M1 General Physics **2022-2023**

Particles

Exam

January 4th 2023

Documents allowed

Notes:
- One may use the usual system of units in which $c = 1$ and $\hbar = 1$.
- Space coordinates may be freely denoted as (x, y, z) or (x^1, x^2, x^3).
- Any drawing, at any stage, is welcome, and will be rewarded!

1 Study of the decay $\pi^0 \to \gamma\gamma$

1. Express the angle θ between the momenta of the two photons in the reaction $\pi^0 \to \gamma\gamma$ as a function of their energies and of the π^0 mass.

Hint: compute the scalar product of the 3-momenta of the two photons.

Solution

The conservation of momentum implies that

$$\vec{p} = \vec{p}_1 + \vec{p}_2 \,.$$

Besides, denoting the relative angle between the two photons as θ, we have

$$\vec{p}^{\,2} = \vec{p}_1^{\,2} + \vec{p}_2^{\,2} + \vec{p}_1 \cdot \vec{p}_2 = E_1^2 + E_2^2 + 2E_1E_2 \cos\theta$$

where we have used the fact that $\|\vec{p}_1\| = E_1$ and $\|\vec{p}_2\| = E_2$. Since

$$\vec{p}^{\,2} = E^2 - m^2 = E_1^2 + E_2^2 + 2E_1E_2 - m^2$$

one finally gets

$$\cos\theta = \frac{2E_1E_2 - m^2}{2E_1E_2} \,.$$

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2. Compute separately $\cos\theta_1$ and $\cos\theta_2$, where θ_1 and θ_2 are the angle of the 3-momenta of each photon with respect to the direction of the incoming pion. One should obtain

$$\cos\theta_i = \frac{E - m^2/(2E_i)}{\sqrt{E^2 - m^2}} \,. \tag{1}$$

Solution

Indirect solution:
The projection of the momenta on the π^0 axis gives

$$E_1 \cos\theta_1 + E_2 \cos\theta_2 = \|\vec{p}\| = \sqrt{E^2 - m^2}$$

while the conservation of momentum on the transverse axis gives

$$E_1 \sin\theta_1 = E_2 \sin\theta_2 \,.$$

Thus, one gets

$$\cos^2\theta_2 = 1 - \sin^2\theta_2 = 1 - \frac{E_1^2}{E_2^2}\sin^2\theta_1$$

so that (among the two photons, at least one of them should have a positive $\cos\theta_i$, due to overall conservation of momenta along the axis of the π^0, so let us label photon 2 to be the one with $\cos\theta_2 \geq 0$)

$$\cos\theta_2 = \frac{\sqrt{E_2^2 - E_1^2\sin^2\theta_1}}{E_2} = \frac{\sqrt{E_2^2 - E_1^2(1-\cos^2\theta_1)}}{E_2} \,.$$

This leads to

$$E_1\cos\theta_1 + \sqrt{E_2^2 - E_1^2(1-\cos^2\theta_1)} = \sqrt{E^2 - m^2}$$

or

$$\sqrt{E_2^2 - E_1^2(1-\cos^2\theta_1)} = \sqrt{E^2 - m^2} - E_1\cos\theta_1$$

which after squaring gives

$$E_2^2 - E_1^2(1-\cos^2\theta_1) = E^2 - m^2 + E_1^2\cos^2\theta_1 - 2\sqrt{E^2 - m^2}\,E_1\cos\theta_1$$

and thus, using $E = E_1 + E_2$,

$$E_2^2 - E_1^2 = E_1^2 + E_2^2 + 2E_1E_2 - m^2 + E_1^2\cos^2\theta_1 - 2\sqrt{E^2 - m^2}\,E_1\cos\theta_1$$

which leads finally to

$$\cos\theta_1 = \frac{E - m^2/(2E_1)}{\sqrt{E^2 - m^2}} \,.$$

More direct method:
since $p - p_1 = p_2$,

$$(p - p_1)^2 = p^2 - 2p\cdot p_1 + p_1^2 = m^2 - 2EE_1 + 2\|\vec{p}\|E_1\cos\theta_1 = p_2^2 = 0$$

and thus

$$m^2 - 2EE_1 + 2\sqrt{E^2 - m^2}\,E_1\cos\theta_1 = 0 \,.$$

This implies finally that

$$\cos\theta_1 = \frac{E - m^2/(2E_1)}{\sqrt{E^2 - m^2}}\,.$$

Similarly, exchanging the role of photon 1 and 2, one gets

$$\cos\theta_2 = \frac{E - m^2/(2E_2)}{\sqrt{E^2 - m^2}}\,.$$

Direct solution:
From $p_2 = p - p - 1$ one gets

$$p_2^2 = 0 = (p - p_1)^2 = m^2 - 2p\cdot p_1 = m^2 - 2EE_1 + 2\|\vec{p}\|E_1\cos\theta_1 = 2\sqrt{E^2 - m^2}E_1\cos\theta_1$$

and thus

$$\cos\theta_1 = \frac{E - m^2/(2E_1)}{\sqrt{E^2 - m^2}}\,.$$

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3. From the above expression, after computing $\sin\theta_i$, finally check your result for $\cos\theta$.

Solution

From the above expression obtained for $\cos\theta_i$, one gets

$$\sin\theta_i = \sqrt{1 - \cos^2\theta_i} = \left[1 - \frac{(E - m^2/(2E_i))^2}{E^2 - m^2}\right]^{1/2} = \sqrt{\frac{4E_1E_2 - m^2}{E^2 - m^2}}\frac{m}{2E_i}\,.$$

Thus,

$$\begin{aligned}
\cos(\theta_1 + \theta_2) &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \\
&= \frac{1}{E^2 - m^2}\left[\left(E - \frac{m^2}{2E_1}\right)\left(E - \frac{m^2}{2E_2}\right) - (4E_1E_2 - m^2)\frac{m^2}{4E_1E_2}\right] \\
&= \frac{1}{(E^2 - m^2)4E_1E_2}\left[(2E_1E - m^2)(2E_2E - m^2) - (4E_1E_2 - m^2)m^2\right] \\
&= \frac{1}{(E^2 - m^2)4E_1E_2}\left[4E_1E_2E^2 - 2m^2E^2 - 4E_1E_2m^2 + 2m^4\right] \\
&= \frac{2E_1E_2 - m^2}{2E_1E_2}
\end{aligned}$$

as expected.

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4. Detailed kinematics of the two photons

(i) Study in detail the variation of θ as a function of the fraction of the total energy carried by one of the photon. Give in particular the minimal value θ_{min} of this relative angle.

(ii) Discuss the range of energy covered by each photon.

Solution

Let us introduce the fraction x of the total energy carried by photon 1, so that $E_1 = xE$ and $E_2 = (1-x)E$. Thus,

$$\cos\theta = 1 - \frac{m^2}{E^2}\frac{1}{2x(1-x)}$$

Introducing $y = x(1-x)$, $\cos\theta$ is clearly an increasing function y. It is thus maximal for $y = 1/4$, i.e. $x = 1/2$. The maximal value of $\cos\theta$ is then $c = 1 - 2\frac{m^2}{E^2}$, so that the minimal angle between the two photons is

$$\theta_{min} = \arccos\left(1 - 2\frac{m^2}{E^2}\right).$$

One gets the following variations:

x	0	x_+	$1/2$	x_-	1
y	0		$1/4$		0
$\cos\theta$	$\nearrow$	$-1 \nearrow$	c	$\searrow -1$	$\searrow$

Indeed, $\cos\theta$ should be in the interval $[-1, 1)$. The upper constraint is obviously satisfied. The lower one gives

$$1 - \frac{m^2}{E^2}\frac{1}{2x(1-x)} \geq -1$$

i.e.

$$x^2 - x + \frac{m^2}{4E^2} \leq 0\,,$$

so that $x \in [x_-, x_+]$ with

$$x_\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{m^2}{E^2}}\,.$$

Thus, both E_1 and E_2 are in the range $[x_-E, x_+E]$. When the border of this domain is reached, $\theta = \pi$: the two photons are emitted back-to-back.

5. Discuss the two extreme limits $E = m$ and $E \gg m$.

Solution

Case $E = m$:
In this case, we are in the CMS of the π^0. Thus one gets $\theta = \pi$. Indeed, inspecting the equation written in question (2) (the one before using $E = E_1 + E_2$) shows that $E_1 = E_2$, thus $E_1 = E_2 = m/2$ so that $\cos\theta = -1$: the two photons share the energy and are emitted back-to-back.

Case $E \gg m$:
From the relation

$$\cos\theta = 1 - \frac{m^2}{E^2}\frac{1}{2x(1-x)}$$

one immediately gets that $\cos\theta \to 1$: the two photon are emitted collinearly, in the direction of the decaying π^0.

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2 Noether theorem

2.1 Current associated to Lagrangians independent of the fields

2.1.1 Scalar case

1. Consider the Lagrangian of a real massless scalar field.

$$\mathcal{L} = \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi)\,. \tag{2}$$

(i) Write the Noether current associated to the transformation

$$\phi \to \phi + \alpha \tag{3}$$

where α is a constant, and explain why $j^\mu = \partial^\mu\phi$ is conserved.

Solution

The Lagrangian (2) is obviously invariant under the transformation (3). Thus, the Noether current

$$j^\mu = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi = (\partial^\mu\phi)\,\alpha$$

is conserved, and since this is valid for any constant α, this implies that

$$j^\mu = \partial^\mu\phi$$

is conserved.

(ii) Check directly that this current is conserved.

Solution

One gets

$$\partial_\mu j^\mu = \Box\phi = 0$$

after using the Euler-Lagrange equation which is just the Klein-Gordon equation.

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2. Suppose that the Lagrangian contains a mass term, i.e.

$$\mathcal{L}_m = \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{1}{2}m^2\phi^2\,. \tag{4}$$

(i) What appends to the above current?

Solution

With a mass term, the Lagrangian is not anymore invariant under the transformation (3), and thus the Noether current is not anymore conserved.

(ii) Compute its derivative in terms of ϕ and m. Comment.

Solution

One has

$$\partial_\mu j^\mu = \Box\phi = -m^2\phi$$

after using the Euler-Lagrange equation which now reads

$$\Box\phi + m^2\phi = 0\,.$$

Obviously, as expected this vanishes in the limit $m = 0$.

2.1.2 The case of QED

3. In the case of QED for free photons without matter, we know that the Lagrangian reads

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}\,F^{\mu\nu} \tag{5}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{6}$$

(i) By considering the global transformation

$$\begin{aligned} \delta x^\mu &= 0 \\ \delta A^\mu(x) &= \text{constant} = \delta A^\mu\,, \end{aligned} \tag{7}$$

show that the current

$$\frac{\delta\mathcal{L}}{\delta(\partial_\mu A_\nu)} \tag{8}$$

is conserved.

Solution

The Lagrangian is invariant under the transformation (7). Thus, the current

$$j^\mu = \frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\nu)} \delta A_\nu$$

is conserved, for any constant δA^μ. This thus leads to the conservation of the current

$$\frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\nu)} \,.$$

(ii) Deduce that $F^{\mu\nu}$ is conserved. Comment.

Solution

The antisymmetry of $\partial_\mu A_\nu - \partial_\nu A_\mu$ allows to rewrite (5) as

$$\mathcal{L}_{QED} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) = -\frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\partial^\mu A^\nu \,,$$

which leads to

$$\frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\nu)} = -F^{\mu\nu} \,.$$

The conservation of the current (8) then reads

$$\partial_\mu F^{\mu\nu} = 0 \,,$$

which is nothing more than the first set of Maxwell's equations in the vacuum.

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4. The QED Lagrangian of photons coupled to an external current reads

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu \,. \tag{9}$$

(i) What appends to the above current?

Solution

With the presence of a term involving the coupling of an external current to the field A_μ, the Lagrangian is not anymore invariant under the transformation (7), and thus the Noether current is not anymore conserved.

(ii) Compute its derivative. Comment.

Solution

One has

$$-\partial_\mu F^{\mu\nu} = -j^\nu$$

after using the Euler-Lagrange equation which are just the first set of Maxwell's equation. Obviously, as expected this vanishes in the limit $j^\mu = 0$.

2.2 Multiple symmetry generators

1. Preliminary

Consider the set $U(N)$, made of $N \times N$ matrices with complex coefficients satisfying

$$U^\dagger \cdot U = U \cdot U^\dagger = \mathrm{Id} \tag{10}$$

where Id is the $N \times N$ identity matrix.

(i) Show that the determinant of these matrices is a phase factor.

Solution

From (10) one gets

$$|\det U|^2 = 1$$

which shows that $|\det U| = 1$ and thus that $\det U$ is phase factor.

(ii) Consider a matrix U of $U(N)$, expanded in the vicinity of Id. For convenience, this expansion is written in the form

$$U = \mathrm{Id} + i\,T + o(T) \tag{11}$$

where $\|T\| \ll 1$ (the precise definition of this norm plays no role here, one should just interpret this as T small with respect to Id).

Show that the matrices T are hermitian.

Solution

From (10) one gets

$$(\mathrm{Id} + iT + o(T))(\mathrm{Id} - iT^\dagger + o(T)) = \mathrm{Id} + i(T - T^\dagger) + o(T) = \mathrm{Id}$$

so that $T = T^\dagger$, hence the result.

(iii) Show that there are N^2 independent $N \times N$ hermitian matrices. In the rest of this exercise, they will be labeled by an index $a \in \{1, \cdots, N^2\}$. A given chosen set of N^2 independent T matrices is called a set of $U(N)$ generators.

Solution

A hermitian matrix is completely fixed by the value of $(N-1)N/2$ complex coefficients (the non-diagonal terms) and N real coefficients (the diagonal terms). Since any complex number is a set of two real numbers (its real and imaginary parts), this means $(N-1)N + N = N^2$ real coefficients.

(iv) The subset $SU(N)$ of $U(N)$ matrices is made of matrices of determinant unity. Besides, one can show that for any $N \times N$ (diagonalizable) matrix X,

$$\det(\mathrm{Id} + \epsilon X) = 1 + \epsilon\, \mathrm{Tr} X + o(\epsilon)\,. \tag{12}$$

Deduce the constraint which should be satisfied by the generators of $SU(N)$, and then the number of generators of $SU(N)$.

Solution

The constraint $\det U = 1$ obviously leads to $\mathrm{Tr}\, X = 0$. This adds one constraint on the real coefficients fixing the value of X, so that there are $N^2 - 1$ independent generators of $SU(N)$.

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2. One can prove that in the case of $U(N)$ (this is valid for any compact group), the whole connected component of Id can be obtained by exponentiating a suitable linear combination of the generators or $U(N)$. It means that any $U(N)$ matrix which belongs to the connected component of Id reads

$$U = e^{i\omega^a T^a} \tag{13}$$

where ω^a are N^2 real numbers, and T^a are the generators.

Consider the Lagrangian

$$\mathcal{L} = (\partial_\mu \Phi)^\dagger \partial_\mu \Phi - m^2 \Phi^\dagger \Phi \tag{14}$$

where

$$\Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_N \end{pmatrix} \tag{15}$$

is a column vector made of N complex scalar fields.

(i) Show that the Lagrangian is symmetric under the variation

$$\Phi \;\to\; e^{i\omega^a T^a}\Phi \tag{16}$$

$$\Phi^\dagger \;\to\; \Phi^\dagger e^{-i\omega^a T^a} \tag{17}$$

Solution

This is obvious from the definition of $U(N)$.

(ii) Write the corresponding set of N^2 conserved currents.

Solution

Consider the infinitesimal transformations

$$\delta\Phi = i\omega^a T^a \Phi \tag{18}$$
$$\delta\Phi^\dagger = -\Phi^\dagger i\omega^a T^a \tag{19}$$

where $\|\omega\| \ll 1$. The Noether theorem reads

$$\begin{aligned} j^\mu &= \frac{\delta\mathcal{L}}{\delta(\partial_\mu\Phi)}\delta\Phi + \delta\Phi^\dagger\frac{\delta\mathcal{L}}{\delta(\partial_\mu\Phi^\dagger)} \\ &= (\partial^\mu\Phi^\dagger)(i\omega^a T^a\Phi) - (\Phi^\dagger i\omega^a T^a)(\partial^\mu\Phi) \\ &= -i(\Phi^\dagger T^a\partial^\mu\Phi - (\partial^\mu\Phi^\dagger)T^a\Phi)\omega^a \end{aligned}$$

which implies that the family of N^2 currents

$$j^{a\mu} = -i(\Phi^\dagger T^a\partial^\mu\Phi - (\partial^\mu\Phi^\dagger)T^a\Phi)$$

are conserved.

(iii) Discuss the special case $N = 1$.

Solution

When $N = 1$ we recover the usual $U(1)$ current

$$j^\mu = -i(\Phi^*\partial^\mu\Phi - (\partial^\mu\Phi^*)\Phi)$$

since there is just one generator, the number 1 which is the only 1×1 hermitian matrix.

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3. Using the fact that each field φ_i can be decomposed into its real and imaginary part, one can rewrite, adapting the notation accordingly,

$$\Phi = \begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \\ \vdots \\ \varphi_{2N-1} + i\varphi_{2N} \end{pmatrix} \tag{20}$$

(i) Show that the Lagrangian can be rewritten as

$$\mathcal{L} = (\partial_\mu\tilde{\Phi})^T\partial^\mu\tilde{\Phi} - m^2\tilde{\Phi}^T\tilde{\Phi} \tag{21}$$

with

$$\tilde{\Phi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_{2N} \end{pmatrix}. \tag{22}$$

Solution

This is obvious from the fact that $\varphi_{2i-1}^2 + \varphi_{2i}^2 = |\varphi_{2i-1} + i\varphi_{2i}|^2$.

(ii) Deduce that the symmetry of the Lagrangian is in fact $O(2N)$.

Solution

This comes from the fact that $O(2N)$ is the set of transformation which leaves the norm of $\tilde{\Phi}$ invariant. This also leaves the norm of $\partial_\mu \tilde{\Phi}$ invariant.

(iii) Repeating the above discussion made for $U(N)$, see question 1., characterize the generators of $O(2N)$ and find their number. Write the corresponding Noether currents.

Solution

A Taylor expansion of the constraint

$$A^T \cdot A = A \cdot A^T = \mathrm{Id}$$

gives now

$$T + T^T = 0$$

i.e. the generators are made of $(2N) \times (2N)$ antisymmetric matrices. These are fixed by the knowledge of $(2N)(2N-1)/2 = N(2N-1)$ real coefficients. Denoting as X^a a set of $N(2N-1)$ independent $(2N) \times (2N)$ antisymmetric matrices, the Noether currents now read

$$j^{a\mu} = -i\tilde{\Phi}^T X^a \partial^\mu \tilde{\Phi}.$$
