gg m$.

2 Noether theorem

2.1 Current associated to Lagrangians independent of the fields

2.1.1 Scalar case

1. Consider the Lagrangian of a real massless scalar field.

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) \, . \tag{2}$$

(i) Write the Noether current associated to the transformation

$$\phi \to \phi + \alpha \tag{3}$$

where α is a constant, and explain why $j^\mu = \partial^\mu \phi$ is conserved.

(ii) Check directly that this current is conserved.

2. Suppose that the Lagrangian contains a mass term, i.e.

$$\mathcal{L}_m = \frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) - \frac{1}{2} m^2 \phi^2 \,. \tag{4}$$

(i) What appends to the above current?

(ii) Compute its derivative in terms of ϕ and m. Comment.

2.1.2 The case of QED

3. In the case of QED for free photons without matter, we know that the Lagrangian reads

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu}\, F^{\mu\nu} \tag{5}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{6}$$

(i) By considering the global transformation

$$\begin{aligned} \delta x^\mu &= 0 \\ \delta A^\mu(x) &= \text{constant} = \delta A^\mu \,, \end{aligned} \tag{7}$$

show that the current

$$\frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\nu)} \tag{8}$$

is conserved.

(ii) Deduce that $F^{\mu\nu}$ is conserved. Comment.

4. The QED Lagrangian of photons coupled to an external current reads

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu}\, F^{\mu\nu} - j^\mu A_\mu \,. \tag{9}$$

(i) What appends to the above current?

(ii) Compute its derivative. Comment.

2.2 Multiple symmetry generators

1. Preliminary

Consider the set $U(N)$, made of $N \times N$ matrices with complex coefficients satisfying

$$U^\dagger \cdot U = U \cdot U^\dagger = \mathrm{Id} \tag{10}$$

where Id is the $N \times N$ identity matrix.

(i) Show that the determinant of these matrices is a phase factor.

(ii) Consider a matrix U of $U(N)$, expanded in the vicinity of Id. For convenience, this expansion is written in the form

$$U = \mathrm{Id} + i\,T + o(T) \tag{11}$$

where $\|T\| \ll 1$ (the precise definition of this norm plays no role here, one should just interpret this as T small with respect to Id).

Show that the matrices T are hermitian.

(iii) Show that there are N^2 independent $N \times N$ hermitian matrices. In the rest of this exercise, they will be labeled by an index $a \in \{1, \cdots, N^2\}$. A given chosen set of N^2 independent T matrices is called a set of $U(N)$ generators.

(iv) The subset $SU(N)$ of $U(N)$ matrices is made of matrices of determinant unity. Besides, one can show that for any $N \times N$ (diagonalizable) matrix X,

$$\det(\mathrm{Id} + \epsilon X) = 1 + \epsilon\,\mathrm{Tr}X + o(\epsilon)\,. \tag{12}$$

Deduce the constraint which should be satisfied by the generators of $SU(N)$, and then the number of generators of $SU(N)$.

2. One can prove that in the case of $U(N)$ (this is valid for any compact group), the whole connected component of Id can be obtained by exponentiating a suitable linear combination of the generators or $U(N)$. It means that any $U(N)$ matrix which belongs to the connected component of Id reads

$$U = e^{i\omega^a T^a} \tag{13}$$

where ω^a are N^2 real numbers, and T^a are the generators.

Consider the Lagrangian

$$\mathcal{L} = (\partial_\mu \Phi)^\dagger \partial_\mu \Phi - m^2 \Phi^\dagger \Phi \tag{14}$$

where

$$\Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_N \end{pmatrix} \tag{15}$$

is a column vector made of N complex scalar fields.

(i) Show that the Lagrangian is symmetric under the variation

$$\begin{aligned} \Phi &\rightarrow e^{i\omega^a T^a}\Phi & (16) \\ \Phi^\dagger &\rightarrow \Phi^\dagger e^{-i\omega^a T^a} & (17) \end{aligned}$$

(ii) Write the corresponding set of N^2 conserved currents.

(iii) Discuss the special case $N = 1$.

3. Using the fact that each field φ_i can be decomposed into its real and imaginary part, one can rewrite, adapting the notation accordingly,

$$\Phi = \begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \\ \vdots \\ \varphi_{2N-1} + i\varphi_{2N} \end{pmatrix} \tag{18}$$

(i) Show that the Lagrangian can be rewritten as

$$\mathcal{L} = (\partial_\mu \tilde{\Phi})^T \partial^\mu \tilde{\Phi} - m^2 \tilde{\Phi}^T \tilde{\Phi} \tag{19}$$

with

$$\tilde{\Phi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_{2N} \end{pmatrix} . \tag{20}$$

(ii) Deduce that the symmetry of the Lagrangian is in fact $O(2N)$.

(iii) Repeating the above discussion made for $U(N)$, see question 1., characterize the generators of $O(2N)$ and find their number. Write the corresponding Noether currents.